Chapter #2

EEE8013

**Linear Controller Design and State Space Analysis**

State space representation using Matlab

1. State space representation, Observability, Controllability

Introduction: In the second chapter, we will expand on the material of the first chapter which focused on the numerical and analytical solution of ODEs. This chapter will transform the ODEs into state space form and we will derive the numerical solution of the state space models. The relation between state space models and Transfer Functions is studied and then the controllability and observability of state space models is presented.

Example 2.1:

Derive a state space representation of the simple mass, spring system:
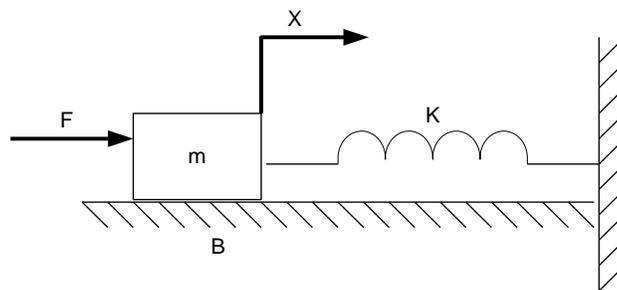
Using Newtonian mechanics we get:

$$m\frac{d^2x}{dt^2} = F - B\frac{dx}{dt} - kx \Rightarrow m\ddot{x} = F - B\dot{x} - kx$$

By choosing the states as $x_1 = x$, $x_2 = \dot{x}$ we have:

$$\left.\begin{aligned}\dot{x}_1 &= \dot{x} = x_2\\\dot{x}_2 &= \ddot{x} = \frac{1}{m}\left(F - Bx_2 - kx_1\right)\end{aligned}\right\} \Rightarrow \begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\-\dfrac{k}{m} & -\dfrac{B}{m}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\F/m\end{bmatrix}$$

Or $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{B}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} F \Leftrightarrow \dot{x} = Ax + Bu$

Let's assume that we can buy two sensors, that measures the displacement and the velocity then the output is:

$$y = x_1 \Leftrightarrow y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \Leftrightarrow y = Cx$$

Now assume that $m=B=k=1$ and the input force F is 0 simulate the numerical solution, the analytical solution, the state space representation assuming zero initial conditions.

Numerical solution:

$$m\frac{d^2 x}{dt^2} = F - B\frac{dx}{dt} - kx \Rightarrow \ddot{x} = -\dot{x} - x$$
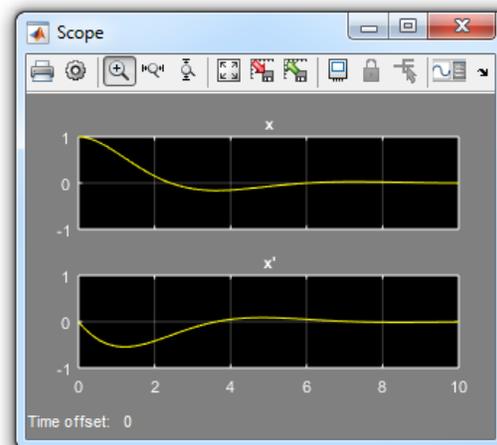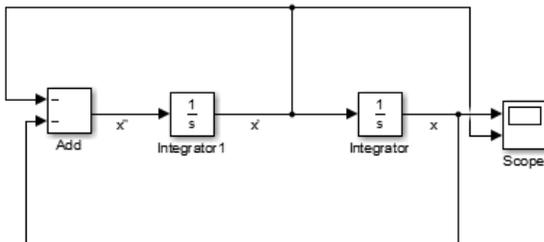
First trace for $x(t)$ or x_1
Second trace for $x'(t)$ or x_2

Analytical solution:

The ODE can be written as: $x'' + x' + x = 0$ where x(0)=0, x'(0)=0

$$x = e^{at}\left(c_1 \cos(bt) + c_2 \sin(bt)\right)$$

The solution can be written as:

$$x = c_1 x_1 + c_2 x_2 = e^{at}\left(c_1 \cos(bt) + c_2 \sin(bt)\right)$$

Now you can calculate c_1 and c_2

x(0)=0, x'(0)=0 => c_1=1, c_2=1/$\sqrt{3}$

$$x(t) = e^{at}\left(c_1 \cos(bt) + c_2 \sin(bt)\right) = e^{-\frac{1}{2}t}\left(\cos(\frac{\sqrt{3}}{2}t) + \frac{1}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}t)\right)$$

Note: since *a* is negative the response is stable and it converges to zero with some oscillations (under damped system).
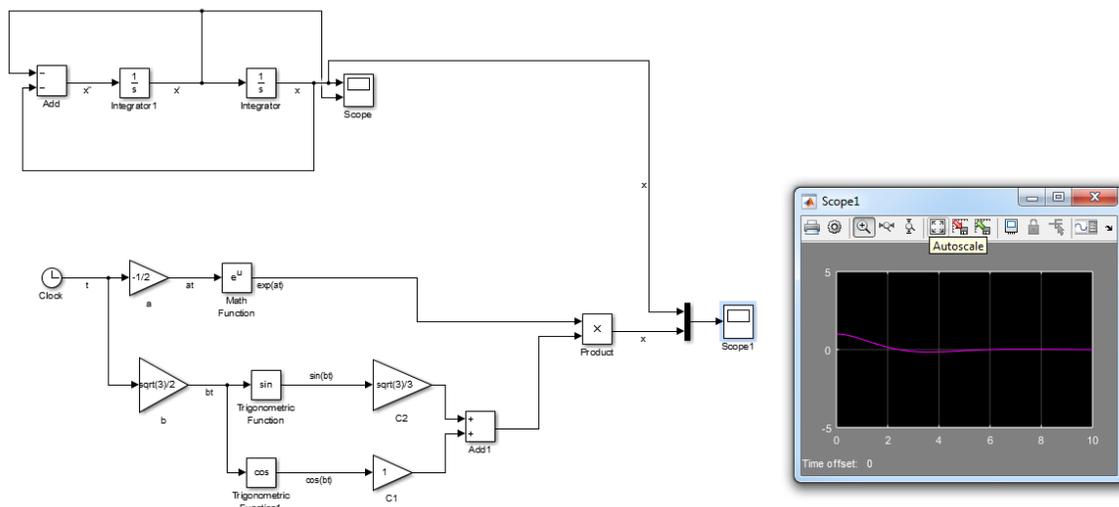
State space representation:

The state space representation of the system is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{B}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} F \Leftrightarrow \dot{x} = Ax + Bu$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F \Leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = x_1 \Leftrightarrow y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \Leftrightarrow y = Cx$$
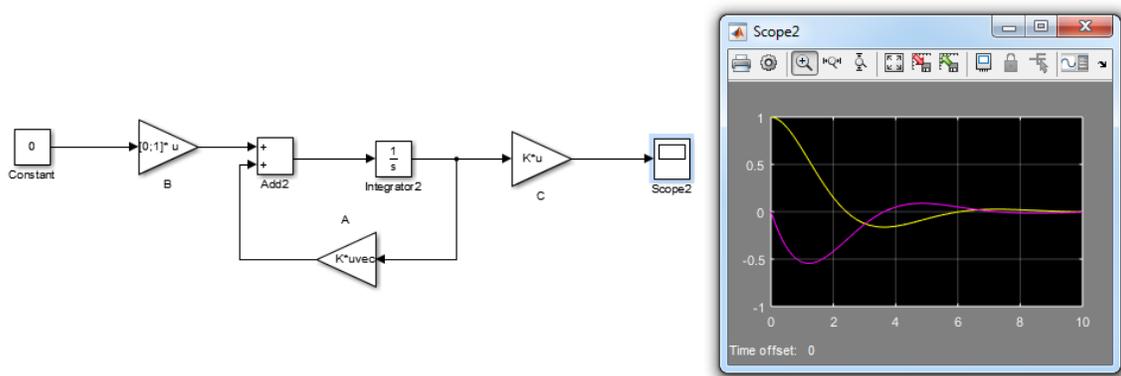
Note that the gains *A*, *B*, *C* are Matrix gains. You need to change the type of multiplication in the gain block parameters to Matrix (*K*u*).
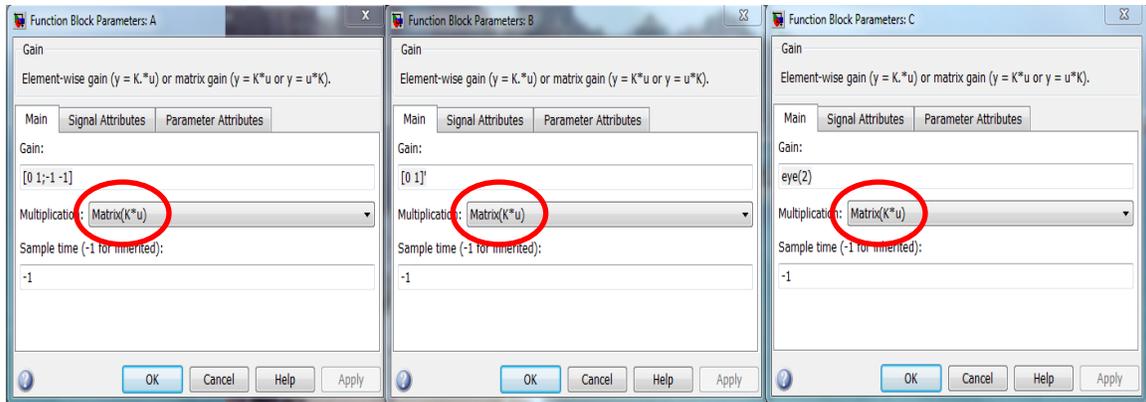
Also the initial conditions of the states should be defined in the integrator block but as a vector $\begin{bmatrix} x_1(0) & x_2(0) \end{bmatrix}$ so in our case it would be:
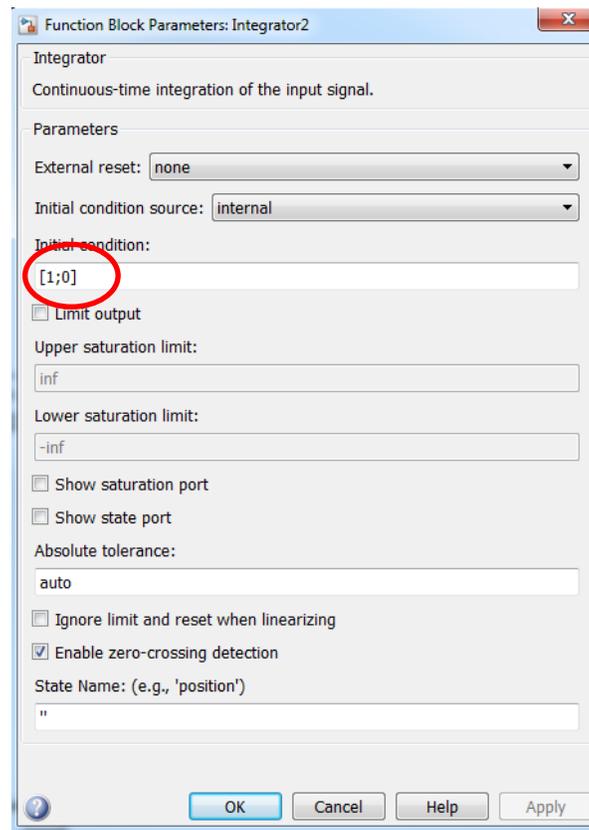
Exercise 2.1: Try to solve the same problem assuming initial conditions of $x(0)=1, x'(0)=1$.

