

## **Chapter #2**

**EEE 2002**

### **Automatic Control**

- **Transfer functions**
- **Pole location and s-plane**
- **Time domain characteristics**
- **Extra poles and zeros**

## Transfer functions

### Laplace Transform

Used only on LTI systems

Differential expression => Polynomial expression

$$\frac{dy(t)}{dt} - y(t) = 10 \Rightarrow sY(s) - Y(s) = 10$$

The LT is transforming a DE from the time domain (domain is a set of values that describe a function, in that case the variable is the time) to another complex domain (i.e. the variable has a real and imaginary part).

$$F(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt, \quad s = \sigma \pm j\omega$$

Use formula tables => Easier

Properties:

#### 1. Differentiation

$$a. \quad L\left\{\frac{df(dt)}{dt}\right\} = sF(s) - f(0)$$

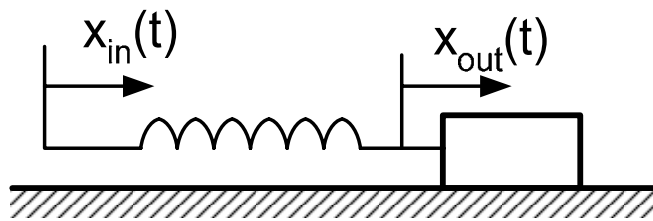
$$b. \quad L\left\{\frac{d^2 f(dt)}{dt^2}\right\} = s^2 F(s) - sf(0) - \frac{df(0)}{dt}$$

2. Final value theorem  $f_{ss} = \lim_{s \rightarrow 0} sF(s)$ , where  $f_{ss}$  is the value of  $f(t)$  after infinite time.

### Transfer functions

The ratio of the Laplace transform of the output over the Laplace transform of the input.

Example:



$$\sum f = ma \Leftrightarrow f_{Spring} = m \frac{d^2 x_{out}}{dt^2}$$

$$K(x_{in} - x_{out}) = m \frac{d^2 x_{out}}{dt^2}$$

$$K(x_{in} - x_{out}) = m \frac{d^2 x_{out}}{dt^2} \xrightarrow[IC=0]{LT}$$

$$KX_{in}(s) - KX_{out}(s) = ms^2 X_{out}(s) \Leftrightarrow$$

$$KX_{in}(s) = X_{out}(s)(ms^2 + K) \Leftrightarrow$$

$$\frac{X_{out}(s)}{X_{in}(s)} = \frac{K}{ms^2 + K}$$

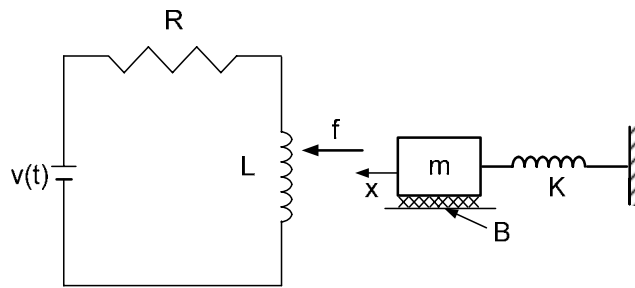
To find the characteristic equation of ODE:

$$Kx_{in} - Kx_{out} = m \frac{d^2 x_{out}}{dt^2} \Rightarrow Kx_{in} = m \frac{d^2 x_{out}}{dt^2} + Kx_{out}$$

The homogeneous system is:  $0 = m \frac{d^2 x_{out}}{dt^2} + Kx_{out}$  and therefore the **CE**

is  $mr^2 + K = 0$  i.e. exactly as the **denominator of TF**.

Example:



$$f = k_A \frac{i^2}{x^2} \Rightarrow f = k_A i$$

$$\left. \begin{array}{l} \frac{di}{dt} = \frac{1}{L}(v - iR) \\ k_A i - kx - B\dot{x} = m\ddot{x} \end{array} \right\} \Rightarrow \left. \begin{array}{l} sI(s) = \frac{1}{L}(V(s) - I(s)R) \\ k_A I(s) - kX(s) - BsX(s) = ms^2 X(s) \end{array} \right\}$$

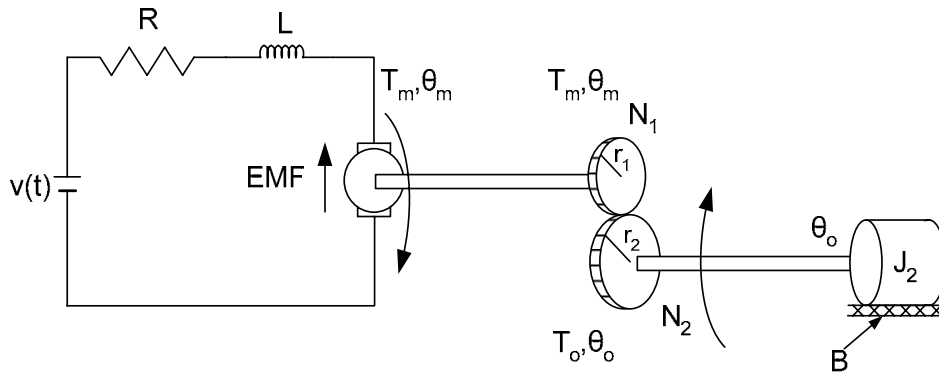
$$\left. \begin{array}{l} sLI(s) + I(s)R = V(s) \\ k_A I(s) - kX(s) - BsX(s) = ms^2 X(s) \end{array} \right\} \Rightarrow$$

$$\left. \begin{aligned} I(s) &= \frac{V(s)}{sL + R} \\ k_A \frac{V(s)}{sL + R} - kX(s) - BsX(s) &= ms^2 X(s) \end{aligned} \right\} \Rightarrow$$

$$k_A \frac{V(s)}{sL + R} = (ms^2 + Bs + k)X(s) \Rightarrow$$

$$\frac{X(s)}{V(s)} = \frac{k_A}{(sL + R)(ms^2 + Bs + k)}$$

Example:



$$\left. \begin{aligned} J \ddot{\theta}_0 &= T_0 - B \dot{\theta}_0 \\ T_0 &= \frac{n_2}{n_1} T_m \\ T_m &= K_T \phi i_a \end{aligned} \right\} \Rightarrow J \ddot{\theta}_0 = \frac{n_2}{n_1} K_T \phi i_a - B \dot{\theta}_0$$

$$\left. \begin{aligned} v_a &= i_a R_a + L_a \frac{di_a}{dt} + K_T \phi \dot{\theta}_m \Leftrightarrow v_a = i_a R_a + L_a \frac{di_a}{dt} + K_T \phi \frac{n_2}{n_1} \dot{\theta}_0 \end{aligned} \right\} \Rightarrow$$

$$\left. \begin{aligned}
 J \ddot{\theta}_0 &= \frac{n_1}{n_2} K_T \dot{i}_a - B \dot{\theta}_0 \\
 v_a &= i_a R_a + L_a \frac{di_a}{dt} + K_T \varphi \frac{n_2}{n_1} \dot{\theta}_0
 \end{aligned} \right\}$$

$$\begin{aligned}
 &\xrightarrow{LT} \left. \begin{aligned}
 Js^2 \Theta_0(s) &= K_2 I_a(s) - B \Theta_0(s) \\
 K_1 = K_T \varphi \frac{n_2}{n_1}, K_2 = K_T \varphi \frac{n_1}{n_2} V_a(s) &= I_a(s) R_a + L_a s I_a(s) + K_1 \Theta_0(s)
 \end{aligned} \right\}
 \end{aligned}$$

$$\left. \begin{aligned}
 I_a(s) &= \frac{(Js^2 + B)}{K_2} \Theta_0(s) \\
 V_a(s) &= (R_a + L_a s) I_a(s) + K_1 \Theta_0(s)
 \end{aligned} \right\}$$

$$\Rightarrow V_a(s) = (R_a + L_a s) \frac{(Js^2 + B)}{K_2} \Theta_0(s) + K_1 \Theta_0(s)$$

$$\frac{\Theta_0(s)}{V_a(s)} = \frac{1}{\left( (R_a + L_a s) \frac{(Js^2 + B)}{K_2} + K_1 \right)} =$$

$$\frac{K_2}{(R_a + L_a s)(Js^2 + B) + K_1 K_2}$$

**Matlab and TF**

- Define numerator
- Define denominator
- Use the command `tf( , )`

Example:

$$G(s) = \frac{5}{s + 2}$$

```
>> num=5;
>> den=[1 2];
>> g=tf(num,den)
```

Transfer function:

```
  5
-----
s + 2
```

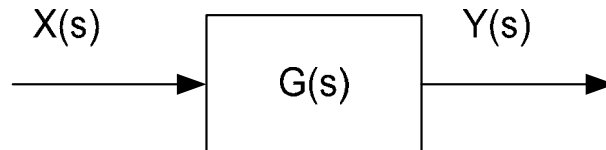
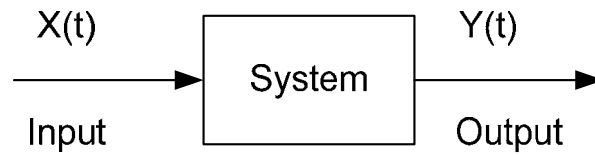
$$G(s) = \frac{s + 4}{s^2 + 3s - 6}$$

```
>> num=[1 2];
>> num=[1 4];
>> den=[1 3 -6];
>> g=tf(num,den)
```

