Chapter #4

EEE 2002

Automatic Control

- **Root Locus**

Introduction

Response depends on system's and controller's parameters =>

Closed loop pole location depends on system's and controller's parameters.

Task: Find the controller gain(s) such as we have a satisfactory performance => Find the controller gain(s) such as we have a satisfactory pole location.

To do that we need a good understating of how poles move in the s-plane. We have already seen three cases:

$$G_{OL} = \frac{k}{s+a}, \; G_{OL} = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \; G_{OL} = \frac{k}{(s+a)(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$
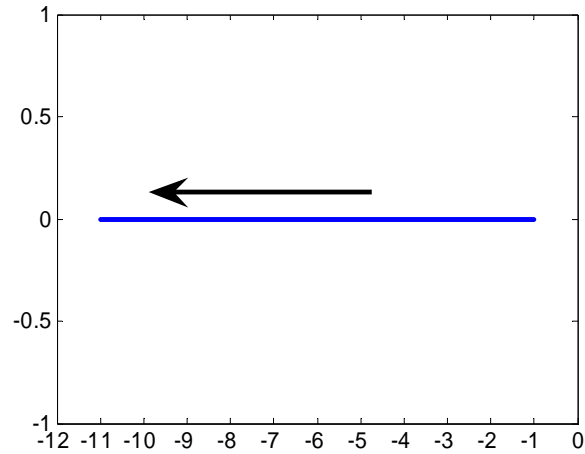
Example (H=1):

$$G_{OL} = \frac{k}{s+1} \Rightarrow G_{CL} = \frac{k}{s+1+k} \text{ so pole location: } s+1+k=0 \Leftrightarrow s=-1-k:$$

```
cnt=1;
for k=0:0.01:10;
r(cnt)=roots([1 1+k]);
cnt=cnt+1;
end


plot(real(r),imag(r),'.')
```
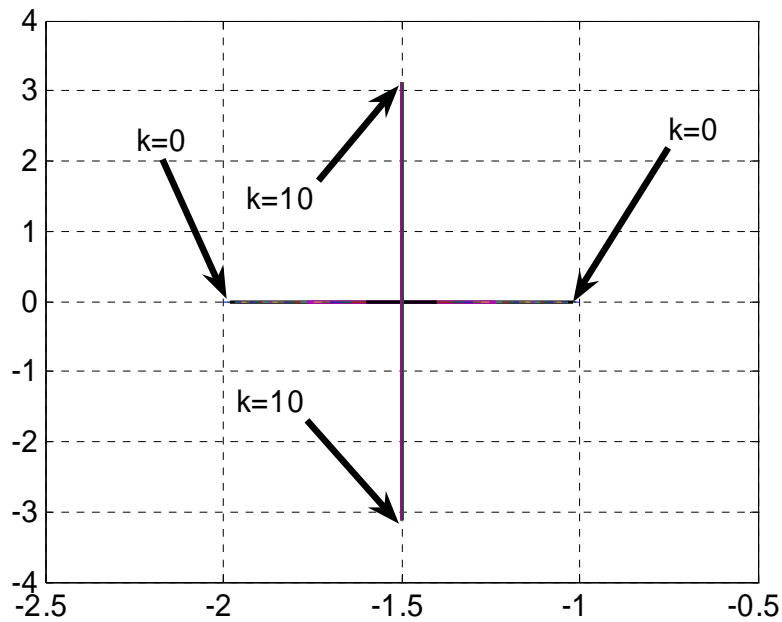
Properties:

- Starts from the OL pole location.

- Ends at $-\infty$.

Example (H=1):

$$G_{OL} = \frac{k}{s^2 + 3s + 2} \Rightarrow G_{CL} = \frac{k}{s^2 + 3s + 2 + k}$$ so   pole   location:

$s^2 + 3s + 2 + k = 0$:

```
clear, clc
cnt=1;
for k=0:0.01:10;
r(1:2,cnt)=roots([1 3 2+k]);
cnt=cnt+1;
end
plot(real(r),imag(r))
```

Properties:

- Starts from the OL pole location.

- Collides at a point.

- Moves on a straight line perpendicular to the real axis.

- Symmetrical with respect to the real axis
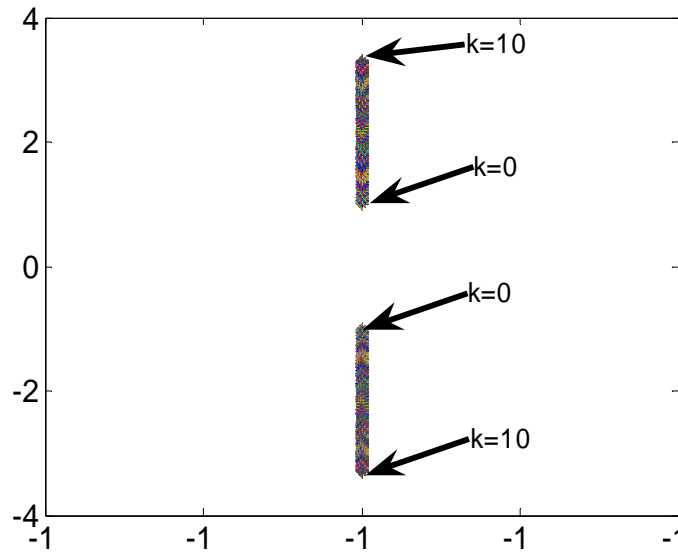
- Ends at $-1.5 \pm \infty$.

Example (H=1):

$$G_{OL} = \frac{k}{s^2 + 2s + 2} \Rightarrow G_{CL} = \frac{k}{s^2 + 2s + 2 + k}$$ so   pole   location:

$$s^2 + 2s + 2 + k = 0:$$

```
clear, clc
cnt=1;
for k=0:0.01:10;
r(1:2,cnt)=roots([1 2 2+k]);
cnt=cnt+1;
end
plot(real(r),imag(r),'*')
```
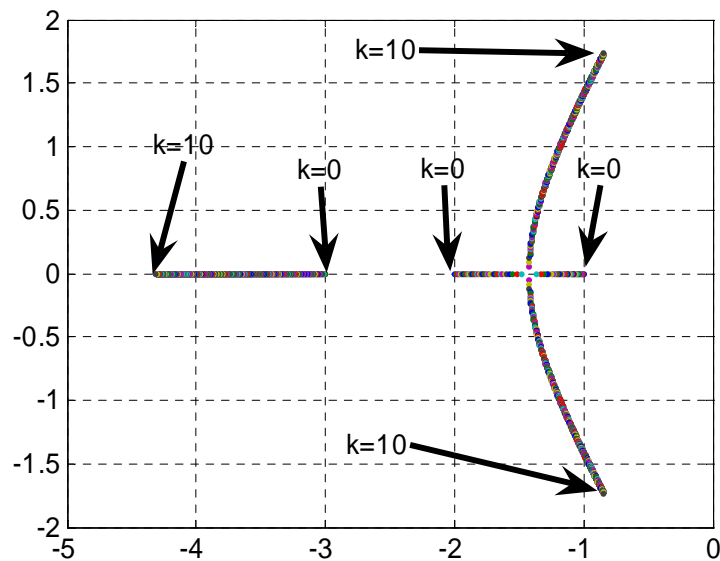

Properties:

- Starts from the OL pole location.

