Chapter #1

EEE8013 – EEE3001

Linear Controller Design and State Space Analysis

Ordinary Differential Equations

1. Introduction

To understand the properties (dynamics) of a system, we can model (represent) it using differential equations (DEs). The response/behaviour of the system is found by solving the DEs. In our cases, the DE is an Ordinary DE (ODE), i.e. not a partial derivative. The main purpose of this Chapter is to learn how to solve first and second order ODEs in the time domain. This will serve as a building block to model and study more complicated systems. Our ultimate goal is to control the system when it does not show a "satisfactory" behaviour. Effectively, this will be done by modifying the ODE.

Note for EEE8013 students: There are footnotes throughout the notes, which **is assessed material**!

2. First Order ODEs

The general form of a first order ODE is:

$$\frac{dx(t)}{dt} = f\left(x(t),t\right) \tag{1}$$

where[1] $x,t \in \mathbb{R}$

Analytical solution: Explicit formula for $x(t)$ (a solution which can be found using various methods) which satisfies $\frac{dx}{dt} = f(x,t)$

[1] The proper notation is $x(t)$ and not x but we drop the brackets in order to simplify the presentation.

Example 1.1: Prove that $x = e^{-3t}$ and $x = -10e^{-3t}$ are solutions of $\frac{dx}{dt} = -3x$.

$$\frac{dx}{dt} = -3x \Leftrightarrow \frac{d\left(e^{-3t}\right)}{dt} = -3\left(e^{-3t}\right) \Leftrightarrow -3e^{-3t} = -3e^{-3t}$$

$$\frac{dx}{dt} = -3x \Leftrightarrow \frac{d\left(-10e^{-3t}\right)}{dt} = -3\left(-10e^{-3t}\right) \Leftrightarrow 30e^{-3t} = 30e^{-3t}$$ ■[2]

Obviously there are infinite solutions to an ODE and for that reason the found solution is called the **General Solution** of the ODE.

First order Initial Value Problem : $\frac{dx}{dt} = f(x,t), \quad x(t_0) = x_0$

An initial value problem is an ODE with an initial condition, hence we do not find the general solution but the **Specific Solution** that passes through x_0 at $t=t_0$.

Analytical solution: Explicit formula for $x(t)$ which satisfies $\frac{dx}{dt} = f(x,t)$ and passes through x_0 when $t = t_0$.

Example 1.2: Prove that $x = e^{-3t}$ is a solution, while $x = -10e^{-3t}$ is not a solution of $\frac{dx}{dt} = -3x, x_0 = 1$

Both expressions ($x = e^{-3t}$ and $x = -10e^{-3t}$) satisfy the $\frac{dx}{dt} = -3x$ but at t=0

$$x(t) = e^{-3t} \Rightarrow x(0) = 1$$

$$x(t) = -10e^{-3t} \Rightarrow x(0) = -10 \neq 1$$ ■[3]

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[2]
```
clc, clear all, syms t, x1=exp(-3*t); dx=diff(x1,t); isequal(dx,-3*x1)
x2=-10*exp(-3*t); dx=diff(x2,t); isequal(dx,-3*x2)
```

[3]
```
clc, clear all, syms t, x1=exp(-3*t); x2=-10*exp(-3*t); x0=1;
x0_1=double(subs(x1,t,0)); x0_2=double(subs(x2,t,0)); isequal(x0,x0_1),
isequal(x0,x0_2),
```

For that reason some books use a different symbol for the specific solution: $\phi(t,t_0,x_0)$.

You must be clear about the difference between an ODE and the solution to an IVP! From now on we will just study IVP unless otherwise explicitly mentioned.

Linear First Order ODEs

A linear 1st order ODE is given by:

$$\begin{cases} a(t)x'+b(t)x = c(t), a(t) \neq 0 & \textit{Non autonomous} \\ ax'+bx = c, a \neq 0 & \textit{Autonomous} \end{cases} \tag{2}$$

with $a,b,c \in \mathbb{R}$ and $a \neq 0$.

In engineering books the most common form of (2) is (since $a \neq 0$):

$$x'+k(t)x = u(t) \tag{3}$$

with $k,u \in \mathbb{R}$

Note: We say that *u* is the input to our system that is represented by (3)

The solution of (3) (using the integrating factor) is given by:

$$x(t) = e^{-kt}x(t_0) + e^{-kt}\int_{t_0}^{t} e^{kt_1}u(t_1)\,dt_1$$

The term $e^{-kt}x(t_0)$ is called transient response, while $e^{-kt}\int_{t_0}^{t} e^{kt_1}u(t_1)dt_1$ comes from the input signal *u*.