Transfer function:

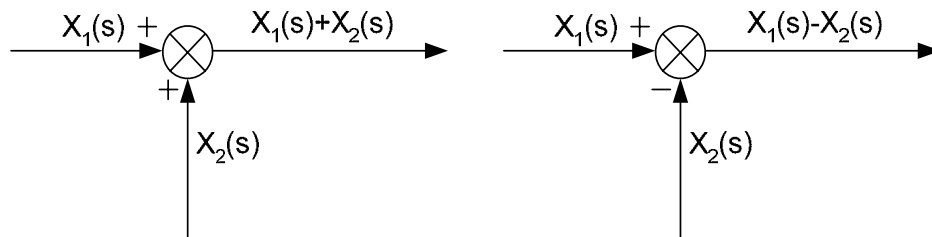
```
  s + 4
-----
s^2 + 3 s - 6
```

## Block diagrams

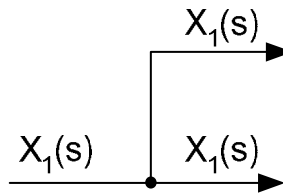


## Block Diagram Algebra

1. To sum (subtract) two signals, we use a summing point:

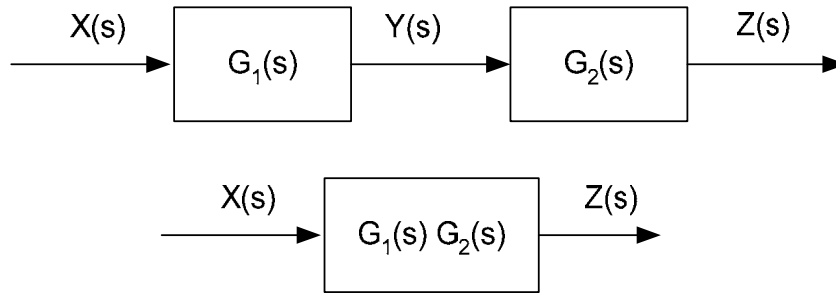


2. To “distribute” a signal, we use a branch point:

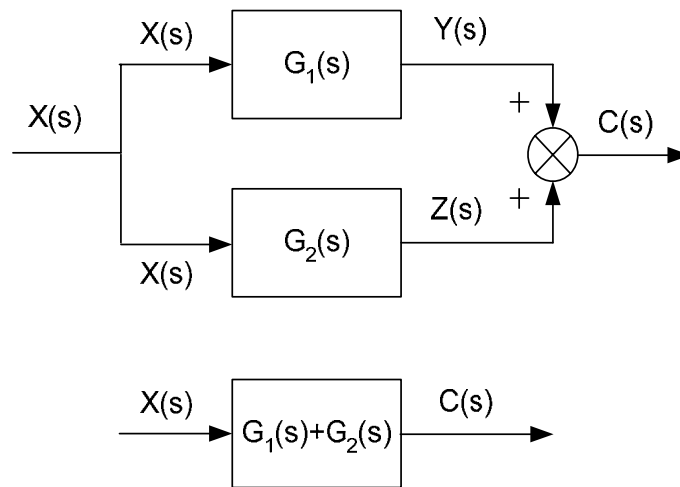




## 3. Series connection:



## 4. Parallel connection

**Matlab and block diagrams:**

```
>> num1=1;
>> den1=[1 1];
>> num2=[1 0];
>> den2=[1 5 6];
>> g1=tf(num1,den1)
```

Transfer function:

```
  1
-----
s + 1
>> g2=tf(num2,den2)
```

Transfer function:

$$\frac{s}{s^2 + 5s + 6}$$

```
>> g=series(g1,g2)
```

Transfer function:

$$\frac{s}{s^3 + 6s^2 + 11s + 6}$$

```
>> g=parallel(g1,g2)
```

Transfer function:

$$\frac{2s^2 + 6s + 6}{s^3 + 6s^2 + 11s + 6}$$

## Pole location / s-plane

In previous example

$$K(x_{in} - x_{out}) = m \frac{d^2 x_{out}}{dt^2} \xrightarrow[IC=0]{LT} \frac{X_{out}(s)}{X_{in}(s)} = \frac{K}{ms^2 + K}$$

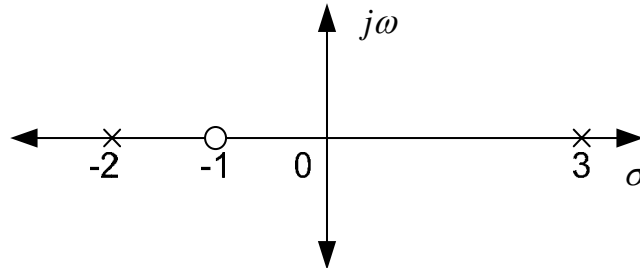
The order of the ODE is 2 = order of the denominator = order of the system.

Example  $\frac{Z(s)}{Y(s)} = \frac{K_1 K_2}{(m_1 s^2 + K_1 + K_2)(K_2 + m_2 s^2) - K_2^2} \Rightarrow \text{order} = 4.$

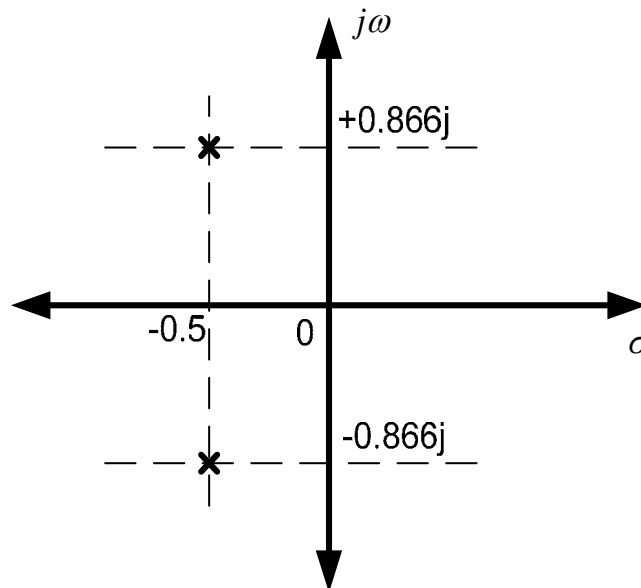
$G(s) = \frac{N(s)}{D(s)}$ : roots of the numerator are called zeros, while the roots of

the denominator are called poles.

$\frac{s+1}{(s+2)(s-3)}$  one zero at  $s=-1$  and two poles at  $s=-2$  and  $s=+3$ :



$$G(s) = \frac{1}{s^2 + s + 1}$$



To find the roots of a polynomial use the command `roots()`:

```
>> num=[1 4];
>> den=[1 3 -6];
>> roots(num)
```

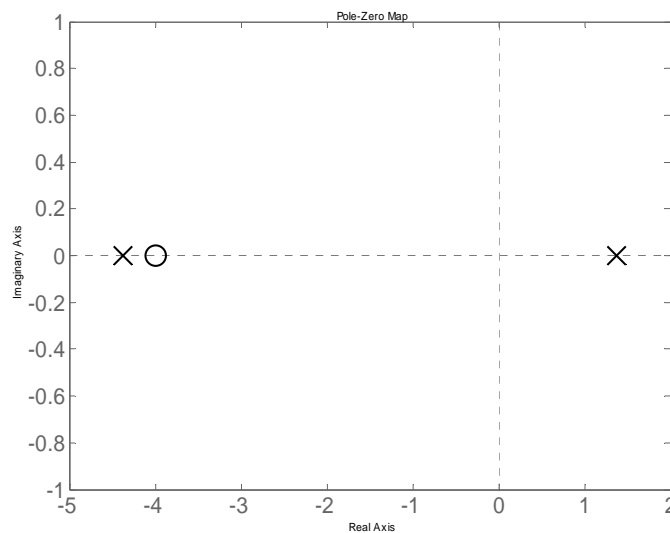
```
ans =  
    -4  
>> roots(den)
```

```
ans =  
  
    -4.3723  
     1.3723
```

To find and plot directly the poles/zeros use the command `pzmap()`:

```
>> num=[1 4];  
>> den=[1 3 -6];  
>> g=tf(num,den);  
>> pzmap(g)
```

```
>> [p,z]=pzmap(g)  
p =  
  
    -4.3723  
     1.3723  
z =  
    -4
```

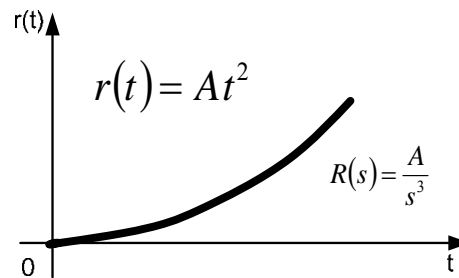
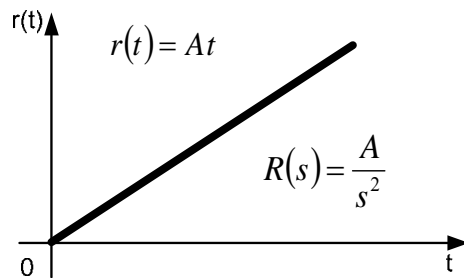
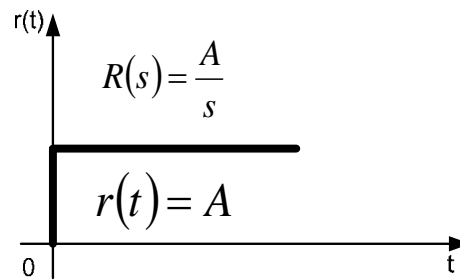
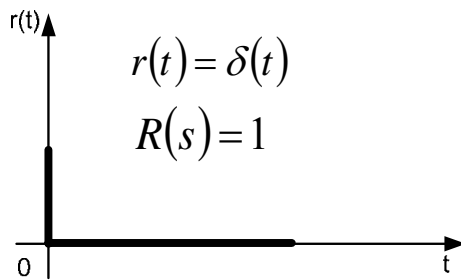


Useful commands: `zgrid`, `sgrid`.

