MAS451 Tutorial Exercises

1. The random variable X has PDF

$$f(x) = \begin{cases} \sin(x), & 0 \le x \le \pi/2, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Derive a transformation method for simulating values of X based on $U(0,1)$ random variates.

   (b) Derive a uniform rejection method for simulating values from X. What is the acceptance probability?

   (c) Derive an envelope rejection method for simulating values of X based on a proposal with density

$$g(x) = \begin{cases} kx, & 0 \le x \le \pi/2, \\ 0, & \text{otherwise,} \end{cases}$$

   for some fixed k. You should use the fact that $\sin(x) \le x, \ \forall x > 0$. What is the acceptance probability?

2. (a) Show that the product of two stochastic matrices is stochastic.

   *(b) Show that for stochastic P, and *any* row vector π, we have $\|\pi P\|_1 \le \|\pi\|_1$, where $\|v\|_1 = \sum_i |v_i|$. Deduce that all eigenvalues, λ, of P must satisfy $|\lambda| \le 1$. As an aside, note that as a consequence of (a), it is clear that for all *probability* row vectors, π, we have $\|\pi P\|_1 = \|\pi\|_1$.

   **(c) Show that a stochastic matrix always has a row eigenvector with eigenvalue 1. Show that this row eigenvector can be chosen to correspond to a stationary distribution of the induced Markov chain. *Hint: It is an easy consequence of the Brouwer fixed point Theorem, which says, inter alia, that any continuous map from a compact convex set to itself must have a fixed point.*

3. For the AR(1) process

$$Z_t = \alpha Z_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim Exp(\lambda), \quad \alpha \in (0,1), \ \lambda > 0,$$

   (a) what is the transition kernel, $p(x, y)$, of the chain?

   *(b) Hence deduce an integral equation satisfied by the stationary distribution, $\pi(x)$.

   ***(c) Show that if Z has density $\pi(x)$, then the moments of Z satisfy the recurrence relation

$$\mathrm{E}(Z^n) = \frac{n \, \mathrm{E}(Z^{n-1})}{\lambda(1-\alpha^n)}$$

   and hence are given by

$$\mathrm{E}(Z^n) = \frac{n!}{\lambda^n \prod_{k=1}^{n} (1-\alpha^k)}.$$

**(d) Using the MA(∞) representation of the AR(1) process, and the MGF of an exponential, show that the MGF of the stationary distribution can be written

$$m(t) = \prod_{k=0}^{\infty} \frac{\lambda}{\lambda - \alpha^k t}.$$

*(e) Is this chain reversible? *Hint: You don't need to work out the stationary density!*

4. Identify the full conditionals of the bivariate density

$$\pi(x, y) \propto x^2 \exp\{-xy^2 - y^2 + 2y - 4x\}, \quad x > 0, y \in \mathbb{R}$$

and use them to construct a Gibbs sampler which has this stationary distribution.

5. Let X be as for question 1.

   (a) i. Construct a Metropolis-Hastings independence sampler based on a $U(0, \pi/2)$ proposal.

   *ii. If the chain is currently at θ, what is the probability that the chain will move (unconditional on the proposed value)?

   **iii. When the chain reaches equilibrium, use the previous result to show that the overall acceptance probability of the chain is $2(\pi - 2)/\pi$.

   (b) i. Construct a Metropolis-Hastings independence sampler for X based on the proposal from 1.(c).

   **ii. If the chain is currently at θ, what is the probability that the chain will move?

   **iii. When the chain reaches equilibrium, show that the overall acceptance probability of the chain is $8(\pi - 2)/\pi^2$.

   (c) i. Construct a Metropolis random walk sampler for X based on $U(-\varepsilon, \varepsilon)$ innovations ($0 < \varepsilon < \pi/4$).

   ***ii. If the chain is currently at θ, what is the probability that the chain will move?

   ***iii. When the chain reaches equilibrium, show that the overall acceptance probability of the chain is

$$1 - \frac{1 - \cos(\varepsilon)}{\varepsilon}.$$

   iv. In practice, how would you go about choosing ε?