If we assume that *u* is constant:

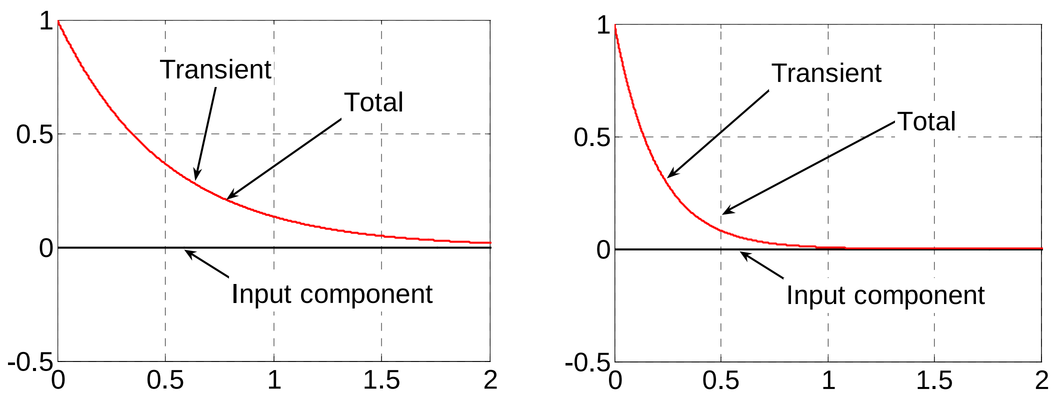
$$x(t) = e^{-kt}x(t_0) + e^{-kt}\int_{t_0}^{t} e^{kt_1}u\,dt_1 \Leftrightarrow x(t) = e^{-kt}x(t_0) + u\frac{1}{k}\left(1 - e^{-k(t-t_0)}\right)$$

Hence: $\lim_{t\to\infty} x(t) = \begin{cases} 0 + u\frac{1}{k}(1-0) = u/k, & k > 0 \\ \pm\infty, & k < 0 \end{cases}$

Thus we say that if *k*>0 the system is stable (and the solution converges exponentially at *u*/*k*) while if *k*<0 the system is unstable (and the solution diverges exponentially to ±∞,).

Example 1.3: *u*=0 and *k*=2 & 5, x_0=1

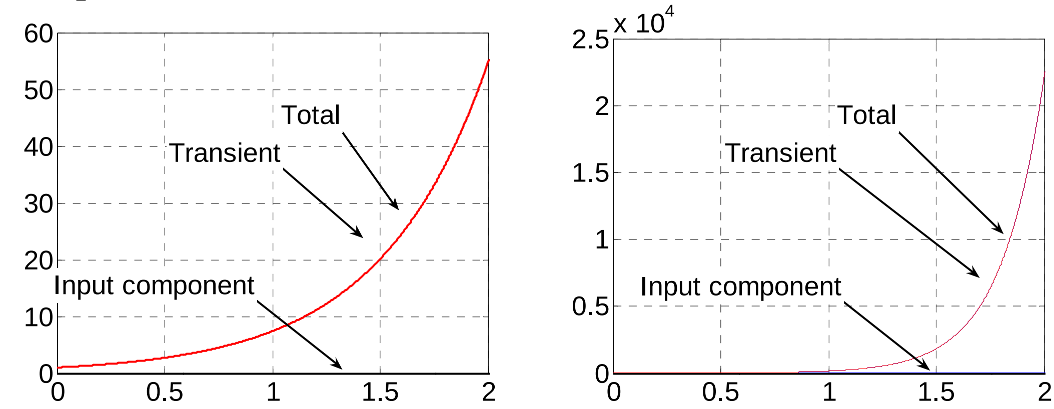
$x(t) = e^{-2t}\cdot 1 + 0$, $\lim_{t\to\infty} x(t) = 0$, as 2>0



■[4]

[4] `clc, clear all, syms x(t), dx=diff(x); dsolve(dx+2*x, x(0)==1)`

Example 1.4: u=0 and k=-2 & 5, x_0=1



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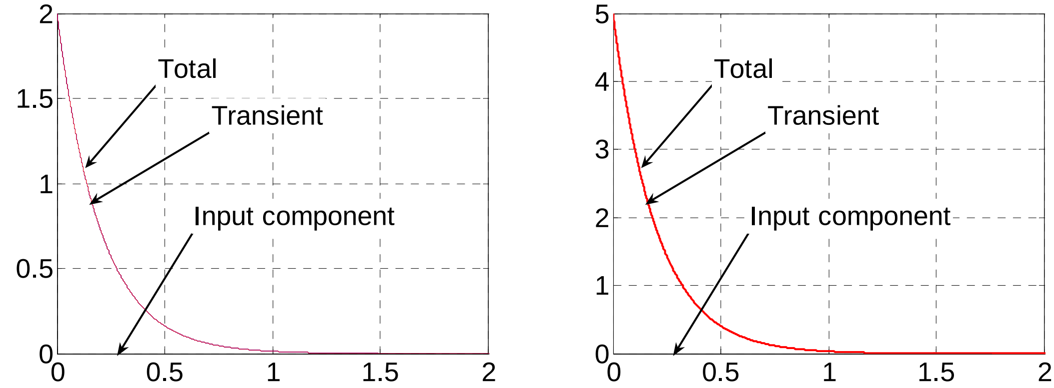
Example 1.5: u=0 and k=5, x_0=1 & 5