## Time domain characteristics

### Typical input signals

1. The Dirac function
2. The step or the pulse function (!!!)
3. The Ramp function
4. Parabolic function



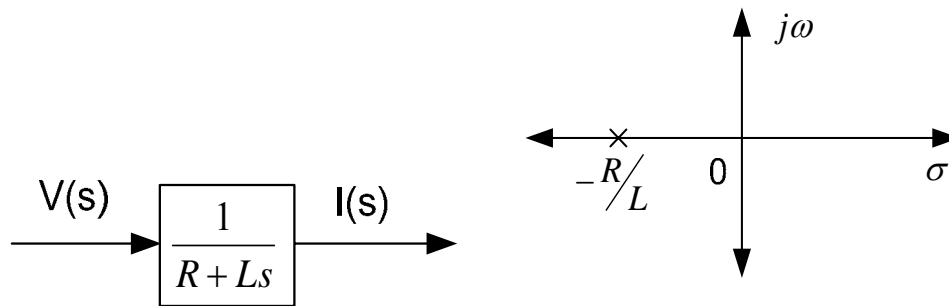
### First order systems

Remember:  $x' + ax = b$ ; HE:  $x' + ax = 0$ , CE:  $m + a = 0$  a solution of

HE:  $x = C_1 e^{at}$  hence stable solution if  $a < 0$  or the pole is at the LHS.

The same at s-plane:

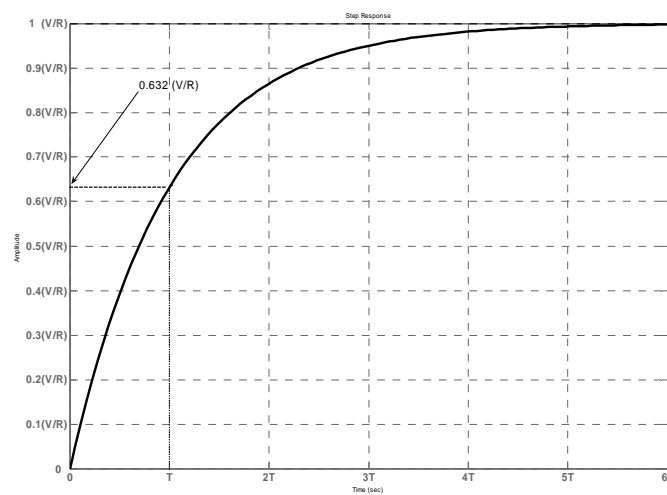
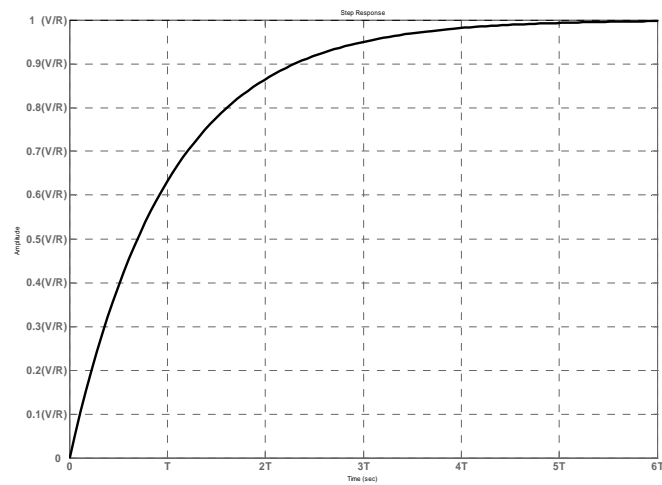
$$\frac{I(s)}{V(s)} = \frac{1}{R + Ls}$$



$$\frac{I(s)}{V(s)} = \frac{K}{1 + \tau s}, \quad K=1/R \text{ and } \tau=L/R. \Rightarrow I(s) = V(s) \frac{K}{\tau s + 1}$$

- Step response:  $I(s) = K \frac{V}{s} \frac{1}{\tau s + 1} \Rightarrow i(t) = \frac{V}{R} \left( 1 - e^{-t/\tau} \right)$
- $i_F = \lim_{t \rightarrow \infty} \frac{V}{R} \left( 1 - e^{-t/\tau} \right) = \frac{V}{R} - 0 = \frac{V}{R}$  or

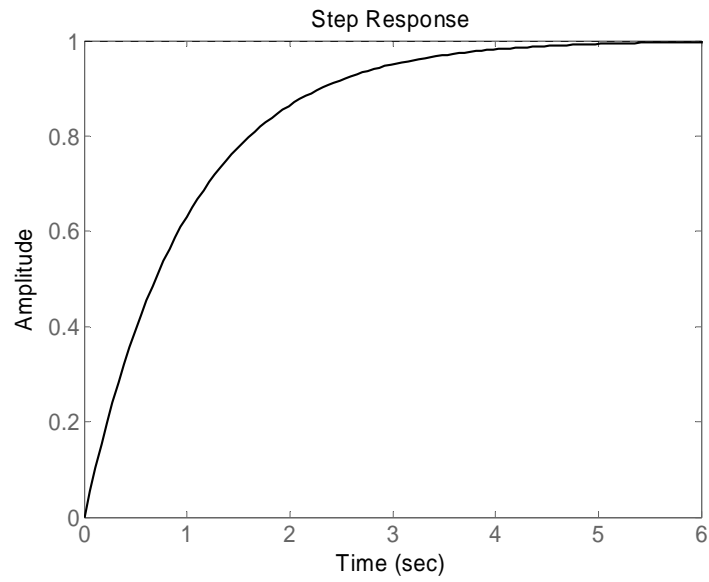
$$I_{ss} = \lim_{s \rightarrow 0} s \frac{V}{s} \frac{K}{\tau s + 1} = VK = \frac{V}{R}$$



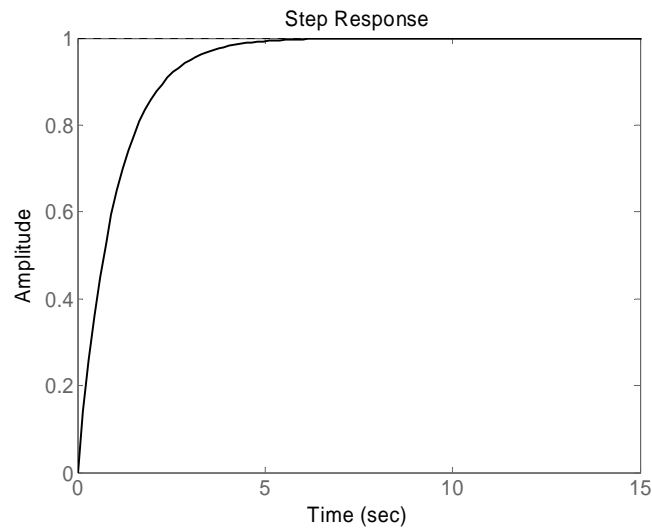
To find the step response (apart from using Simulink), use the command:

`step( , ):`

```
>> num=1;  
>> den=[1 1];  
>> step(num,den)
```



Or: `>> step(num,den,15)`



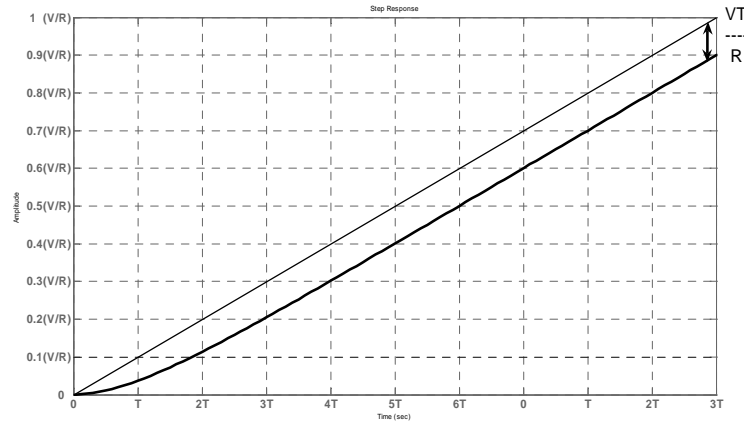
- Ramp response:  $C(s) = K \frac{V}{s^2} \frac{1}{\tau s + 1} \Rightarrow$

$$c(t) = \frac{V}{R} \left( t - \tau + \tau e^{-t/\tau} \right)$$

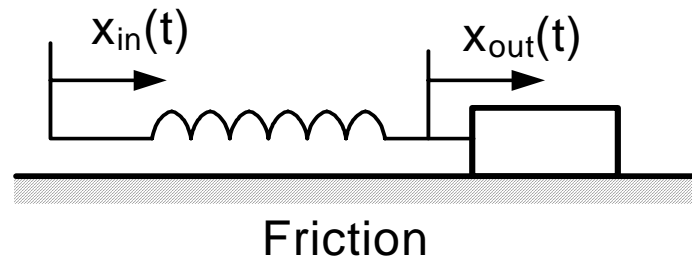


- The final value of the error is

$$e_F = v - i = \lim_{x \rightarrow \infty} \frac{V}{R} \tau \left( 1 - e^{-t/\tau} \right) = \frac{V}{R} \tau$$