Example 2.2: Find the state space model of the following system:

$$\ddot{x} + 3\dot{x} + 2x = u(t)$$
$$y = x$$

We start by choosing our states as follow:

$x_1 = x \Rightarrow \dot{x}_1 = \dot{x} = x_2$

$x_2 = \dot{x} \Rightarrow \dot{x}_2 = \ddot{x} = u(t) - 3\dot{x} - 2x = u(t) - 3x_2 - 2x_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) \Leftrightarrow \dot{x} = Ax + Bu$$

$$y = x = x_1 \Leftrightarrow y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Leftrightarrow y = Cx$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Exercise 2.2: Assume $u(t)=0$, simulate the numerical solution, the analytical solution and the state space model assuming $x(0)=1, x'(0)=1$.

Example 2.3: Find the state space model of the system (Tutorial II Q.4):

$\ddot{x} = -3\ddot{x} - 2\dot{x} - 5x + u_1 - 6u_2 + 7u_3$

$y_1 = \ddot{x} + u_2$

$y_2 = \ddot{x} + 3x + 5u_1$

$y_3 = x - 3\ddot{x} + 2u_2$

Using the classical modelling strategy:

$$x = x_1 \Rightarrow \dot{x} = \dot{x}_1 = x_2$$

$$\dot{x} = x_2 \Rightarrow \ddot{x} = \dot{x}_2 = x_3$$

$$\ddot{x} = x_3 \Rightarrow \dddot{x} = -3\ddot{x} - 2\dot{x} - 5x + u_1 - 6u_2 + 7u_3$$

$$= -3x_3 - 2x_2 - 5x_1 + u_1 - 6u_2 + 7u_3$$

Hence: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -6 & 7 \end{bmatrix}$

The output eqautions are:

$$y_1 = x_3 + u_2$$

$$y_2 = x_3 + 3x_1 + 5u_1$$

$$y_3 = x_1 - 3x_3 + 2u_2$$

And hence: $C = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 0 & 1 \\ 1 & 0 & -3 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 1 & 0 \\ 5 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

Now simulate both numerical and state space representation asumming zero initial conditions and $u_1=u_3=1$ and $u_2=2$.

Example 2.4: Simulate the homogeneous system:

$$\mathbf{A} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Assuming [1 0] IC. Take care how to define D as [0;0]

Check the ICs

$$\frac{1}{s}$$

Integrator2

Scope1

K*uve

A

0

Constant

x' = Ax+Bu
y = Cx+Du

State-Space

Scope2

Yellow is the trace for x_1
Magenta is the trace for x_2
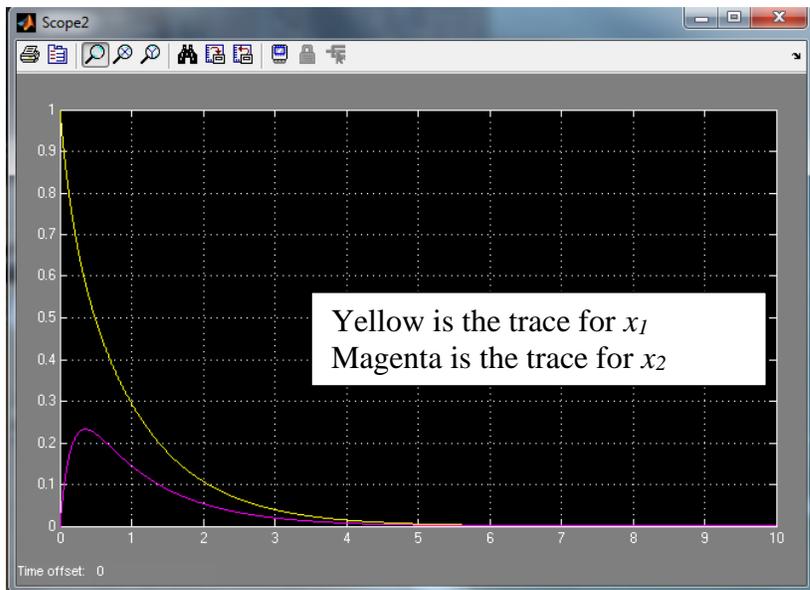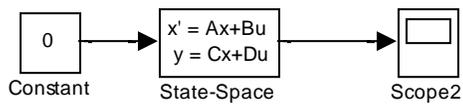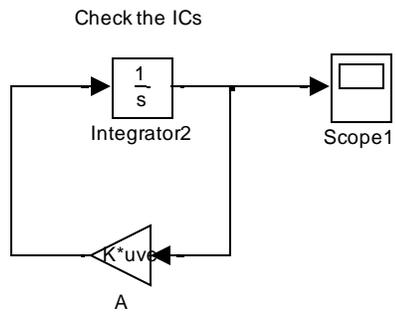
2. Relation of state space and TF using Matlab