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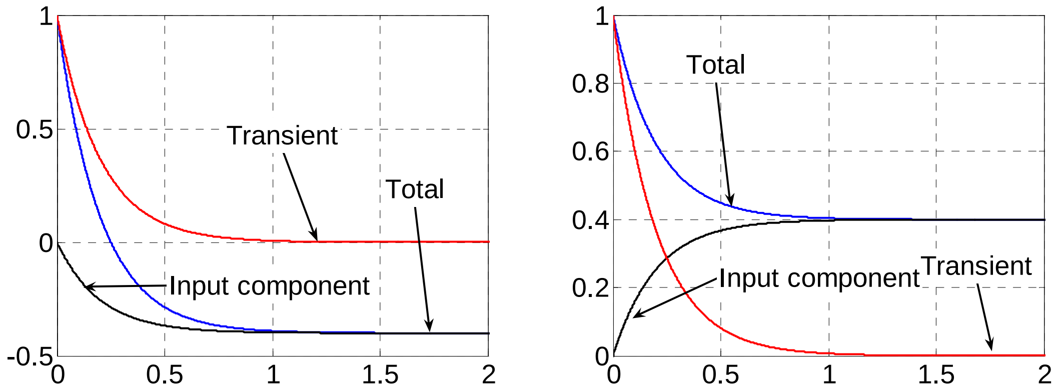
Example 1.6: u=-2 & 2 and k=5, x_0=1



■[5]

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[5]
```
clc, clear all, close all, syms t t1, x0=1; k=5; t0=0; u=2;
t2=0:0.01:2; x_x0=exp(-k*t)*x0; x_u=exp(-k*t)*int(exp(k*t1)*u,t0,t);
x_x0_t=double(subs(x_x0,t,t2)); x_u_t=double(subs(x_u,t,t2));
hold on, plot(t2,x_x0_t), plot(t2,x_u_t), plot(t2,x_u_t+x_x0_t)
```

Comments:

- In real systems we cannot have a state (say the speed of a mass-spring system) that becomes infinite, obviously the system will be destroyed when x gets to a high value.
- For the dynamics (settling time, stability…) of the system we should only focus on the homogenous ODE: $x'+k(t)x=0$

3. Second Order ODEs

3.1 General Material

A second order ODE has as a general form:

$$\frac{d^2x(t)}{dt^2}=f\left(x'(t),x(t),t\right) \tag{4}$$

A linear 2nd order ODE is given by:

$$\begin{cases} x''(t)+A(t)x'(t)+B(t)x(t)=u(t), & \textit{Non autonomous} \\ x''(t)+Ax'(t)+Bx(t)=u(t), & \textit{Autonomous} \end{cases} \tag{5}$$

And again we focus on autonomous homogeneous systems:

$$x''(t)+A(t)x'(t)+B(t)x(t)=0 \tag{6}$$

Again we define as an analytical solution of (6) an expression that satisfies it.

Example 1.7: Given $x''-2x'-3x=0$ prove that $x=e^{3t}$ and $x=e^{-t}$ are two solutions:

$$\left(e^{3t}\right)''-2\left(e^{3t}\right)'-3\left(e^{3t}\right)=0\Leftrightarrow$$
$$9e^{3t}-6e^{3t}-3e^{3t}=0\Leftrightarrow$$
$$0=0$$
$$\left(e^{-t}\right)''-2\left(e^{-t}\right)'-3\left(e^{-t}\right)=0\Leftrightarrow$$
$$e^{-t}+e^{-t}-3e^{-t}=0\Leftrightarrow$$
$$0=0$$

■[6]