## Second order systems



$$K(x_{in} - x_{out}) - B \frac{dx_{out}}{dt} = m \frac{d^2 x_{out}}{dt^2} \xrightarrow[IC=0]{LT}$$

$$KX_{in}(s) - KX_{out}(s) - BsX_{out}(s) = ms^2 X_{out}(s)$$

$$KX_{in}(s) = (K + Bs + ms^2)X_{out}(s)$$

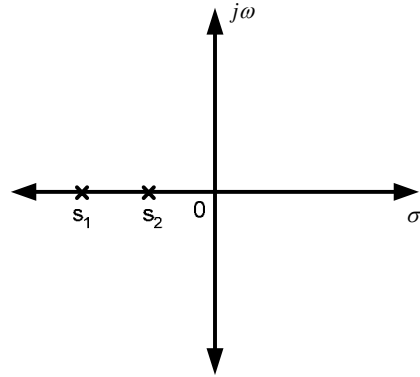
$$\frac{X_{out}(s)}{X_{in}(s)} = \frac{K}{ms^2 + Bs + K} \Rightarrow \frac{X_{out}(s)}{X_{in}(s)} = \frac{K/m}{s^2 + B/m s + K/m}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2s\zeta\omega_n + \omega_n^2}, \quad 2\zeta\omega_n = B/m, \quad \omega_n^2 = K/m$$

$$\text{Roots of CE: } \Rightarrow s = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

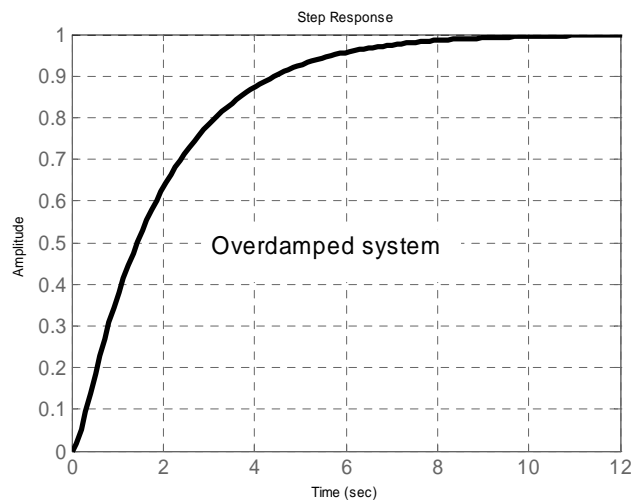
Case 1:  $\zeta > 1$

Then the system has two negative real roots and is called overdamped:



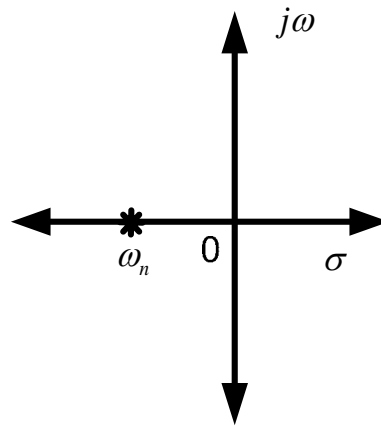
$$c(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left( \frac{e^{+s_1 t}}{s_1} - \frac{e^{+s_2 t}}{s_2} \right)$$

$\zeta \gg 1 \Rightarrow c(t) = 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \Rightarrow$  Overdamped system will be like a very slow response of a first order system

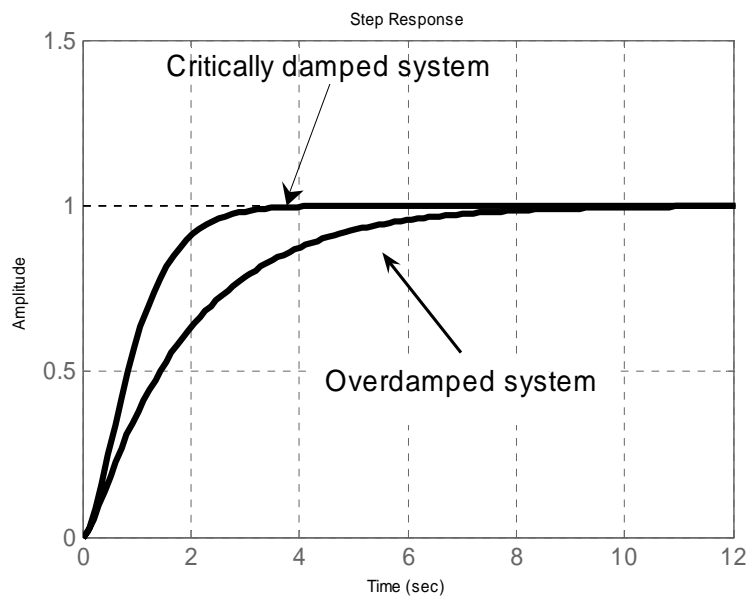


Case 2:  $\zeta = 1$

The system has two equal real roots at  $s = -\omega_n$ :

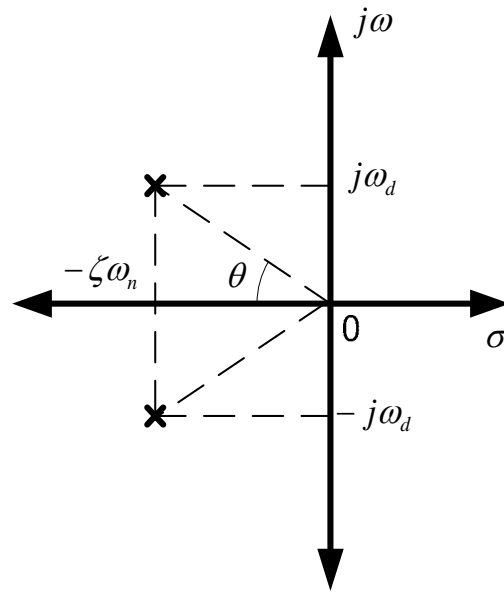


$$c(t) = 1 - e^{-\omega_n t} (1 + \omega_n t):$$



Case 3:  $0 < \zeta < 1$

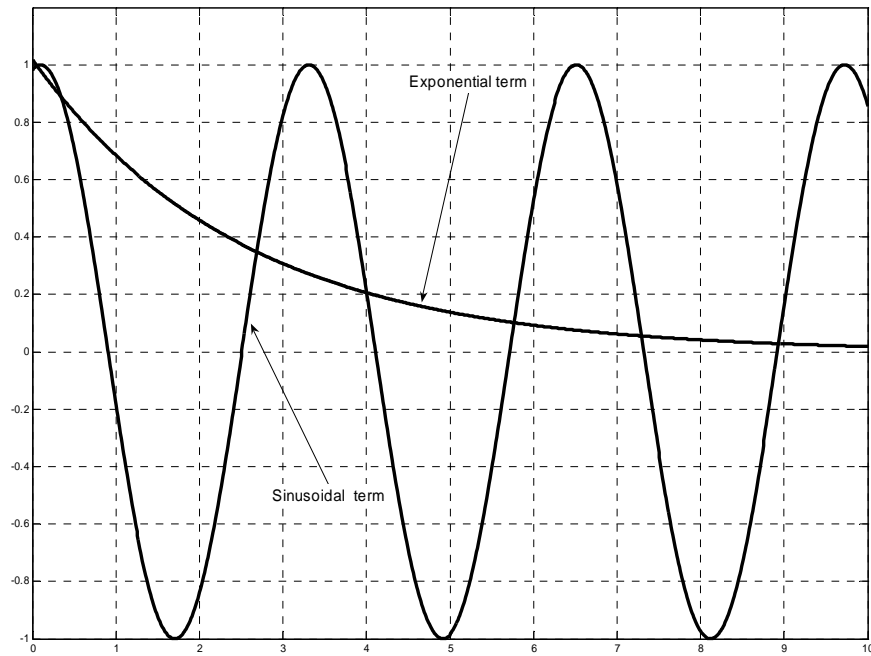
$$\Rightarrow s = \zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \quad \text{or} \quad \Rightarrow s = \zeta\omega_n \pm j\omega_d$$