- Starts already with complex poles so no collision.

- Moves on a straight line perpendicular to the real axis.

- Symmetrical with respect to the real axis

- Ends at $-1 \pm \infty$.

Example (H=1):

$$G_{OL} = \frac{k}{s^3 + 6s^2 + 11s + 6} \Rightarrow G_{CL} = \frac{k}{s^3 + 6s^2 + 11s + 6 + k} \quad \text{so pole location:}$$

$$s^3 + 6s^2 + 11s + 6 + k = 0:$$

```
clear, clc
cnt=1;
for k=0:0.01:10;
r(1:3,cnt)=roots([1 6 11 6+k]);
cnt=cnt+1;
end
plot(real(r),imag(r),'.')
```

Properties:

- Starts from the OL pole location.
- Two poles collide and then they follow asymptotically a line which is not perpendicular to the real axis.

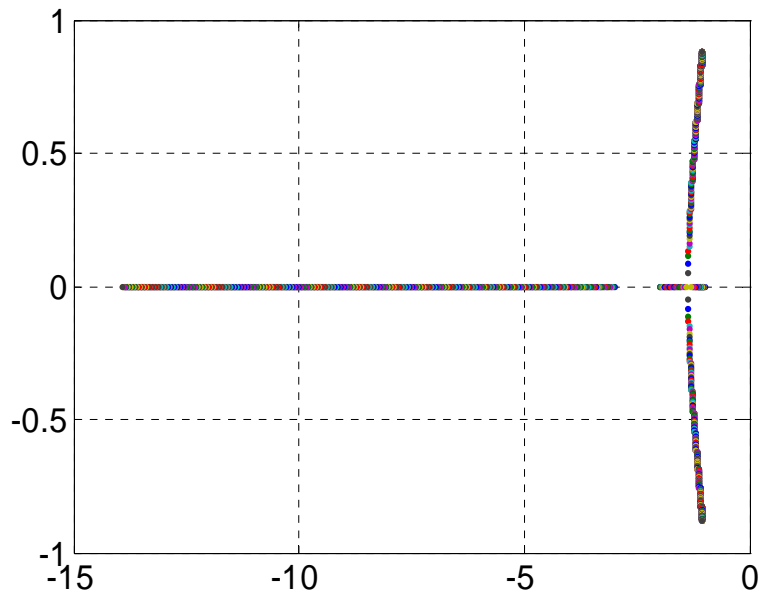- Symmetrical with respect to the real axis

- The other pole goes to $-\infty$.

Example (H=1):

$$G_{OL} = \frac{k\left(s^2 + 2s + 2\right)}{s^3 + 6s^2 + 11s + 6} \Rightarrow G_{CL} = \frac{k\left(s^2 + 2s + 2\right)}{s^3 + 6s^2 + 11s + 6 + k\left(s^2 + 2s + 2\right)} \qquad \text{so}$$

pole location:

$$s^3 + 6s^2 + 11s + 6 + k\left(s^2 + 2s + 2\right) = 0 \Leftrightarrow s^3 + (6 + k)s^2 + (11 + 2k)s + 6 + 2k = 0$$

```
clear, clc
cnt=1;
for k=0:0.01:10;
r(1:3,cnt)=roots([1 6+k 11+2*k 6+2*k]);
cnt=cnt+1;
end
plot(real(r),imag(r),'.')
```

Properties:

- Starts from the OL pole location.
- Two poles collide and then they follow asymptotically a line which is not perpendicular to the real axis.
- They do not diverge to infinity but they converge to the two complex zeros.
- Symmetrical with respect to the real axis
- The other pole goes to $-\infty$.

Hence it can be seen that the root location depends on the order and the number of zeros of the system.

The procedure of finding the closed loop pole location for various values of the proportional gain, k, is known as root locus method.

We can find that location numerically or graphically (W. R. Evans). Matlab provides the command rlocus(num_OL,den_OL) but it is better to have a deeper understanding of the graphical method. This method can also be used for other parameters (apart from the proportional controller) but is rather tricky. In EEE205 we will superficially cover this analysis method. In EEE301 you will cover this topic extensively with Prof. J.W. Finch.

To include the effect of the feedback TF we name as OLTF the $G(s)H(s)$:
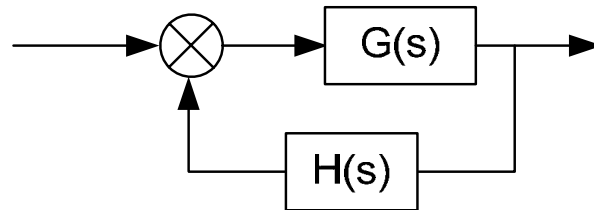
We always start from the location of the OL poles (i.e. poles of G(s)H(s)):

$$G_{CL} = \frac{G}{1+GH} = \frac{k\dfrac{N_G}{D_G}}{1+k\dfrac{N_G}{D_G}\dfrac{N_H}{D_H}} = \frac{kN_G}{D_G + kN_G\dfrac{N_H}{D_H}} = \frac{kN_G D_H}{D_G D_H + kN_G N_H}$$
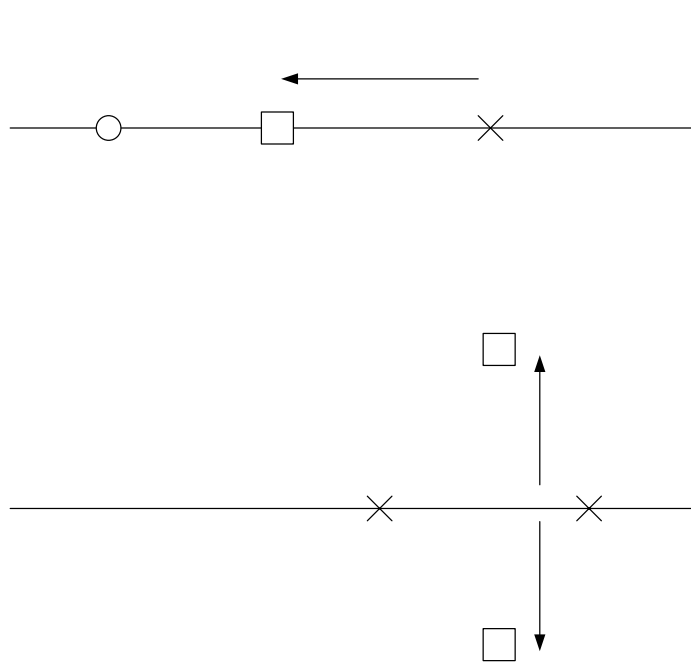
$\Rightarrow CE : D_G D_H + kN_G N_H = 0$. For k=0 (first value): $CE : D_G D_H = 0$, so poles of G(s)H(s).