Assume that you have 2 solutions for a 2nd order ODE x_1 and x_2 (we will see later how to get these two solutions), then:

$$\left.\begin{aligned} x_1''(t)+A(t)x_1'(t)+B(t)x_1(t)=0\\ x_2''(t)+A(t)x_2'(t)+B(t)x_2(t)=0\end{aligned}\right\}$$

obviously I can multiply these two equations with arbitrary constants:

$$\left.\begin{aligned} C_1x_1''(t)+C_1A(t)x_1'(t)+C_1B(t)x_1(t)=0\\ C_2x_2''(t)+C_2A(t)x_2'(t)+C_2B(t)x_2(t)=0\end{aligned}\right\}$$

and now I can add them and collect similar terms:

$$\underbrace{\left(C_1x_1(t)+C_2x_2(t)\right)}_{\text{Common Term}}''+A(t)\underbrace{\left(C_1x_1(t)+C_2x_2(t)\right)}_{\text{Common Term}}'+B(t)\underbrace{\left(C_1x_1(t)+C_2x_2(t)\right)}_{\text{Common Term}}=0$$

which means that $C_1x_1(t)+C_2x_2(t)$ (i.e. the linear combination of x_1 and x_2) is also a solution of the ODE.

[6]
```
clc, clear all, close all, syms x(t) t
Dx=diff(x); D2x=diff(x,2); ODE=D2x-2*Dx-3*x;
subs(ODE, x, exp(-t)), subs(ODE, x, exp(3*t))
```

Example 1.8: Given $x''-2x'-3x=0$ prove that $x=e^{3t}+2e^{-t}$ is a solution:

$$\left(e^{3t}+2e^{-t}\right)''-2\left(e^{3t}+2e^{-t}\right)'-3\left(e^{3t}+2e^{-t}\right)=0 \Leftrightarrow$$

$$9e^{3t}+2e^{-t}-2\left(3e^{3t}-2e^{-t}\right)-3e^{3t}-6e^{-t}=0 \Leftrightarrow$$

$$9e^{3t}+2e^{-t}-6e^{3t}+4e^{-t}-3e^{3t}-6e^{-t}=0 \Leftrightarrow$$

$$9e^{3t}-6e^{3t}-3e^{3t}+2e^{-t}+4e^{-t}-6e^{-t}=0 \Leftrightarrow$$

$$0=0$$

■[7]

Now, the question is, if we have x_1 and x_2, can ALL other solutions of the ODE, be expressed as a linear combination of x_1 and x_2? So assume a third solution $\varphi(t)$:

$$\varphi''(t)+A(t)\varphi'(t)+B(t)\varphi(t)=0$$

Now, the question can be written as, can we find constants C_1 and C_2 such as:

$$\begin{Bmatrix} \varphi(t)=C_1x_1(t)+C_2x_2(t) \\ \varphi'(t)=C_1x_1'(t)+C_2x_2'(t) \end{Bmatrix}$$

This equation can be seen as a 2by2 system with unknowns C_1 and C_2 as:

$$\begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}=\begin{bmatrix} \varphi(t) \\ \varphi'(t) \end{bmatrix}$$

From linear algebra this system of equations has a unique solution if:

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[7]
```
clc, clear all, close all, syms x(t) t; Dx=diff(x); D2x=diff(x,2);
ODE=D2x-2*Dx-3*x; subs(ODE, x, exp(3*t)+exp(-t))
```

$$\begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = x_1(t)x_2'(t) - x_2(t)x_1'(t) \neq 0$$

Note: The matrix $W(x_1(t), x_2(t)) = \begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix}$ is called the Wronskian[8] of the ODE.

We also know from linear algebra that the determinant is not zero if:

$$\begin{bmatrix} x_1(t) \\ x_1'(t) \end{bmatrix} \neq C \begin{bmatrix} x_2(t) \\ x_2'(t) \end{bmatrix}$$

So if the two solutions x_1 and x_2 are linear independent (LI) then ANY other solution can be described by the linear combination of x_1 and x_2. So now we have to look for two LI solutions for the 2nd order ODE.

Example 1.9: Prove that two solutions of $x'' - 2x' - 3x = 0$, $x_1 = e^{3t}$ and $x_2 = e^{-t}$ are linear independent.

$$W(x_1(t), x_2(t)) = \begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix} = \begin{bmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{bmatrix} \Rightarrow |W| = \begin{vmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{vmatrix} \Rightarrow$$

$$|W| = e^{3t}(-e^{-t}) - 3e^{3t}e^{-t} = -e^{2t} - 3e^{2t} = -4e^{2t}$$ ■[9]

[8] From the Polish mathematician Józef Maria Hoëne-Wroński

[9]
```
clc, clear all, close all, syms t, x1=exp(3*t); x2=exp(-t);
Dx1=diff(x1); Dx2=diff(x2);  W=[x1, x2; Dx1, Dx2], det(W)
```

Example 1.10: Prove that two solutions of $x''-2x'-3x=0$, $x_1=e^{3t}$ and $x_2=2e^{3t}$ are NOT linear independent.

