Supplementary material for Fundamental solution matrices

For second order systems $\dot{x} = Ax$ we have seen that we have 2 solutions (x_1, x_2) depending on the eigenvalues of A (real and distinct, repeated and complex):

$$x_1 = e_1 e^{\lambda_1 t},\ x_2 = e_2 e^{\lambda_2 t} \text{ if } \lambda_1 \neq \lambda_2,\ \lambda_1, \lambda_2 \in \mathbb{R}.$$

$$x_1 = e e^{\lambda t},\ x_2 = (et + b)e^{\lambda t} \text{ if } \lambda_1 = \lambda_2 = \lambda,\ \lambda \in \mathbb{R}.$$

$$x_2 = \mathrm{Re}\left(e e^{\lambda t}\right),\ x_2 = \mathrm{Im}\left(e e^{\lambda t}\right) \text{ if } \lambda_1 = \overline{\lambda_2} = \lambda,\ \lambda \in \mathbb{C}.$$

Now any combination $x = c_1 x_1 + c_2 x_2$ is also a solution (principle of superposition) and also any other solution can be expressed by the above combination. This effectively means that to describe the behaviour of a 2nd order system we just need x_1 and x_2. When we are given an initial condition x_0 effectively we are asked to find a specific solution that passes (starts) through x_0, and this can be done by finding the appropriate values of c_1, c_2 (this is what we have before).

Now, x_1 and x_2 are 2 2by1 column vectors. If we put them together in one matrix (the fundamental solution matrix) we have $X = [x_1 \quad x_2]$ which is 2by2. It will be better if we write us: $X(t) = [x_1(t) \quad x_2(t)]$.

Thus $x = c_1 x_1 + c_2 x_2$ can be written as: $x(t) = X(t) \times C$, where $C = [c_1 \quad c_2]^T$

We are given the value at t=0 as x_0: $x_0 = X(0) \times C$ or $C = X^{-1}(0) x_0$. Hence going back to $x(t) = X(t) \times C$ we have: $\boxed{x(t) = X(t) \times X^{-1}(0) x_0}$

The product $X(t) \times X^{-1}(0)$ is called the State Transition Matrix.

This means that if found or we are given $X(t)$ we can easily find $X(0)$ and $X^{-1}(0)$. Then using $x(t) = X(t) \times X^{-1}(0) x_0$ we can find any solution given the initial conditions. This is effectively what we previously did but now it is in a more compact form, it can easily be extended to high order systems and above all it can be used in time varying systems (i.e. where the state matrix A is not constant). Before we see how it can be used for time varying systems let's see how it can be used for the systems that we previously studied:

Example 1:

$x_1 = e_1 e^{\lambda_1 t}$, $x_2 = e_2 e^{\lambda_2 t} \Rightarrow X(t) = \begin{bmatrix} e_1 e^{\lambda_1 t} & e_2 e^{\lambda_2 t} \end{bmatrix} \Rightarrow X(0) = \begin{bmatrix} e_1 & e_2 \end{bmatrix}$

Hence $x = \begin{bmatrix} e_1 e^{\lambda_1 t} & e_2 e^{\lambda_2 t} \end{bmatrix} \times \begin{bmatrix} e_1 & e_2 \end{bmatrix}^{-1} x_0$

Example 2:

$x_1 = e e^{\lambda t}$, $x_2 = (et + b) e^{\lambda t} \Rightarrow X(t) = \begin{bmatrix} e e^{\lambda t} & (et + b) e^{\lambda t} \end{bmatrix} \Rightarrow X(0) = \begin{bmatrix} e & b \end{bmatrix}$

Hence $x = \begin{bmatrix} e e^{\lambda t} & (et + b) e^{\lambda t} \end{bmatrix} \times \begin{bmatrix} e & b \end{bmatrix}^{-1} x_0$

Example 3:

$x_2 = \mathrm{Re}\left(e e^{\lambda t}\right), x_2 = \mathrm{Im}\left(e e^{\lambda t}\right) \Rightarrow X(t) = \begin{bmatrix} \mathrm{Re}\left(e e^{\lambda t}\right) & \mathrm{Im}\left(e e^{\lambda t}\right) \end{bmatrix} \Rightarrow X(0) = \begin{bmatrix} \mathrm{Re}(e) & \mathrm{Im}(e) \end{bmatrix}$

Hence $x = \begin{bmatrix} \mathrm{Re}\left(e e^{\lambda t}\right) & \mathrm{Im}\left(e e^{\lambda t}\right) \end{bmatrix} \times \begin{bmatrix} \mathrm{Re}(e) & \mathrm{Im}(e) \end{bmatrix}^{-1} x_0$

Example 4:

We know that for $A = \begin{bmatrix} -8 & -4 \\ 1.5 & -3 \end{bmatrix}$ we have $\begin{cases} \lambda_1 = -6, e_1 = \begin{bmatrix} -2 & 1 \end{bmatrix}^T \\ \lambda_2 = -5, e_2 = \begin{bmatrix} 1 & -3/4 \end{bmatrix}^T \end{cases}$. Hence the FSM is: $X(t) = \begin{bmatrix} e_1 e^{\lambda_1 t} & e_2 e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} -2e^{-6t} & e^{-5t} \\ e^{-6t} & -\frac{3}{4} e^{-5t} \end{bmatrix}$

$$\Rightarrow X(0) = \begin{bmatrix} -2 & 1 \\ 1 & -\frac{3}{4} \end{bmatrix} \Rightarrow X^{-1}(0) = \begin{bmatrix} -1.5 & -2 \\ -2 & -4 \end{bmatrix}$$

Thus if $x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$:

$$x(t) = X(t) X^{-1}(0) = \begin{bmatrix} -2e^{-6t} & e^{-5t} \\ e^{-6t} & -\frac{3}{4} e^{-5t} \end{bmatrix} \begin{bmatrix} -1.5 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -5.5 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-6t} - 10 \begin{bmatrix} 1 \\ -3/4 \end{bmatrix} e^{-5t}$$

Unfortunately we cannot follow a similar strategy when A is time varying, for example $A(t)=\begin{bmatrix} -t & 1 \\ -e^{-t} & -e^{-2t} \end{bmatrix}$. In these cases we have to rely on numerical solutions.

Even though we cannot find x_1 and x_2 we know that they exist. Hence we know that the FSM exists as well: $X(t)=[x_1(t) \quad x_2(t)]=\begin{bmatrix} x_{1_A}(t) & x_{2_A}(t) \\ x_{1_B}(t) & x_{2_B}(t) \end{bmatrix}$ and of course at *t*=0 we have a constant matrix $X(0)=\begin{bmatrix} x_{1_A}(t) & x_{2_A}(t) \\ x_{1_B}(t) & x_{2_B}(t) \end{bmatrix}$ with the inverse $X^{-1}(0)=\begin{bmatrix} x_{1_A}(0) & x_{2_A}(0) \\ x_{1_B}(0) & x_{2_B}(0) \end{bmatrix}^{-1}$ also being constant. Let's assume that for our case $X^{-1}(0)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for some constants *a, b, c, d*. Then:

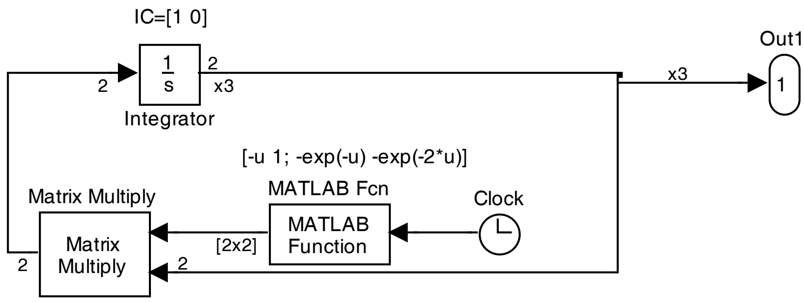
$$X(t)\times X^{-1}(0)=\begin{bmatrix} x_{1_A}(t) & x_{2_A}(t) \\ x_{1_B}(t) & x_{2_B}(t) \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} ax_{1_A}(t)+cx_{2_A}(t) & bx_{1_A}(t)+dx_{2_A}(t) \\ ax_{1_B}(t)+cx_{2_B}(t) & bx_{1_B}(t)+dx_{2_B}(t) \end{bmatrix}$$

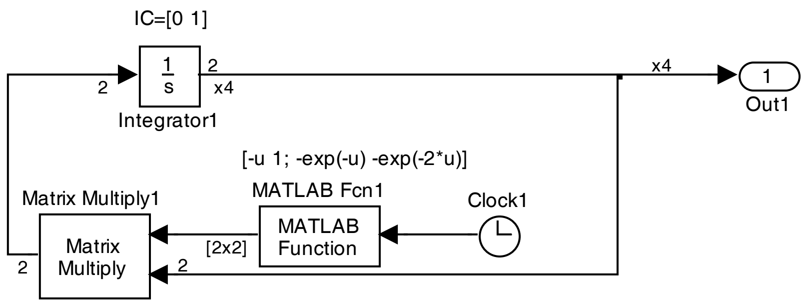
Or: $[ax_1(t)+cx_2(t) \quad bx_1(t)+dx_2(t)]$

Now, since x_1 and x_2 are solutions of $\dot{x}=Ax$ then so must be $x_3=ax_1(t)+cx_2(t)$ and $x_4=bx_1(t)+dx_2(t)$. This means that $\dot{x}_3=Ax_3$ and $\dot{x}_4=Ax_4$.