By definition TF G(s): $C(sI-A)^{-1}B+D$ and $C(sI-A)^{-1}x(0)$ is the response to the IC.

Transformation from state space to transfer function:

In Matlab we can use the command *ss2tf* to get the TF from the state space.

- **If the system has only one input:**

[num,den]=ss2tf(A,B,C,D)

- **For multi input system u₁, u₂,…uₘ**

[num,den]=ss2tf(A,B,C,D,1) returns the TFs due to 1st input (u₁)

[num,den]=ss2tf(A,B,C,D,2) returns the TFs due to 2nd input (u₂)

Example 2.6:

Find the TF of $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$ and $y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (Tutorial II Q.7)

Analytical solution:

$G(s) = C(sI-A)^{-1}B+D$

$$\mathbf{G}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{G}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 1 & s+0.5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s(s+0.5)+1} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+0.5 & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{G}(s) = \frac{1}{s(s+0.5)+1} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} = \frac{1}{s(s+0.5)+1} \times 1 = \frac{1}{s(s+0.5)+1}$$

Using Matalb:

```
>> clear all
>> A=[0 1;-1 -0.5]; B=[0 1]'; C=[1 0]; D=0;
>> [num,den]=ss2tf(A,B,C,D)
num =

          0    -0.0000     1.0000
den =

     1.0000     0.5000     1.0000
```

This means that:

$$G(s) = \frac{1}{s^2 + 0.5s + 1}$$

Example .7:

Find the transfer function of:

$$A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}, D \text{ is the zero matrix.}$$

Analytical solution: This system has 1 input ($m=1$) and 2 outputs ($p=2$)

$$\mathbf{G}(s) = \frac{1}{(s-2)(s-1)}\begin{bmatrix} 3(s+1) \\ 3(s+3) \end{bmatrix}$$

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{3(s+1)}{(s-2)(s-1)} \\ \dfrac{3(s+3)}{(s-2)(s-1)} \end{bmatrix}$$

By using Matlab:

>> A=[2 0;4 1];B=[1;1];C=[1 2;-1 4];D=[0;0];

>> [num,den]=ss2tf(A,B,C,D)

num =

     0   3.0000   3.0000   (1st output to input numerator)

     0   3.0000   9.0000   (2nd output to input numerator)

den =

    1   -3   2   (Same denominator)

$$Y(s) = G(s)U(s) \Rightarrow \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) \\ G_{21}(s) \end{bmatrix} U(s)$$

$$G_{11}(s) = \frac{Y_1(s)}{U(s)} = \frac{3s+3}{s^2-3s+2}$$

$$G_{21}(s) = \frac{Y_2(s)}{U(s)} = \frac{3s+9}{s^2-3s+2}$$

Example 2.8:

Find the transfer function of the following system:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u$$

Try to find the TF matrix using analytical solution.

Using Matlab:

This system has 2 inputs and 2 outputs hence 4 TFs are required:

```
>> A=[0 1;-25 -4];B=[1 1;0 1];C=[1 0;0 1];D=[0 0;0
0];
```

1st input

```
>> [num,den]=ss2tf(A,B,C,D,1)
```

num =

        0   1.0000   4.0000   (Y_1 and U_1)