$$W\left(x_1(t),x_2(t)\right)=\begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix}=\begin{bmatrix} e^{3t} & 2e^{3t} \\ 3e^{3t} & 6e^{3t} \end{bmatrix}\Rightarrow$$

$$|W|=\begin{vmatrix} e^{3t} & 2e^{3t} \\ 3e^{3t} & 6e^{3t} \end{vmatrix}=6e^{6t}-6e^{6t}=0$$ ■

Example 1.11: For the ODE $x''-2x'-3x=0$ prove that the solution $x=-e^{3t}+2e^{t}$ cannot be written as any combination of $x_1=e^{3t}$ and $x_2=2e^{3t}$.

$$x=C_1x_1+C_2x_2 \Leftrightarrow -e^{3t}+2e^{t}=C_1e^{3t}+C_2e^{3t}=\left(C_1+C_2\right)e^{3t}$$

From this expression we have that $C_1+C_2=-1$ (and hence we have the term $-e^{3t}$) but there is no term e^{t} for $2e^{t}$. ■

But how can we find two LI solutions? For homogeneous 1st order ODEs with u=0 the solution was: $x(t)=e^{-kt}C$ so we will try a similar approach for 2nd order ODEs:

$x''+Ax'+Bx=0$, assume[10] $x=e^{rt} \Rightarrow x'=re^{rt}$ & $x''=r^2e^{rt} \Rightarrow$

$$x''+Ax'+Bx=0 \Leftrightarrow r^2e^{rt}+Are^{rt}+Be^{rt}=0 \Leftrightarrow$$

$$r^2+Ar+B=0 \tag{7}$$

This is called the Characteristic Equation (CE) and we have to check its roots:

$r=\dfrac{-A\pm\sqrt{A^2-4B}}{2}$, these are the Characteristic values or Eigenvalues.

[10] Notice that we do NOT know what is the value of r.

3.2 Roots are real and unequal

If $A^2 > 4B$ the system is called **Overdamped** and the two roots are r_1 and r_2 with $r_1 \neq r_2$, r_1, $r_2 \in \mathbb{R}$. Then $x_1 = e^{r_1 t}$ and $x_2 = e^{r_2 t}$ are two linear independent solutions as:

$$\begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = e^{r_1 t} r_2 e^{r_2 t} - e^{r_2 t} r_1 e^{r_1 t} \neq 0$$

hence the general solution is $x = C_1 x_1 + C_2 x_2 = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ (8)

If r_1 and $r_2 < 0$ then $x \to 0$ and the system is stable.

If r_1 or $r_2 > 0$ then $x \to \pm\infty$ and the system is unstable.

Example 1.12: The CE of $x'' + 11x' + 30x = 0$ is $r^2 + 11r + 30 = 0$ which means that the two roots are: $r_{1,2} = \frac{-11 \pm \sqrt{11^2 - 4 \cdot 1 \cdot 30}}{2} = \frac{-11 \pm 1}{2} \Rightarrow \begin{cases} r_1 = -5 \\ r_2 = -6 \end{cases}$ [11]

and hence the 2 LI solutions are $\begin{cases} x_1 = e^{r_1 t} = e^{-5t} \\ x_2 = e^{r_2 t} = e^{-6t} \end{cases}$

This means that the general solution is $x = C_1 e^{-5t} + C_2 e^{-6t}$ and hence the ODE is stable[12]. The Wronskian is

$$\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} e^{-5t} & e^{-6t} \\ -5e^{-5t} & -6e^{-6t} \end{vmatrix} = -6e^{-5t} e^{-6t} + 5e^{-6t} e^{-5t} = -\mathrm{e}^{-11t} \neq 0$$

If the initial condition is $x(0) = 1, x'(0) = 0$ then:

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[11] `roots([1 11 30])`

[12]
```
clc, clear all, close all, syms x(t) Dx=diff(x,1); D2x=diff(x,2);
ODE=D2x+11*Dx+30*x; dsolve(ODE)
```

$$\left.\begin{array}{l} C_1 + C_2 = 1 \\ -5C_1 - 6C_2 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} C_1 = 6 \\ C_2 = -5 \end{array}\right\} \Rightarrow x = 6e^{-5t} - 5e^{-6t}$$ ■[13]

3.4 Roots are Complex (and hence not equal)

If $A^2 < 4B$ then the system is called **Underdamped** and the two roots are $r_1 = a + bj$ and $r_2 = \overline{r_1} = a - bj$ with $r_1 \neq r_2$, $r_1, r_2 \in \mathbb{C}$. Then $x_1 = e^{r_1 t} = e^{(a+bj)t}$ and $x_2 = e^{r_2 t} = e^{(a-bj)t}$ are two linear independent solutions as

$$\begin{vmatrix} e^{(a+bj)t} & e^{(a-bj)t} \\ (a+bj)e^{(a+bj)t} & (a-bj)e^{(a-bj)t} \end{vmatrix} = e^{(a+bj)t}(a-bj)e^{(a-bj)t} - e^{(a-bj)t}(a+bj)e^{(a+bj)t} =$$
$$(a-bj)e^{2at} - (a+bj)e^{2at} = e^{2at}(a-bj-a-bj) = -2e^{2at}bj \neq 0$$

Hence the general solution is

$$x = C_1 x_1 + C_2 x_1 = C_1 e^{rt} + C_2 e^{\overline{r}t} \qquad (9)$$

but remember that C_1 and C_2 are complex now variables such as $x \in \mathbb{R}$.