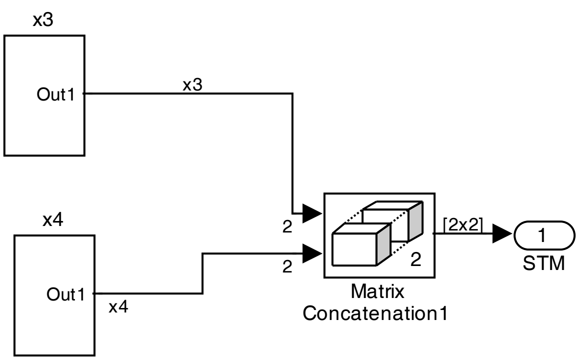
Also $X(0)\times X^{-1}(0)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and hence $[x_3(0) \quad x_4(0)]=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $x_3(0)=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x_4(0)=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. **Be careful** we do not yet know the functions $x_3(t)$ and $x_4(t)$ since we do not know $x_1(t)$ and $x_2(t)$.

In order for us to find $x_3(t)$ and $x_4(t)$ we simply have to numerically solve $\dot{x}_3=Ax_3$ and $\dot{x}_4=Ax_4$ for $x_3(0)=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x_4(0)=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

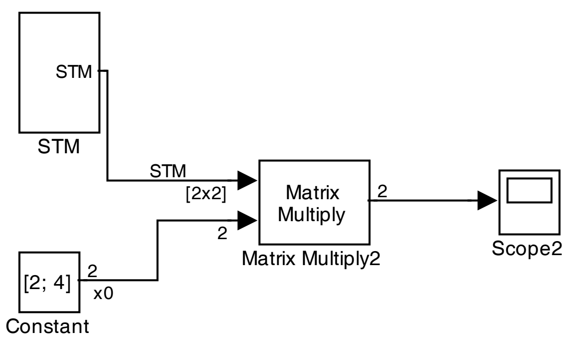




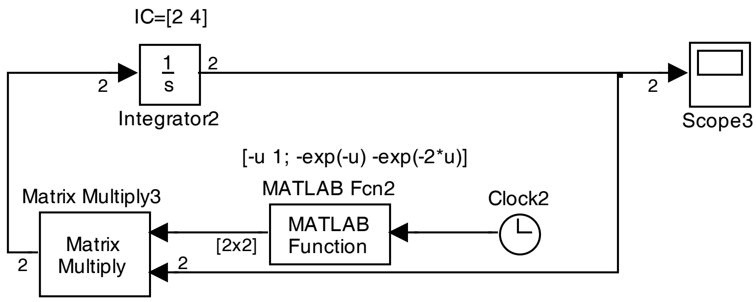
No we need to combine them to get the STM:



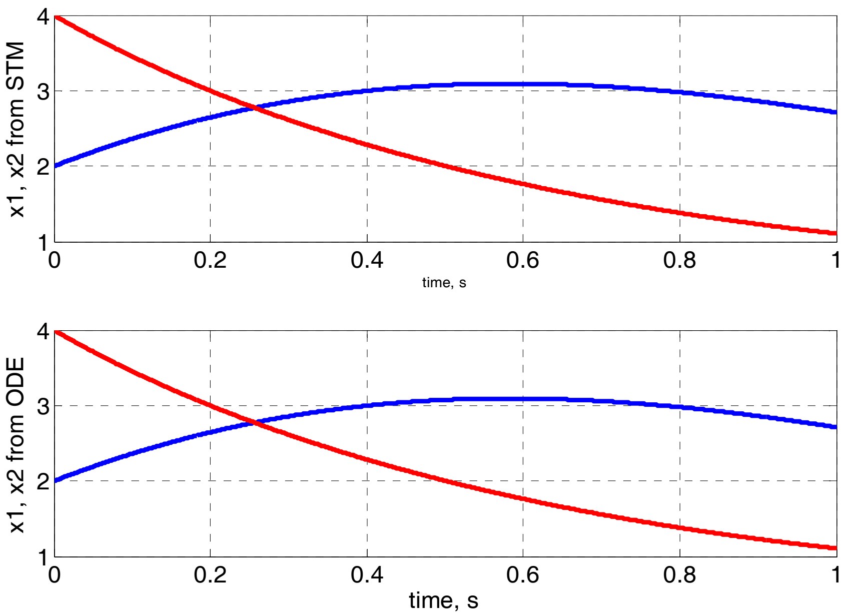
Now if we assume that we are given the IC as [2 4]:



To crosscheck our answer the numerical solution of the original state space model is:



And the results that we got are:



Now, in general our solutions $x_1(t)$ and $x_2(t)$ also depend on t_0 which may not be zero as in the previous case, hence we should have written $x_1(t,t_0)$ and $x_2(t,t_0)$. To avoid confusion and to comply with various other authors we will use $x_1(t)$ and $x_2(t)$ for most cases and $\phi_1(t,t_0)$ and $\phi_2(t,t_0)$ when we want to say that our solutions also depend on the initial time. Hence the FSM is $\Phi(t,t_0)$ and not $X(t)$. Also since $x(t) = \Phi(t,t_0) \times \Phi^{-1}(t_0,t_0)x_0$, i.e. our solution to the IVP also depend on the initial condition: $\phi(t,t_0,x_0) = \Phi(t,t_0) \times \Phi^{-1}(t_0,t_0)x_0$.