We use the symbol "x" to denote OL poles.

We use the symbol "o" to denote OL zeros.

We use the symbol "□" to denote CL poles.

No need for CL zeros as they do not move.
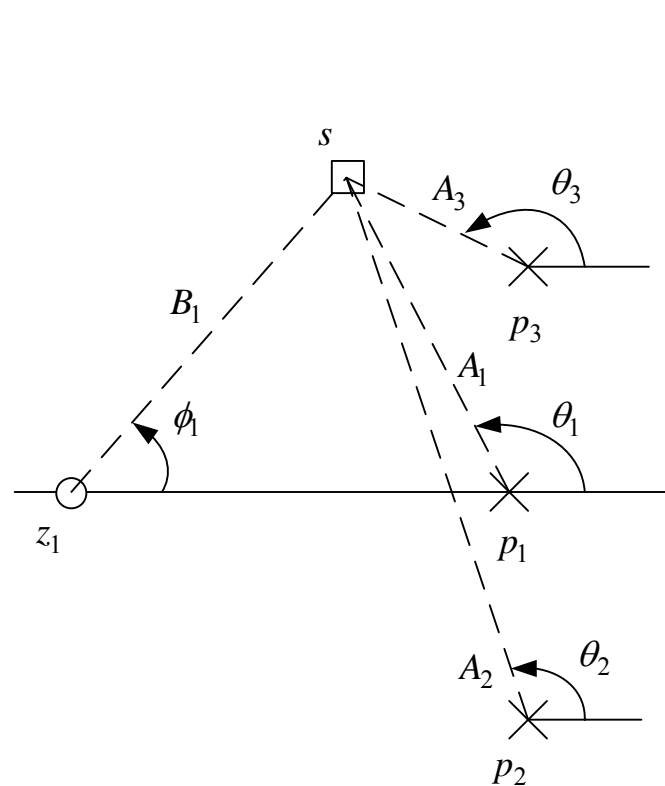
Angle and magnitude conditions

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \Rightarrow CE:1+G(s)H(s)=0 \Leftrightarrow G(s)H(s)=-1$$

This implies that $|G(s)H(s)|=1$ and $\arg(G(s)H(s))=\pm180(2n+1)$, $n=0,1,2...$

For example: $G(s)H(s)=\dfrac{k(s+z_1)}{(s+p_1)(s+p_2)(s+p_3)}$

$$\arg(s) = -\theta_1 - \theta_2 - \theta_3 + \phi_1$$

$$|G(s)H(s)| = \frac{kB_1}{A_1 A_2 A_3}$$

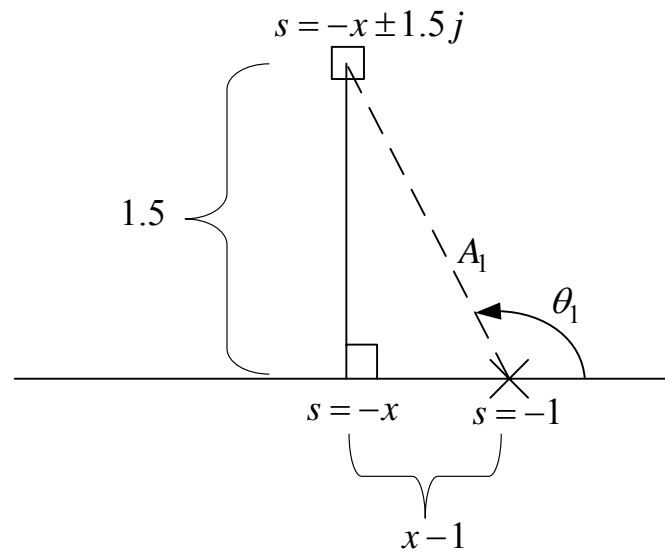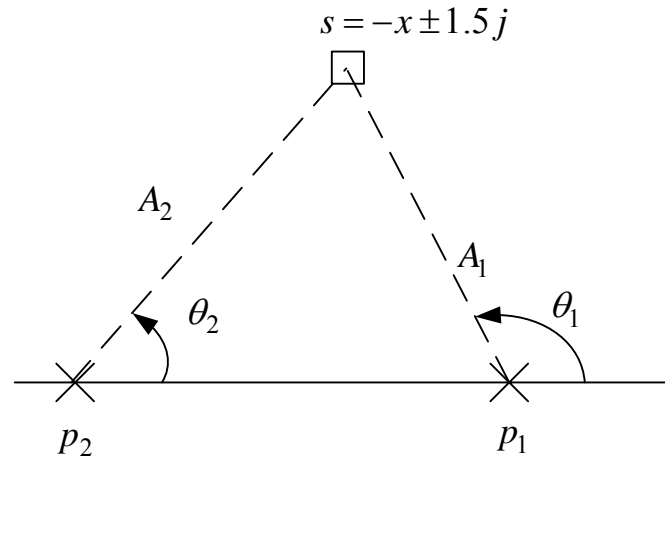Hence every point that belongs onto the Root Locus (RL) satisfies these two criteria.