Example 1.13: The CE of $x'' + 2x' + 5x = 0$ is $r^2 + 2r + 5 = 0$ which means that the two roots are: $r_{1,2} = \dfrac{-2 \pm \sqrt{-16}}{2} = \dfrac{-2 \pm 4j}{2} = -1 \pm 2j \Rightarrow \begin{cases} r_1 = -1 + 2j \\ r_2 = -1 - 2j \end{cases}$

and hence the 2 LI solutions are $\begin{cases} x_1 = e^{r_1 t} = e^{(-1+2j)t} \\ x_2 = e^{r_2 t} = e^{(-1-2j)t} \end{cases}$

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[13]
```
clc, clear all, close all, syms x(t) Dx=diff(x,1); D2x=diff(x,2);
ODE=D2x+11*Dx+30*x; dsolve(ODE, x(0)==1, Dx(0)==0)
```

This means that the general solution is $x = C_1 e^{(-1+2j)t} + C_2 e^{(-1-2j)t}$ and hence the ODE is stable. The Wronskian is

$$\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} e^{(-1+2j)t} & e^{(-1-2j)t} \\ (-1+2j)e^{(-1+2j)t} & (-1-2j)e^{(-1-2j)t} \end{vmatrix} =$$

$$(-1-2j)e^{(-1+2j)t}e^{(-1-2j)t} - (-1+2j)e^{(-1+2j)t}e^{(-1-2j)t} =$$

$$(-1-2j)e^{-2t} - (-1+2j)e^{-2t} = (-1-2j+1-2j)e^{-2t} =$$

$$-4je^{-2t} \neq 0$$

If the initial condition is $x(0)=1, x'(0)=0$ then:

$$\left.\begin{array}{l} C_1 + C_2 = 1 \\ (-1+2j)C_1 + (-1-2j)C_2 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} C_1 = \dfrac{1}{2} + \dfrac{1}{4}j \\ C_2 = \dfrac{1}{2} - \dfrac{1}{4}j \end{array}\right\} \Rightarrow$$

$$x = \left(\frac{1}{2} + \frac{1}{4}j\right)e^{(-1+2j)t} + \left(\frac{1}{2} - \frac{1}{4}j\right)e^{(-1-2j)t} \qquad \blacksquare$$

An alternative approach is not to use x_1 & x_2 but a linear combination of them:

$$y_1 = e^{rt} + e^{\bar{r}t},\ y_2 = e^{rt} - e^{\bar{r}t}$$

Note that $\begin{vmatrix} e^{rt} + e^{\bar{r}t} & e^{rt} - e^{\bar{r}t} \\ re^{rt} + \bar{r}e^{\bar{r}t} & re^{rt} - \bar{r}e^{\bar{r}t} \end{vmatrix} \neq 0$

Using Euler's formula: $e^{(a+bj)t} = e^{at}(\cos bt + j\sin bt)$ and hence:

$$y_1 = e^{(a+bj)t} + e^{(a-bj)t} = e^{at}(\cos bt + j\sin bt + \cos bt - j\sin bt) = 2e^{at}\cos bt$$

$$y_2 = e^{(a+bj)t} - e^{(a-bj)t} = e^{at}(\cos bt + j\sin bt - \cos bt + j\sin bt) = j2e^{at}\sin bt$$

As y_1 and y_2 are solutions so do $y_1 \times \frac{1}{2}, y_2 \times \frac{1}{2j}$. So the general solution when we have complex roots is:

$$x(t) = e^{at}\left(C_1 \cos bt + C_2 \sin bt\right), C_1, C_2 \in \mathbb{R} \tag{10}$$

Example 1.14: The CE of $x'' + 2x' + 5x = 0$ is $r^2 + 2r + 5 = 0$ which means that the two roots are: $r_{1,2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4j}{2} = -1 \pm 2j \Rightarrow \begin{cases} r_1 = -1 + 2j \\ r_2 = -1 - 2j \end{cases}$

and hence the 2 LI solutions are $\begin{cases} x_1 = e^{-t}\cos(2t) \\ x_2 = e^{-t}\sin(2t) \end{cases}$