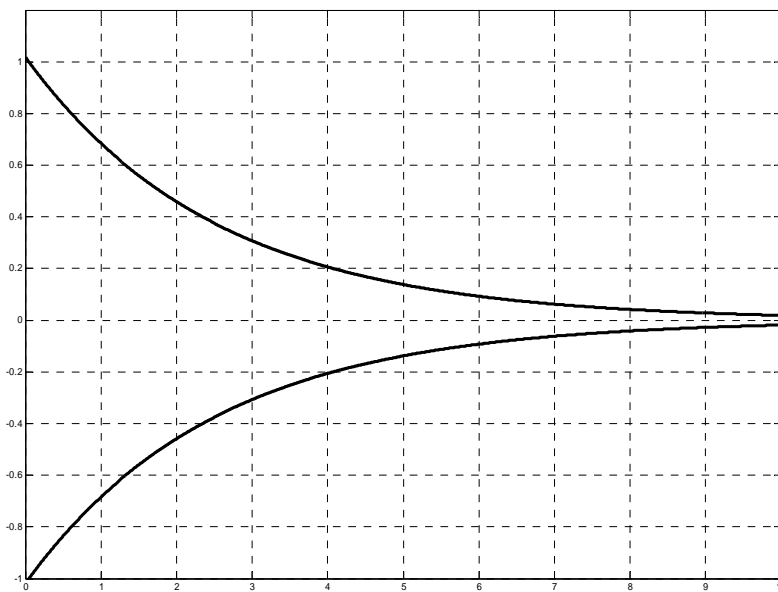
The line between the origin and the pole is:  $d = \sqrt{\omega_d^2 + \zeta^2 \omega_n^2} = \sqrt{\omega_n^2(1 - \zeta^2) + \zeta^2 \omega_n^2} = \omega_n$

The angle is  $\cos \theta = \frac{\omega_n \zeta}{\omega_n} \Leftrightarrow \theta = \cos^{-1}(\zeta)$ .

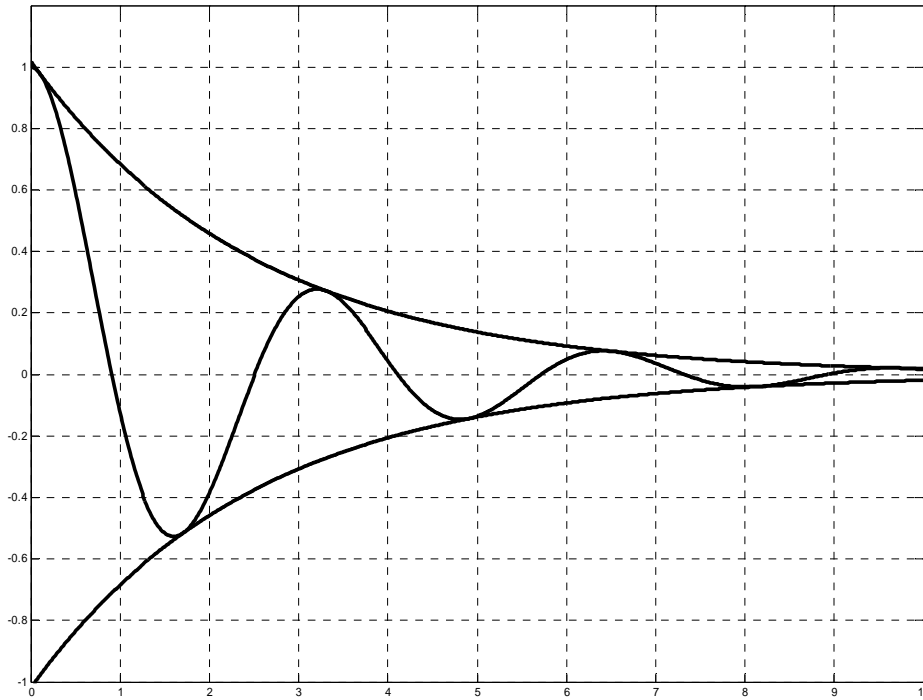
$$c(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1 - \zeta^2}} \sin \left( \omega_d t + \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta} \right)$$



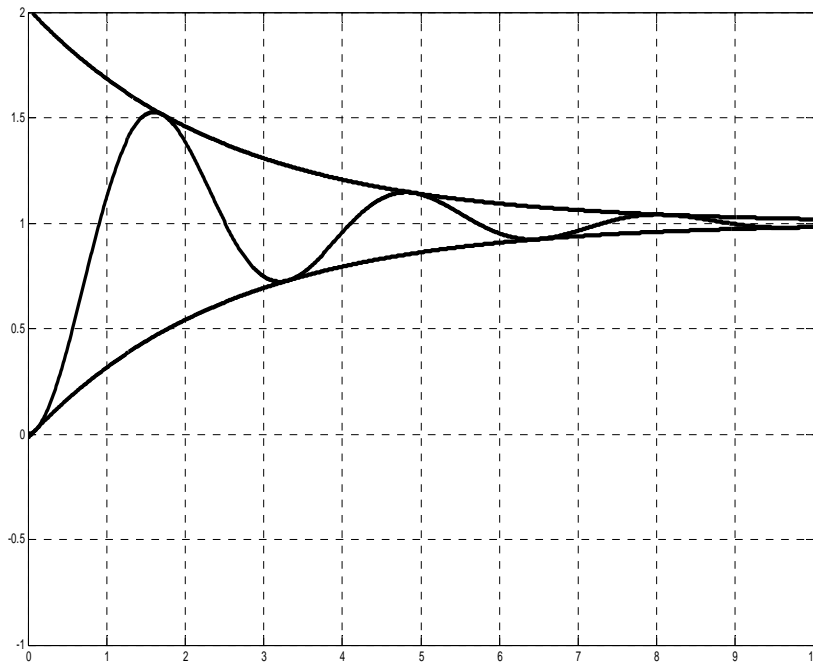
The envelope that will be created from the exponential terms is:

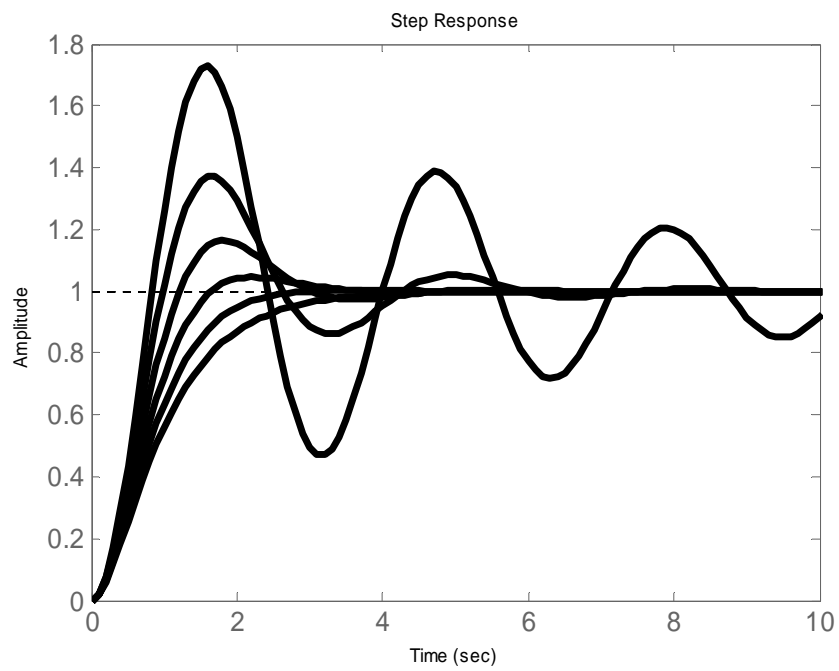
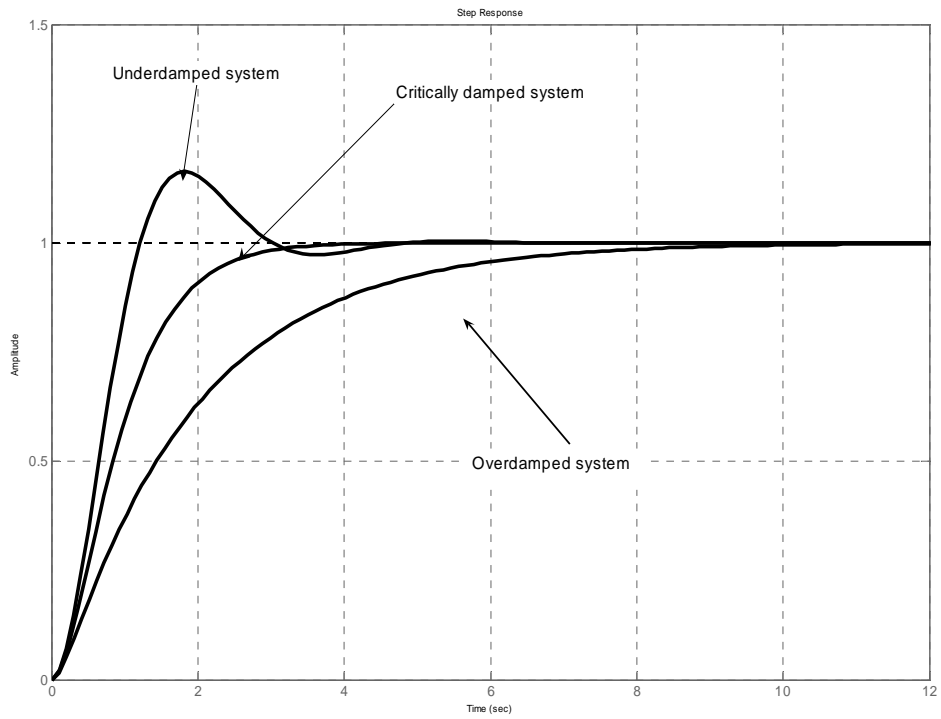