        0   0   -25.0000   (Y_2 and U_1)

den =

    1.0000   4.0000   25.0000

2nd input

```
>> [num,den]=ss2tf(A,B,C,D,2)

num =

          0     1.0000      5.0000          (Y₁ and U₂)

          0     1.0000    -25.0000          (Y₂ and U₂)

den =

     1.0000      4.0000     25.0000
```

$$\frac{Y_1}{U_1} = G_{11} = \frac{s+4}{s^2 + 4s + 25}$$

$$\frac{Y_2}{U_1} = G_{21} = \frac{-25}{s^2 + 4s + 25}$$

$$\frac{Y_1}{U_2} = G_{12} = \frac{s+5}{s^2 + 4s + 25}$$

$$\frac{Y_2}{U_2} = G_{22} = \frac{s-25}{s^2 + 4s + 25}$$

Transformation from TF to state space:

$$\frac{Y(s)}{U(s)} = G(s) = \frac{num}{den}$$

We use the Matlab command *tf2ss*:

[A,B,C,D]=tf2ss(num,den)

Note that there are many different state space representations of the same system. This Matlab command will give one possible representation.

Example 2.9: Find a state space representation of the following TF:

$$\frac{Y(s)}{U(s)} = G(s) = \frac{10s + 10}{s^3 + 6s^2 + 5s + 10}$$

>> num=[10 10];

>> den=[1 6 5 10];

>> [A,B,C,D]=tf2ss(num,den)

A =

  -6   -5 -10

   1   0   0

   0   1   0

B =

   1

   0

   0

C =

   0   10   10

D =   0

3. Controllability and Observability

A state space system is completely observable if the rank of the observability

matrix (M_o), $M_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$, is n (n is the system order). This implies that this

matrix contains n linearly independent row or column vectors. (that is, the matrix M_o is non-singular having non-zero determinant). If the rank of this matrix M_o is less than n then the system is unobservable. To check if the system is observable or not in Matlab we use the command `rank(obsv(A,C))`. The answer is the rank of the observability matrix M_o which should be n for the system to be observable.

Example 2.10:

Check if the following state space system $\left.\begin{array}{l} \dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 3 & 0 \end{bmatrix} x \end{array}\right\}$ is observable.

As in the lecture notes we can check the system observability by checking the determinant of the observability matrix M_o as follow:

$M_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -6 & 0 \end{bmatrix} \Rightarrow |M_o| = \begin{vmatrix} 3 & 0 \\ -6 & 0 \end{vmatrix} = 0$. Which means that the matrix M_o is

singular and the system is unobservable. If we want to use Matlab:

```
>> A=[-2 0;0 -1];
```

```
>> C=[3 0];
```

```
>> rank(obsv(A,C))
```

```
ans =

      1
```

This means that the rank of the observability matrix is 1 and since the system order $n = 2$ then the system is unobservable.

A state space system is completely controllable if the rank of the controllability matrix (M_c), $M_C = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$, is n (n is the system order). This implies that this matrix contains n linearly independent row or column vectors. (that is, the matrix M_c is non-singular having non-zero determinant). If the rank of this matrix M_c is less than n then the system is uncontrollable. To check if the system is controllable or not in Matlab we use the command `rank(ctrb(A,B))`. The answer is the rank of the controllability matrix M_c which should be n for the system to be controllable.

Example 2.11:

Check if the following state space system $\begin{aligned} \dot{x} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & 2 \end{bmatrix} x \end{aligned}$ is controllable.

As in the lecture notes we can check the system controllability by checking the determinant of the controllability matrix M_c as follow:

$M_C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix} \Rightarrow |M_C| = \begin{vmatrix} 2 & -4 \\ 0 & 0 \end{vmatrix} = 0$. Which means that the matrix M_c is singular and the system is uncontrollable. If we want to use Matlab:

```
>> A=[-2 0;0 -1];

>> B=[2 0]';
```

```
>> rank(ctrb(A,B))

ans =

     1
```

This means that the rank of the controllability matrix is 1 and since the system order n =2 then the system is uncontrollable.

Example 2.12:

Show that the following system $\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 3 & 2 \end{bmatrix} x$ is minimal realisation.

```
>> clear all

>> A=[-2 0;0 -1];B=[2 1]';C=[3 2];

>> rank(ctrb(A,B))

ans =

     2

>> rank(obsv(A,C))

ans =

     2
```