This means that the general solution is $x = e^{-t}\left(C_1 \cos 2t + C_2 \sin 2t\right)$ and hence the ODE is stable. The Wronskian is

$$\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} e^{-t}\cos(2t) & e^{-t}\sin(2t) \\ -e^{-t}\cos(2t) - 2e^{-t}\sin(2t) & -e^{-t}\sin(2t) + 2e^{-t}\cos(2t) \end{vmatrix} \neq 2e^{-2t}$$

If the initial condition is $x(0) = 1, x'(0) = 0$ then:

$$\left.\begin{matrix} C_1 = 1 \\ -C_1 + 2C_2 = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} C_1 = 1 \\ C_2 = 0.5 \end{matrix}\right\} \Rightarrow$$

$$x = e^{-t}\left(\cos 2t + 0.5 \sin 2t\right)$$ ■

3.3 Roots are real and equal

If $A^2 = 4B$ then the system is called **Critically damped** and the two roots are $r = r_1 = r_2$ with $r \in \mathbb{R}$. One solution is $x_1 = e^{rt}$ but how about x_2? We can use $x_2 = te^{rt}$ and the general solution:

$$x = C_1 x_1 + C_2 x_2 = C_1 e^{r_1 t} + C_2 t e^{r_1 t} \tag{11}$$

The Wronskian is:

$$\begin{vmatrix} e^{r_1 t} & te^{r_1 t} \\ r_1 e^{r_1 t} & r_1 te^{r_1 t} + e^{r_1 t} \end{vmatrix} = e^{r_1 t}\left(r_1 te^{r_1 t} + e^{r_1 t}\right) - r_1 e^{r_1 t} te^{r_1 t} = r_1 te^{2r_1 t} + e^{2r_1 t} - r_1 te^{2r_1 t} = e^{2r_1 t} \neq 0$$

Example 1.15: The CE of $x''+2x'+x=0$ is $r^2+2r+1=0$ which means that the two roots are: $r_{1,2} = \dfrac{-2\pm\sqrt{0}}{2} \Rightarrow \begin{cases} r_1 = -1 \\ r_2 = -1 \end{cases}$

and hence the 2 LI solutions are $\begin{cases} x_1 = e^{-t} \\ x_2 = te^{-t} \end{cases}$

This means that the general solution is $x = C_1 e^{-t} + C_2 te^{-t}$ and hence the ODE is stable. The Wronskian is

$$\begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t} + e^{-t} \end{vmatrix} = e^{2t} \neq 0$$

If the initial condition is $x(0)=1, x'(0)=0$ then:

$$\left.\begin{array}{l} C_1 = 1 \\ -C_1 + C_2 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} C_1 = 1 \\ C_2 = 1 \end{array}\right\}$$

$x = e^{-t} + te^{-t}$ ■

Not assessed material

To see why $x_2 = te^{rt}$ is the 2nd solution go to the ODE and place $x = e^{rt}$:

$$\left(e^{rt}\right)'' + A\left(e^{rt}\right)' + Bx = e^{rt}\left(r^2 + Ar + B\right)$$

Since r_1 is a double root of the CE: $r^2 + Ar + B = a\left(r - r_1\right)^2$ for some constant a. So: $\left(e^{rt}\right)'' + A\left(e^{rt}\right)' + Bx = e^{rt} a\left(r - r_1\right)^2$

Taking the time derivative wrt r:

$$\frac{d\left(\left(e^{rt}\right)''\right)}{dr}+A\frac{d\left(\left(e^{rt}\right)'\right)}{dr}+B\frac{d\left(e^{rt}\right)}{dr}=\frac{d\left(e^{rt}a\left(r-r_1\right)^2\right)}{dr}$$

And as we can change the sequence of the differentiation:

$$\left(\frac{d\left(e^{rt}\right)}{dr}\right)''+A\left(\frac{d\left(e^{rt}\right)}{dr}\right)'+B\frac{d\left(e^{rt}\right)}{dr}=\frac{d\left(e^{rt}a\left(r-r_1\right)^2\right)}{dr}$$

By using simple calculus:

$$\left(e^{rt}t\right)''+A\left(e^{rt}t\right)'+Be^{rt}t=\frac{d\left(e^{rt}\right)}{dr}a\left(r-r_1\right)^2+e^{rt}\frac{d\left(a\left(r-r_1\right)^2\right)}{dr}\Leftrightarrow$$

$$\left(e^{rt}t\right)''+A\left(e^{rt}t\right)'+Be^{rt}t=e^{rt}ta\left(r-r_1\right)^2+e^{rt}2a\left(r-r_1\right)$$