And their product:



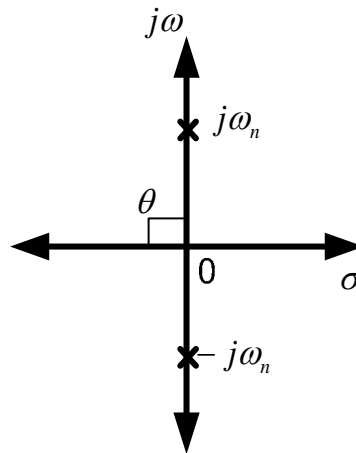
And by adding the constant factor:



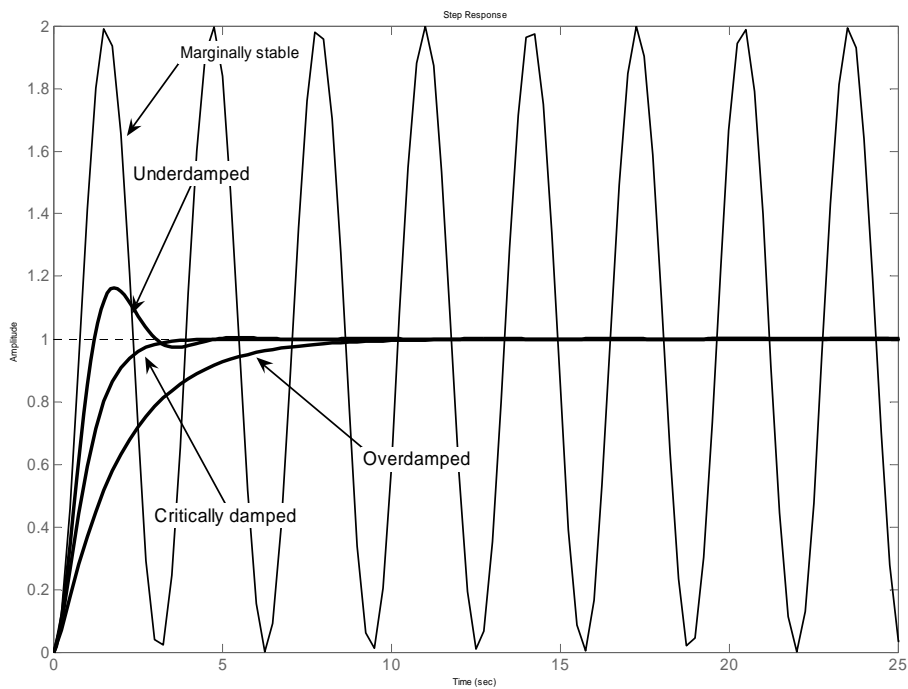




Case 4:  $\zeta = 0$

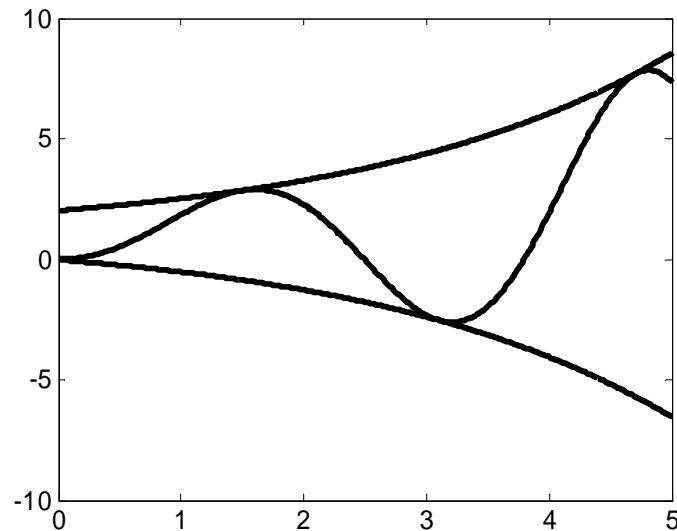


Note: The system is called marginally stable because the solutions do not diverge to infinity. Hence if the previous four cases are combined to one graph:



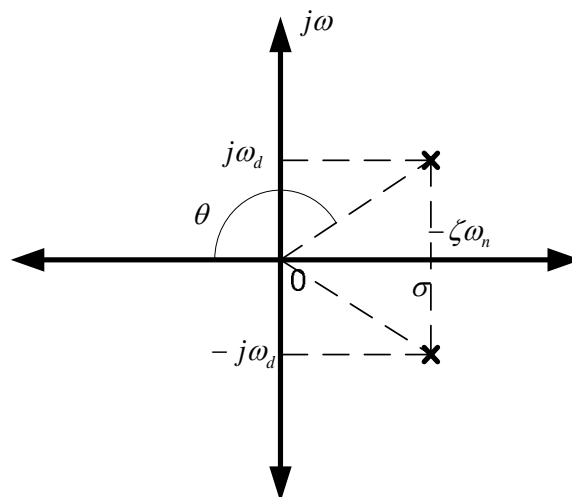
Case 5:  $\zeta < 0$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