For example for the system that we studied before $G_{OL} = \dfrac{k}{s^2 + 3s + 2}$

$\Rightarrow G_{CL} = \dfrac{k}{s^2 + 3s + 2 + k}$ the point $s = -1.8 \pm 1.5 j$ does not belong while the point $s = -1.5 \pm 1.5 j$ does.
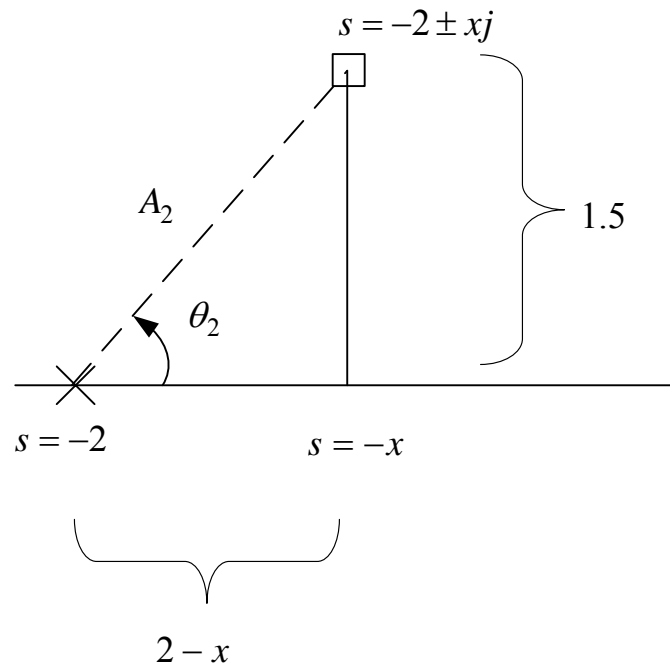
The OL case is $G_{OL} = \dfrac{k}{s^2 + 3s + 2}$ so I have two poles at -1 and -2 and no

zeros:

$$s = -x \pm 1.5j$$

$$A_2$$

$$A_1$$

$$\theta_2 \qquad \theta_1$$

$$p_2 \qquad p_1$$

$$s = -x \pm 1.5j$$

$$1.5$$

$$A_1$$

$$\theta_1$$

$$s = -x \qquad s = -1$$

$$x - 1$$

$$\tan(180 - \theta_1) = \frac{1.5}{x-1} \Rightarrow \theta_1 = 180 - \operatorname{atan}\left(\frac{1.5}{x-1}\right)$$

$$A_1 = \sqrt{1.5^2 + (x-1)^2}$$

$$\tan(\theta_2) = \frac{1.5}{2-x} \Rightarrow \theta_2 = \operatorname{atan}\left(\frac{1.5}{2-x}\right)$$

$$A_2 = \sqrt{1.5^2 + (2-x)^2}$$

For the point $s = -1.8 \pm 1.5j$:

$$\theta_1 = 180 - \operatorname{atan}\left(\frac{1.5}{1.8-1}\right) = 118.07^\circ$$

$$\theta_2 = \operatorname{atan}\left(\frac{1.5}{2-1.8}\right) = 82.40^\circ$$

So $-\theta_1 - \theta_2 = -200.47 \neq -180$ so that point does not belong to the root locus.

For the point $s = -1.5 \pm 1.5j$:

$$\theta_1 = 180 - \operatorname{atan}\left(\frac{1.5}{1.8-1}\right) = 108.43^\circ$$

$$\theta_2 = \operatorname{atan}\left(\frac{1.5}{2-1.8}\right) = 71.56^\circ$$

So $-\theta_1 - \theta_2 = -180$ so that point belongs to the root locus.

By using the magnitude condition we can find the gain:

$$A_1 = \sqrt{1.5^2 + (x-1)^2} = \sqrt{1.5^2 + (1.5-1)^2} = 1.58$$
$$A_2 = \sqrt{1.5^2 + (2-x)^2} = \sqrt{1.5^2 + (2-1.5)^2} = 1.58$$

$$\frac{k}{A_1 A_2} = 1 \Rightarrow k = A_1 A_2 = 2.5$$

Analytically:

$$s = -1.8 \pm 1.5j$$

$$1 + GH = 1 + \frac{k}{s^2 + 3s + 2} = 0 \Rightarrow s^2 + 3s + 2 + k = 0 \Rightarrow \text{-2.41-}0.9j + k = 0 \Rightarrow$$

WRONG since k is real.

$$s = -1.5 \pm 1.5j$$

$$1 + GH = 1 + \frac{k}{s^2 + 3s + 2} = 0 \Rightarrow s^2 + 3s + 2 + k = 0 \Rightarrow \text{-2.5} + k = 0$$

```
x=1.5

th1=180-atand(1.5/(x-1))
th2=atand(1.5/(2-x))
th1+th2

A1=sqrt(1.5^2+(x-1)^2)
A2=sqrt(1.5^2+(2-x)^2)
k=A1*A2

s=-x+1.5*j;
K=-s^2-3*s-2
```

Simple root locus

Assume that the OLTF is $G(s) = \dfrac{k}{s(s+1)(s+2)}$. There are three real OL poles with angles: $\arg(s), \arg(s+1), \arg(s+2)$.

First of all the number of branches equals the number of poles. The root locus is always symmetrical with respect to the real axis and the branches start from poles and end up to zeros or infinity (infinite zero).

In this specific case I have three braches, which will diverge to imaginary zeros at infinity.

If s>0 then $\arg(s)=0$, $\arg(s+1)=0$, $\arg(s+2)=0$ and therefore that area does not belong to the root locus as $-\arg(s)-\arg(s+1)-\arg(s+2)=0$.

If s<0 and s>-1 then $\arg(s)=180$, $\arg(s+1)=0$, $\arg(s+2)=0$ and therefore that area belongs to the root locus as

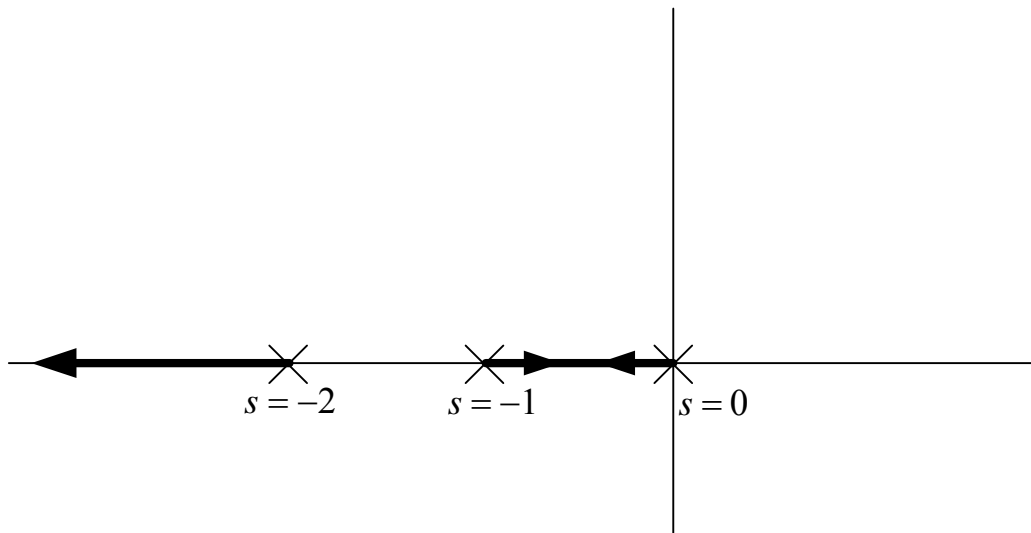$-\arg(s) - \arg(s+1) - \arg(s+2) = -180 - 0 - 0 = -180$.