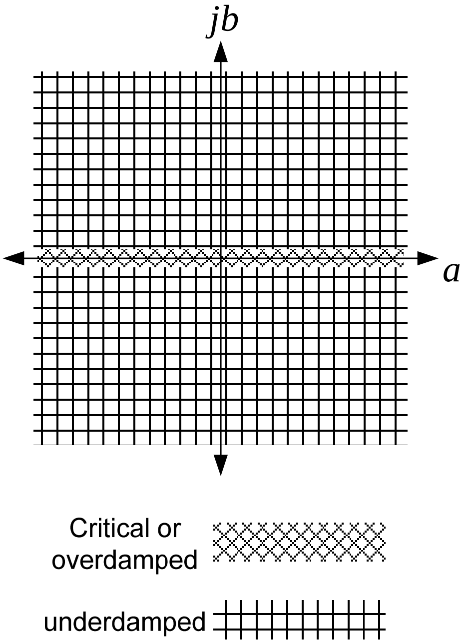
By placing now where r=r_1: $\left(e^{rt}t\right)''+A\left(e^{rt}t\right)'+Be^{rt}t=0$

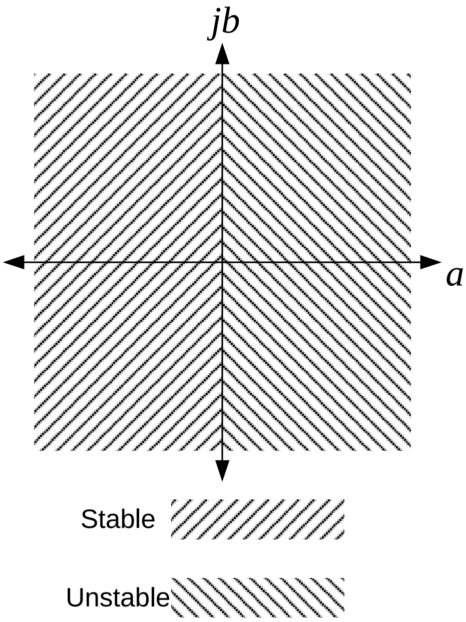
Which means that $e^{rt}t$ must be a solution of my ODE and:

$$\begin{vmatrix} e^{r_1t} & te^{r_1t} \\ r_1e^{r_1t} & tr_1e^{r_1t}+e^{r_1t} \end{vmatrix}=e^{r_1t}\cdot\left(tr_1e^{r_1t}+e^{r_1t}\right)-te^{r_1t}\cdot r_1e^{r_1t}=tr_1e^{2r_1t}+e^{2r_1t}-tr_1e^{2r_1t}=e^{2r_1t}\neq 0$$

And hence $x_2\left(t\right)=e^{rt}t$ is my second solution.

Root Space





Name	Oscillations?	Components of solution
Overdamped	No	Two exponentials: $e^{k_1 t}, e^{k_2 t},\ k_1, k_2 < 0$
Critically damped	No	Two exponentials: $e^{kt}, te^{kt},\ k < 0$
Underdamped	Yes	One exponential and one cosine $e^{kt}, \cos(\omega t),\ k < 0$
Undamped	Yes	one cosine $\cos(\omega t)$

4. Tutorial Exercise I

1. By using the general form of the analytic solution try to predict the response of the following systems. Your answer must describe the system as stable/unstable, convergent to zero/nonzero value. Crosscheck your answer by solving the DE:
   - $5\frac{dx}{dt}+6x=0, \quad x(0)=0, x(0)=1, x(0)=-1$
   - $5\frac{dx}{dt}-6x=0, \quad x(0)=0, x(0)=1, x(0)=-1$
   - $5\frac{dx}{dt}+6x=1, \quad x(0)=0, x(0)=1, x(0)=-1$
   - $5\frac{dx}{dt}+6x=-1, \quad x(0)=0, x(0)=1, x(0)=-1$
   - $\frac{dx}{dt}-3=0, \quad x(0)=0, x(0)=1, x(0)=-1$
2. Find the solution of $\ddot{x}+6\dot{x}+5x=0, x(0)=2, \dot{x}(0)=3$. Briefly describe how the solution behaves for these initial conditions. Draw a sketch of the response.
3. Find the solution of $\ddot{x}+2\dot{x}+6x=0, x(0)=1, \dot{x}(0)=0$. Briefly describe how the solution behaves for these initial conditions. Draw a sketch of the response.
4. Find the solution of $\ddot{x}-\dot{x}+0.25x=0, x(0)=2, \dot{x}(0)=1/3$. Briefly describe how the solution behaves for these initial conditions. Draw a sketch of the response.
5. Find the Wronskian matrices of the solutions of Q2-5.