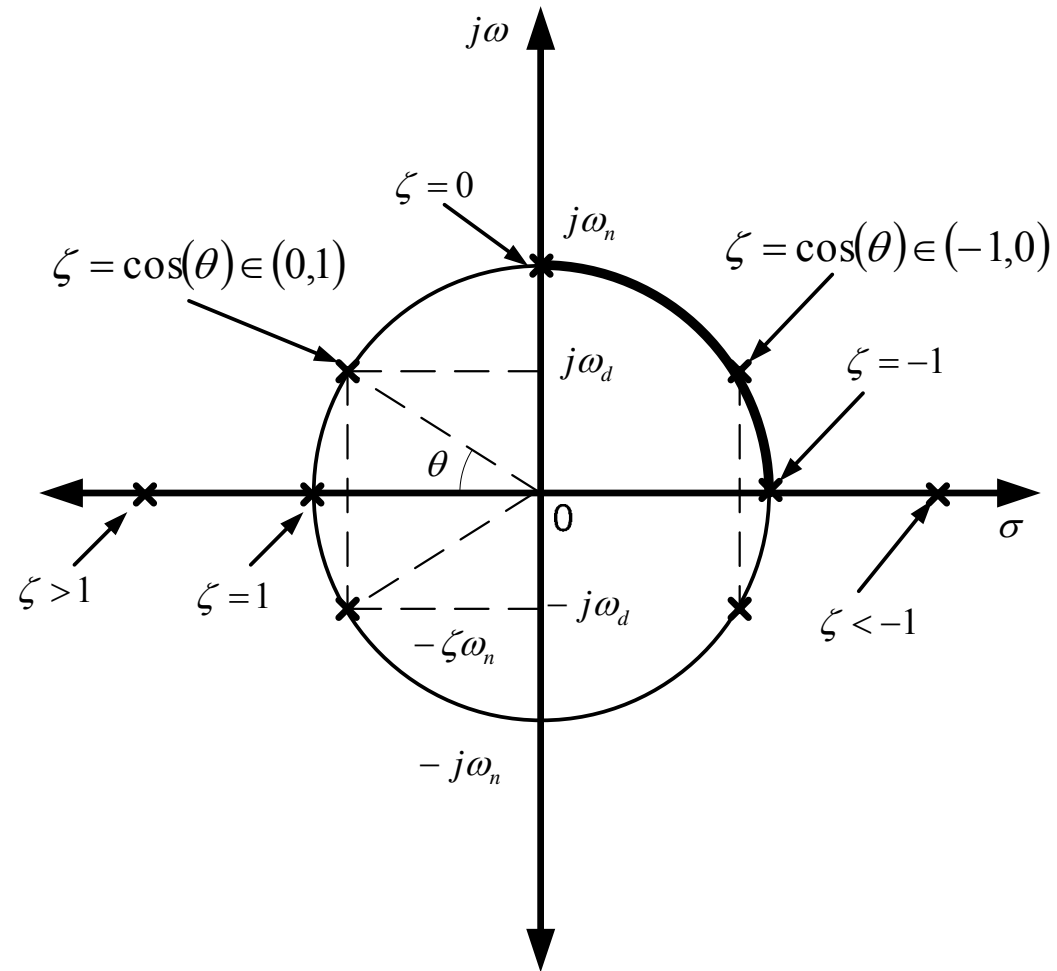


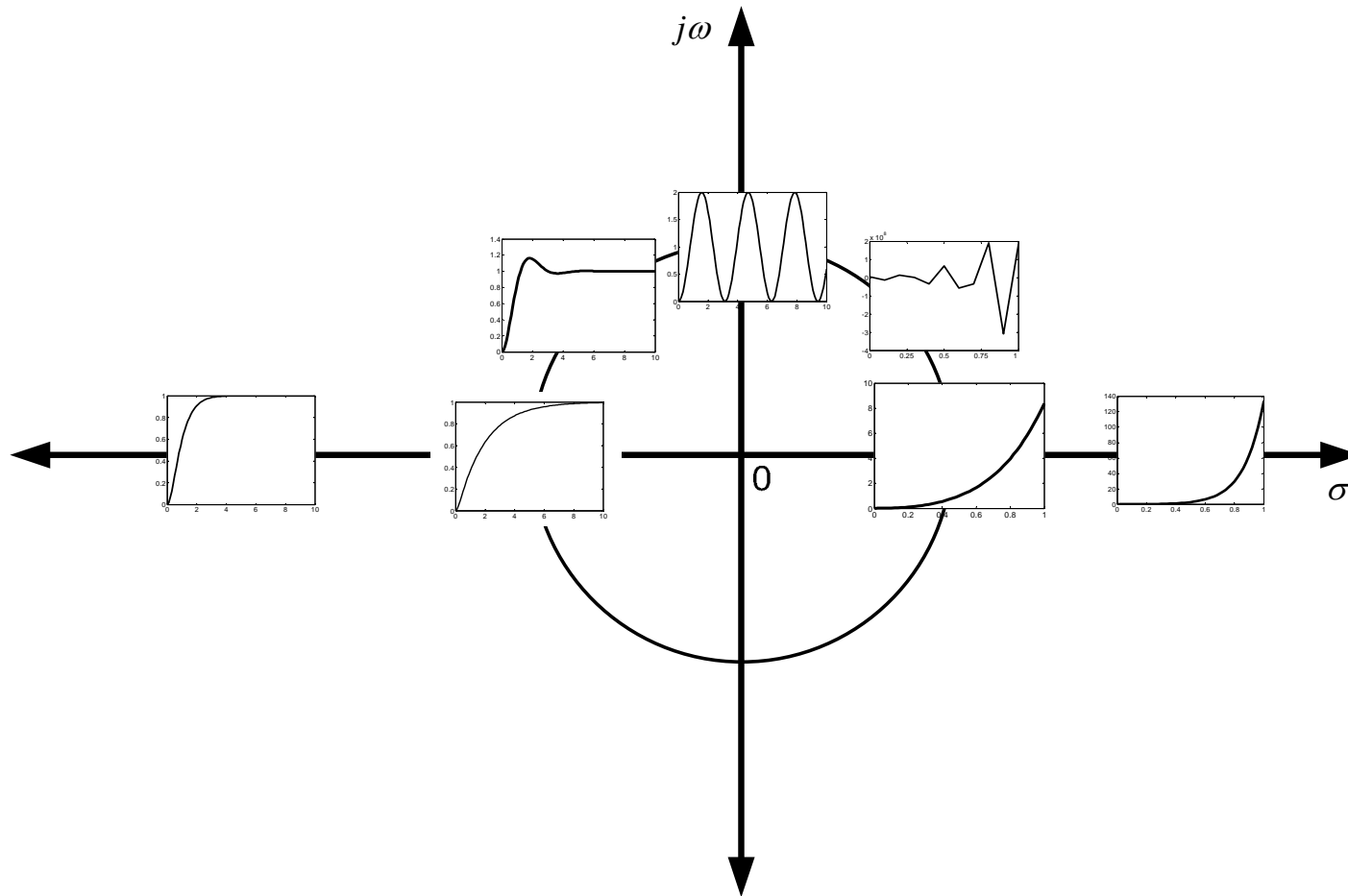
Since  $\zeta < 0$  then angle  $\theta$  defined in the s-plane

$(\cos \theta = \frac{\omega_n \zeta}{\omega_n} \Leftrightarrow \theta = \cos^{-1}(\zeta))$  has to be greater than  $90^\circ$ :

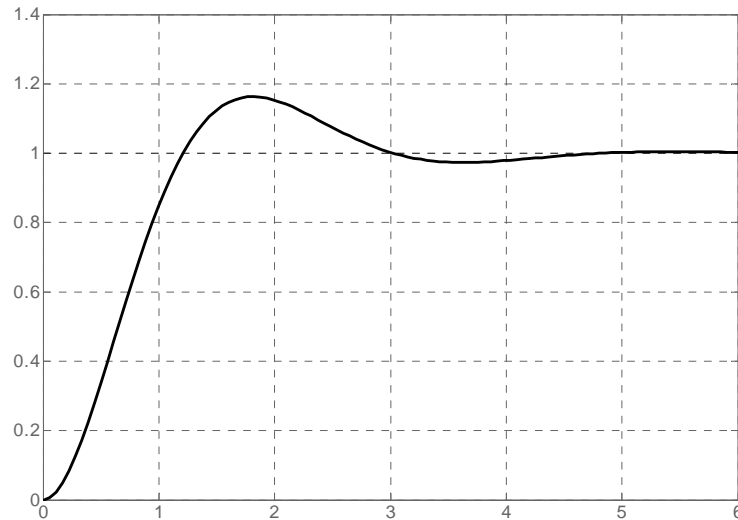


By combining the previous s-planes we have:

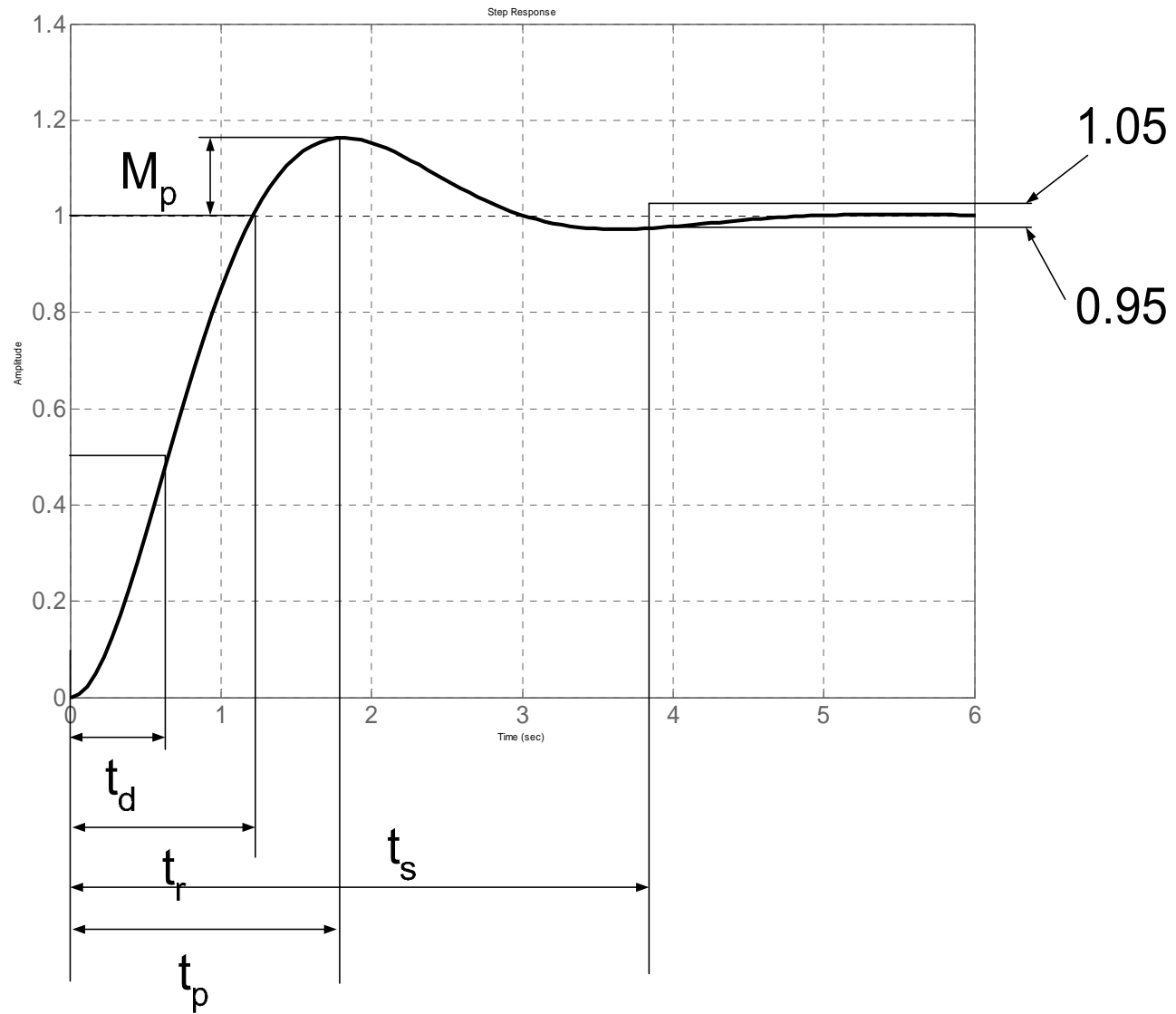




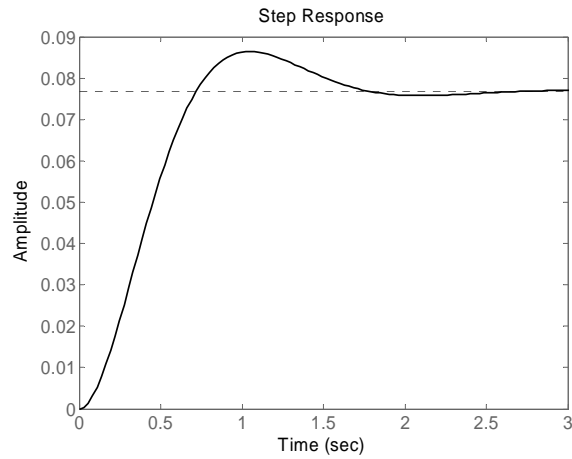
A general response is:



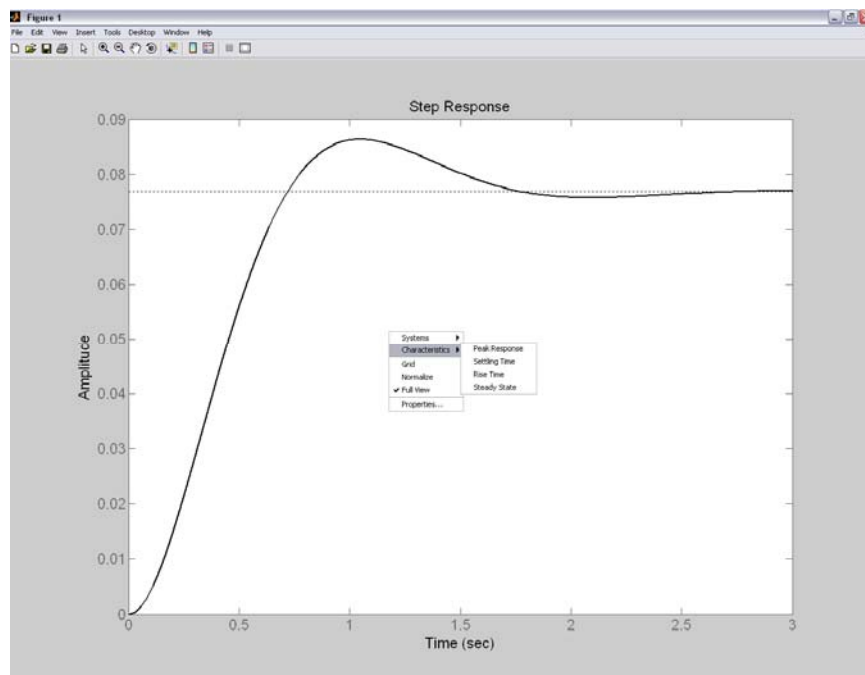
- Time that the system needs to reach half of its final value:
- Rise time (10%-90% or 5%-95% or 0%-100%)  $t_r = \frac{\pi - \theta}{\omega_d}$
- Peak time:  $t_p = \frac{\pi}{\omega_d}$
- Maximum overshoot:  $Mp = e^{-\left(\zeta/\sqrt{1-\zeta^2}\right)\pi}$
- Settling time:  $t_{s5\%} = \frac{3}{\zeta\omega_n}$  and  $t_{s2\%} = \frac{4}{\zeta\omega_n}$

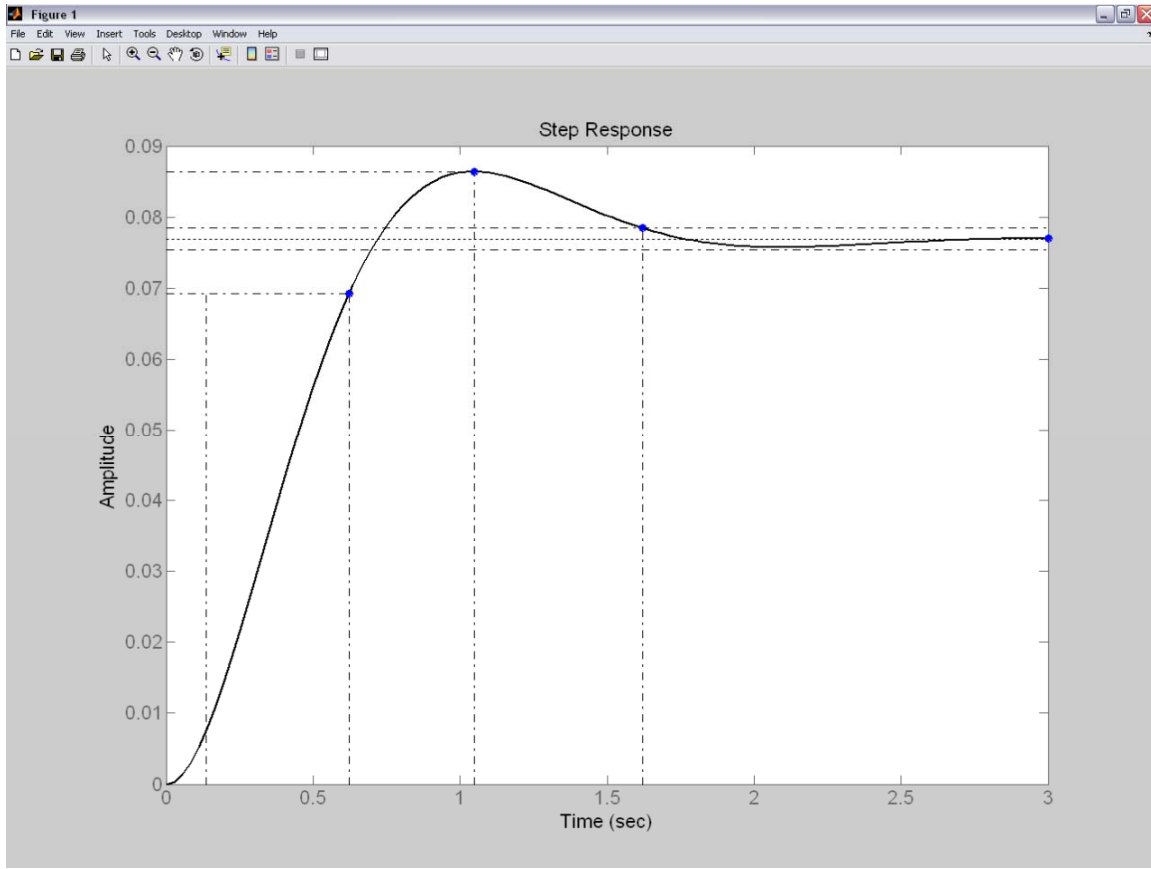


```
>> num=1;  
>> den=[1 4 13];  
>> g=tf(num,den);  
>> step(g)
```



Right click on figure and choose the characteristics that you would like to see.





## Extra poles and zeros

General form of a TF: 
$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + \dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

For a step input:

$$C(s) = \frac{1}{s} + \sum_{j=1}^q \frac{a_j}{s + p_j} + \sum_{k=1}^r \frac{b_k (s + \zeta_k \omega_k) + c_k \omega_k \sqrt{1 - \zeta_k^2}}{s^2 + 2\zeta_k \omega_k s + \omega_k^2}$$

$(q + 2r = n)$ , i.e. combination of first and second order systems.



$$\text{Example: } \frac{1}{s^3 + as^2 + bs + c} = \frac{1}{(s + f)(s^2 + ds + e)} \Rightarrow$$

$$s^3 + as^2 + bs + c = (s + f)(s^2 + ds + e) \Leftrightarrow$$

$$\Leftrightarrow s^3 + as^2 + bs + c = s^3 + (d + f)s^2 + (e + fd)s + fe$$

$$\Leftrightarrow \begin{cases} 1 = 1 \\ a = d + f \\ b = e + fd \\ c = fe \end{cases}$$

- The response of a higher order system is the sum of exponential and damped sinusoidal curves.
- Assuming that all poles are at the left hand side then the final value of the output is “1” since all exponential terms will converge to 0.
- Let’s assume that some poles have real parts that are far away from the imaginary axis=>

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin\left(\omega_d t + \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta}\right) \Rightarrow e^{-\zeta\omega_n t} \rightarrow 0$$

- Overall performance is characterised by the isolated (far away from zeros) poles that are close to the imaginary axis.
- If we have only one pole (or a pair for complex roots) that is closed to the real axis then we say that this pole (or pair of poles) is (are) the DOMINANT pole(s) for the system.

- A simple rule is that the dominant poles must be at least five to ten times closer to the imaginary axis than the other ones.

$$C(s) = \frac{1}{s} + \sum_{j=1}^q \frac{a_j}{s + p_j} + \sum_{k=1}^r \frac{b_k (s + \zeta_k \omega_k) + c_k \omega_k \sqrt{1 - \zeta_k^2}}{s^2 + 2\zeta_k \omega_k s + \omega_k^2}$$

$$c(t) = 1 + \sum_{j=1}^q a_j e^{-p_j t} + \sum_{k=1}^r b_k e^{-\zeta_k \omega_k t} \cos(\omega_k \sqrt{1 - \zeta_k^2} t) + \sum_{k=1}^r c_k e^{-\zeta_k \omega_k t} \sin(\omega_k \sqrt{1 - \zeta_k^2} t)$$

The values of b (numerator coefficients) determine the amplitude of the oscillations of the system.