If s<-1 and s>-2 then $\arg(s) = 180$, $\arg(s+1) = 180$, $\arg(s+2) = 0$ and therefore that area does not belong to the root locus as

$-\arg(s) - \arg(s+1) - \arg(s+2) = -180 - 180 - 0 = -360$.

If s<-3 then $\arg(s) = 180$, $\arg(s+1) = 180$, $\arg(s+2) = 180$ and therefore that area belongs to the root locus as

$-\arg(s) - \arg(s+1) - \arg(s+2) = -180 - 180 - 180 = -180 \times (2 \times 1 + 1)$ with $n = 1$.

So the root locus on the real axis is:

We have seen that when we numerically calculate the root locus.

At some point the poles from -1 and -2 will collide. To find this break-out point we rearrange the CE as:
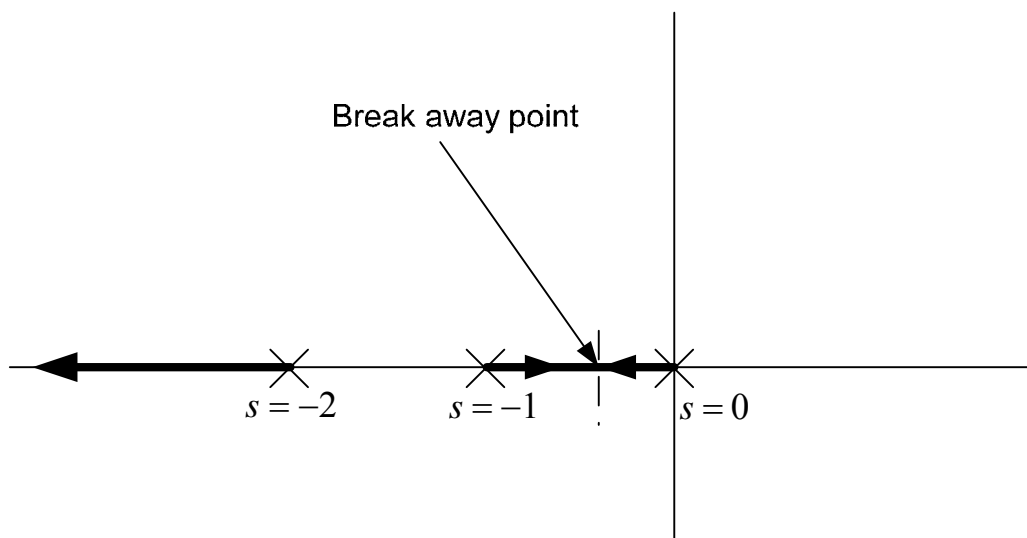
$$CE: f(s) = 0 \Leftrightarrow B(s) + kA(s) = 0 \Leftrightarrow k = -\frac{B(s)}{A(s)}$$ and then we find the 1st

derivative with respect to s and we equate the result with zero:

$$\frac{dk}{ds} = 0.$$ Hence in this case: $s(s+1)(s+2) + k = 0 \Leftrightarrow k = -s(s+1)(s+2)$ and

hence $\frac{dk}{ds} = -s(s+1)(s+2) = 3s^2 + 6s + 2 = 0 \Leftrightarrow s = \begin{cases} -1.5 \\ -0.42 \end{cases}$. Since the 2nd

point only belongs to the root locus the other one (-1.5) is ignored:

Break away point

$$s = -2 \qquad s = -1 \qquad s = 0$$

The same procedure can be used to find break-in points.
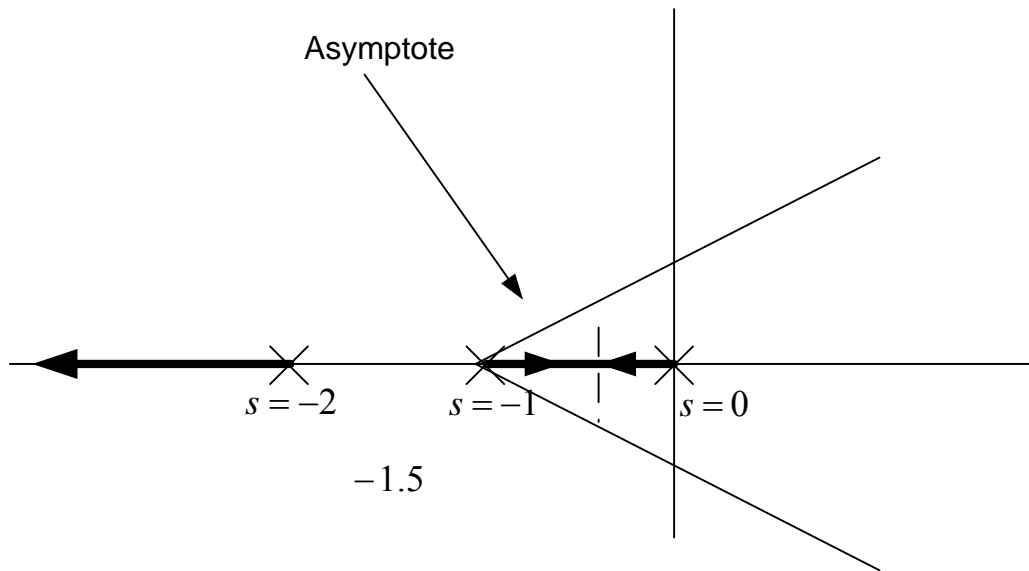
To find the asymptote that the locus of these two poles will follow:

The angle of the asymptote is: $\frac{\pm 180(2n+1)}{n_p - n_z}$, where n_p is the number of

finite poles and n_z is the number of finite zeros. Hence in this case

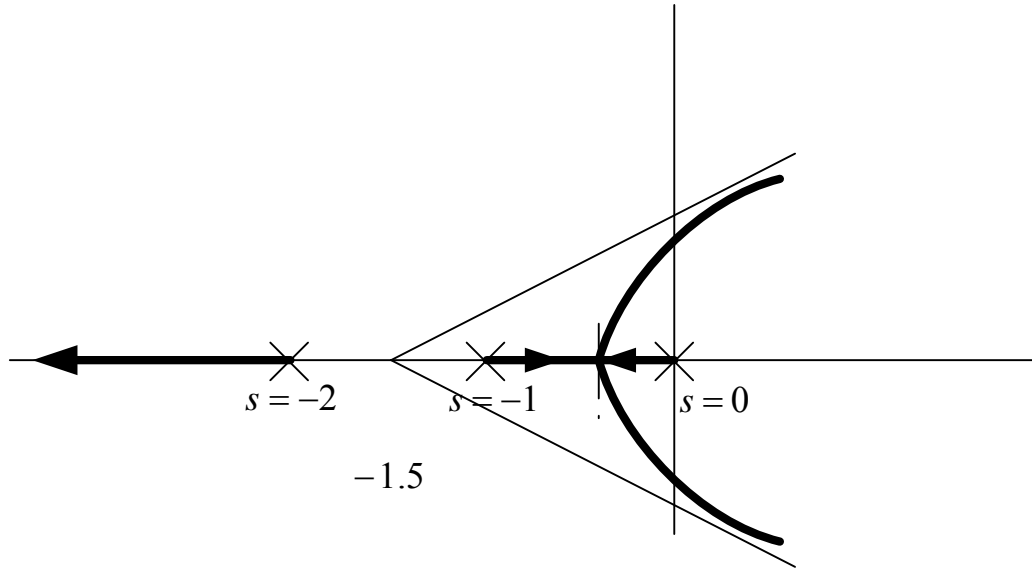$n_p = 3, n_z = 0$ and hence their angle is 180/3=60. The point of intersection with the real axis is:

$$s = -\frac{\sum_{i=1}^{n_p} p_i - \sum_{i=1}^{n_z} z_i}{n_p - n_z} \Rightarrow s = -\frac{0+1+2}{3} = -1:$$

Asymptote

$s = -2$ $s = -1$ $s = 0$

-1.5

The point of intersection of the root locus with the imaginary axis can be calculated by using the CE at $s = j\omega$. So in this case
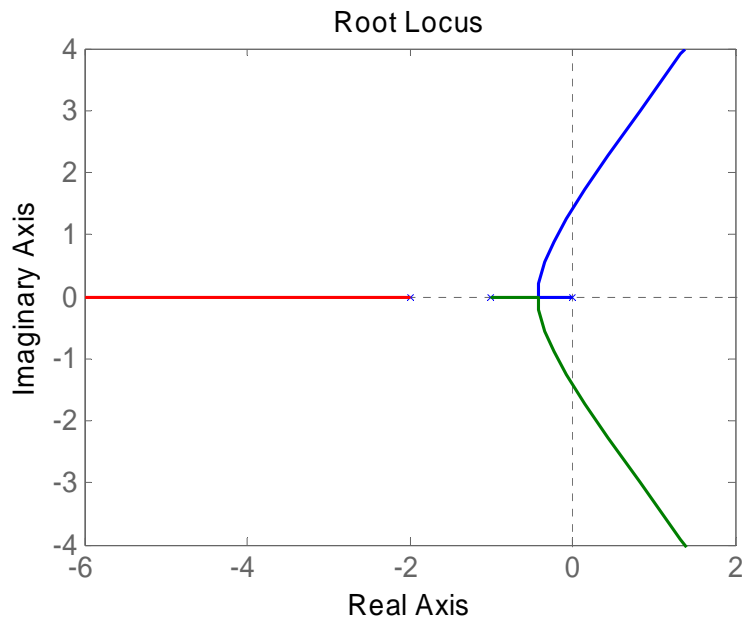
$$s(s+1)(s+2) + k = 0 \Leftrightarrow s^3 + 3s^2 + 2s + k = 0 \overset{s=j\omega}{\Rightarrow} -j\omega^3 - 3\omega^2 + 2j\omega + k = 0$$

So $-\omega^3 + 2\omega = 0 \Rightarrow \omega = \begin{cases} 0 \\ 1.4 \end{cases}$ and $-3\omega^2 + k = 0 \Rightarrow -6 + k = 0 \Leftrightarrow k = 6$.
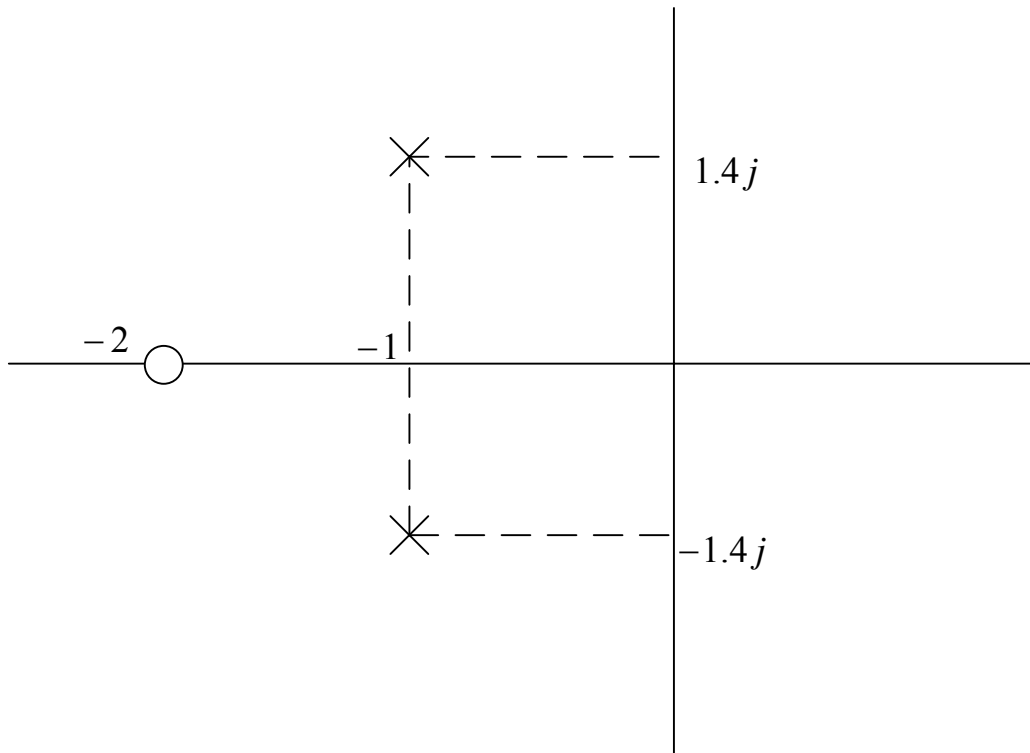
A similar result will be obtained if we use Matlab:

```
>> num=1; den=[1 3 2 0]; rlocus(num,den)
```

Example: Find the root locus of: $G(s) = \dfrac{k(s+2)}{s^2 + 2s + 3}$, $H(s)=1$:

1. I have two complex poles and 1 real zero.

2. Hence I have two branches that are symmetrical with respect to the real axis.

3. Place the OL poles and zeros:

4. Find the loci on the real axis:

    a. For s>-2 I have no poles or zeros so even number and hence that area does not belong to the locus.

    b. From $-\infty$ to -2 I have 1 zeros, i.e. odd number and hence this belongs to the locus:
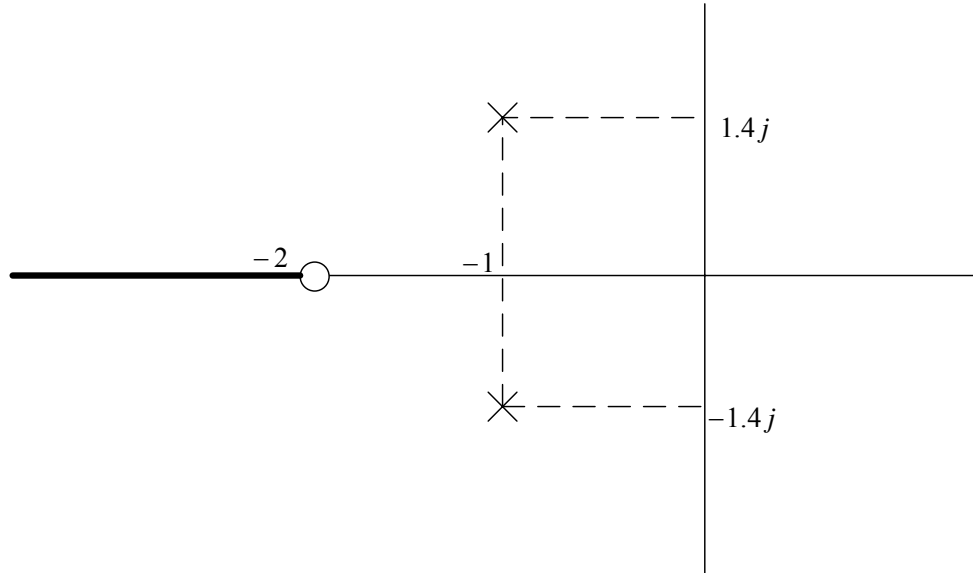
5. Determine break in and out points:

$$G(s) = \frac{k(s+2)}{s^2 + 2s + 3} \Rightarrow CE: s^2 + 2s + 3 + k(s+2) = 0 \Leftrightarrow k = -\frac{s^2 + 2s + 3}{s+2} \Rightarrow$$

$$\frac{dk}{ds} = s^2 + 4s + 1 = 0 \Rightarrow s = \begin{cases} -3.73 \\ -0.268 \end{cases}. \text{ The acceptable value is } s = -3.73$$
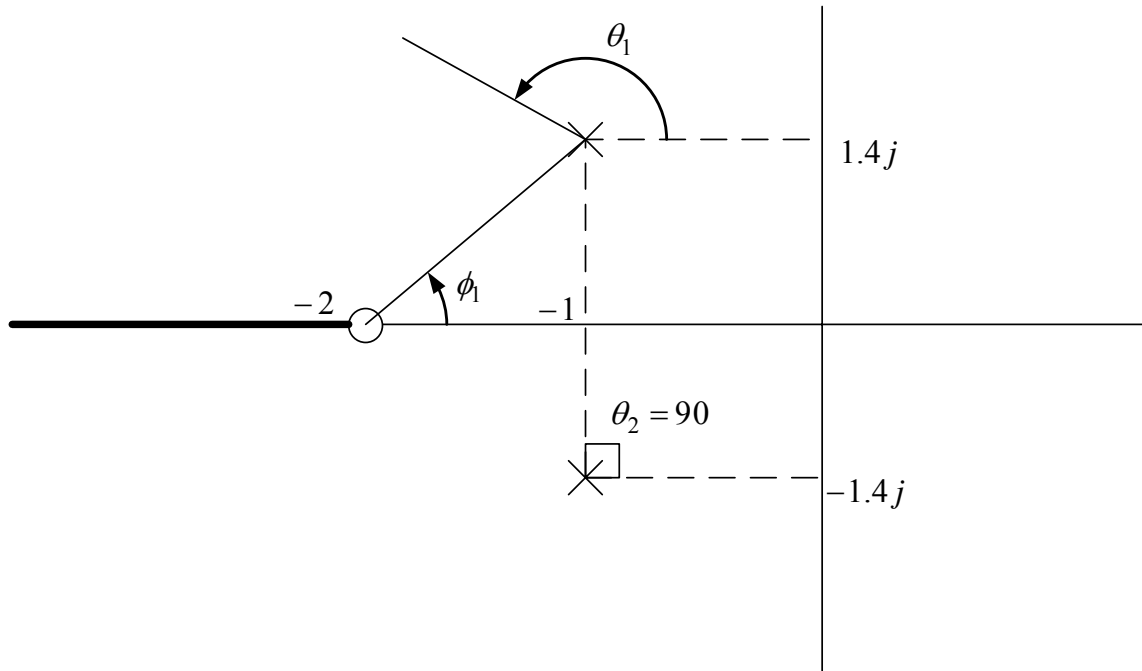
6. Determine behaviour of complex poles:

These poles will move towards the real axis by increasing k. After the collision at -3.73 one will move towards to the zero at -2and the other will diverge to $-\infty$.
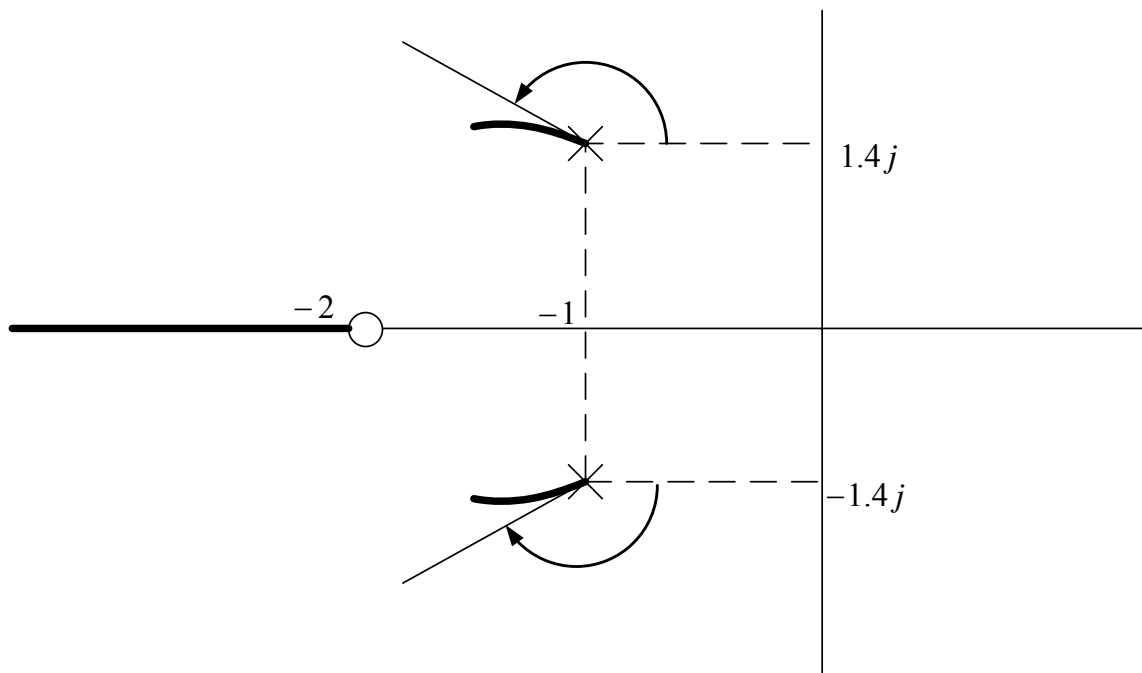
The angle ofdeparture of the complex poles is calculated by

180 -angle with respect to other poles +angle with respect to other zeros: $180 - \sum \theta + \sum \phi$:
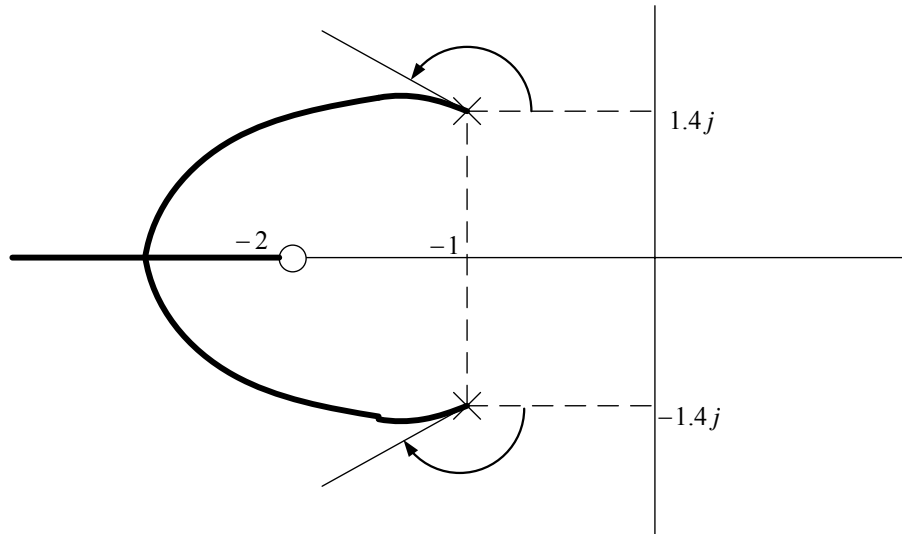
So $\theta_1 = 180 - \theta_2 + \phi_1 = 180 - 90 + 55 = 145$. The angle of departure for
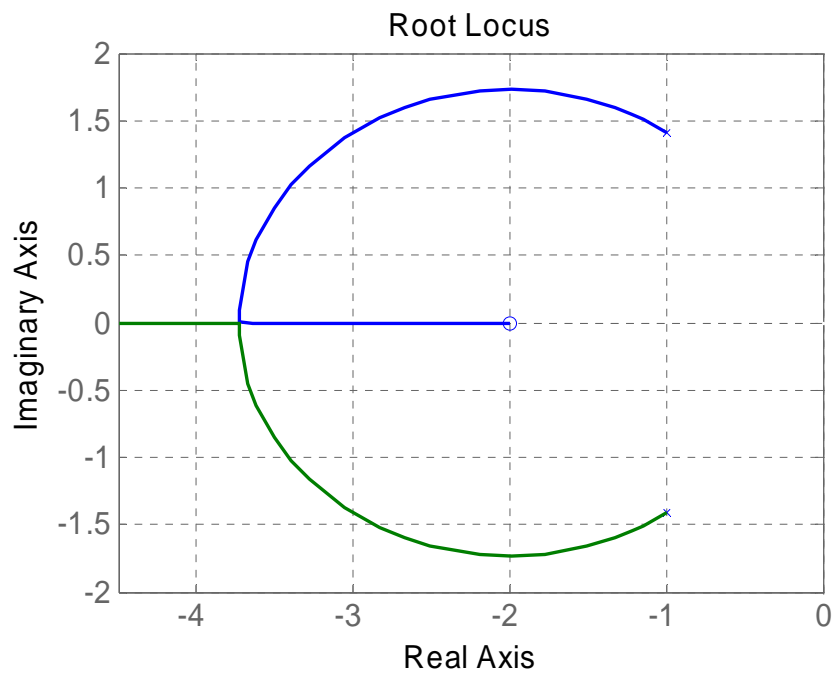
-1-1.4j is -145 since the locus is symmetric.

Now we simply have to draw the locus from the complex poles to the real axis:

With Matlab:

`>> num=[1 2]; den=[1 2 3]; rlocus(num,den)`

Generally the rules are:

1. The branches are always symmetric with respect to the real axis.

2. The number of the branches equals the number of the OL poles.

3. Every branch starts from an OL pole (k=0) and ends at an OL zero (k=∞). If we have more poles than zeros then we assume that there are imaginary zeros at ∞.

4. There are $n_p - n_z$ asymptotes where they intersect on the real axis at:

$$s = -\frac{\sum_{i=1}^{n_p} p_i - \sum_{i=1}^{n_z} z_i}{n_p - n_z}$$

5. The angle between them is $2\pi / (n_p - n_z)$. The angle of the 1st asymptote with the real axis is $\pi / (n_p - n_z)$

6. The branch exists on a point on the real axis if there are odd number of real poles and zeros on its right. Complex poles do not contribute.

7. The break in and out points are found by solving $\frac{dk}{ds} = 0$.

8. The points where we have intersection with the imaginary axis are found by replacing s with $j\omega$ at the CE.

9. The departure angle from a complex pole is $180 - \sum \theta + \sum \phi$.

10. The arrival angle at a complex zero is $180 + \sum \theta - \sum \phi$

Generally the root locus of more complicated systems will be very difficult to be derived and for that reason we use Matlab.