Revision

2nd order

$x = c_1 x_1 + c_2 \cdot x_2$

x_1, x_2 are L.I. sol^s

- $x_1 = e_1 e^{\lambda_1 t}$, $x_2 = e_2 e^{\lambda_2 t}$
- $x_1 = e e^{\lambda t}$, $x_2 = e \cdot t e^{\lambda t}$?
- $x_1 = e e^{\lambda t}$, $x_2 = \bar{e} e^{\bar{\lambda} t}$

or

Re( ) Im( )

or

cos... sin...

F.S.M.

$$x(t) = \underset{2\times 2}{\underbrace{\mathbb{X}(t)}} \cdot c, \quad x(t):\ 2\times 1, \quad \mathbb{X}(t) = [x_1 \quad x_2], \quad c = \underset{2\times 1}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$$

$t=0$ $\quad x(0) = \mathbb{X}(0) \cdot c \Rightarrow c = \mathbb{X}^{-1}(0) \cdot x_0$

$$x(t) = \underbrace{\mathbb{X}(t) \cdot \mathbb{X}^{-1}(0)}_{\text{S.T.M.}} \cdot x_0$$

$\bullet\ \lambda_1 \neq \lambda_2 \quad \mathbb{X}(t) = [e_1 e^{\lambda_1 t} \quad e_2 e^{\lambda_2 t}]$

$\bullet\ \lambda_1 = \lambda_2 = \lambda \quad \mathbb{X}(t) = [e\, e^{\lambda t} \quad (et+b)\, e^{\lambda t}]$

$\bullet\ \lambda = a \pm bi \quad \mathbb{X}(t) = [e\, e^{\lambda t} \quad \bar{e}\, e^{\bar{\lambda} t}]$

or

Re $\qquad$ Im

or

cos $\qquad$ sin

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cont. from previous ex.

4) F.S.M.

$$\mathbb{X} = [e\, e^{\lambda t} \quad (et+b)e^{\lambda t}]$$

$$= \left[\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \quad \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{-t} \right]$$

$$= \begin{bmatrix} e^{-t} & (t+1)e^{-t} \\ -e^{-t} & -te^{-t} \end{bmatrix}$$

$$\mathbb{X}(0) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

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$$\Phi^{-1}(0) = \frac{1}{1}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$x(t) = \Phi(t) \cdot \Phi^{-1}(0) \cdot x_0$$

$$= \begin{bmatrix} e^{-t} & (t+1)e^{-t} \\ -e^{-t} & -te^{-t} \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (t+1)e^{-t} & -e^{-t} + (t+1)e^{-t} \\ -te^{-t} & e^{-t} - te^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (t+1)e^{-t} \\ -te^{-t} \end{bmatrix}$$

$\dot{x} = a \cdot x$, $a, x \in \mathbb{R}$

$x(t) = e^{at} \cdot x_0$

$\dot{x} = Ax$, $x \in \mathbb{R}^{n\times 1}$, $A \in \mathbb{R}^{n\times n}$

$x(t) = e^{At} \cdot x_0$?

Taylor series

$$f(x) \simeq f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}\cdot(x-a)^3 + \cdots$$

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots$$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

$\dot{x} = Ax$
- $\to e_1\ \lambda_1$
- $\to$ Soln M.
- $\to e^{At}$

simple, easy

L.T.V.

~~No~~ No S.C

HOS

$e^{At} = ?$ A simple way,

$\lambda_1 \neq \lambda_2$

$n=2$

$A e_1 = \lambda_1 e_1 \qquad A e_2 = \lambda_2 e_2$

$$\underbrace{A}_{2\times 2} \cdot \underbrace{[e_1 \quad e_2]}_{2\times 2} = \underbrace{[e_1 \quad e_2]}_{2\times 2} \cdot \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{2\times 2}$$

eigenmatrix

$$A \cdot T = T \cdot \Lambda$$

$$A = T \cdot \Lambda \cdot T^{-1}$$

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{3!} + \dots$$

$$= I + (T \cdot \Lambda T^{-1}) \cdot t + \frac{(T\Lambda T^{-1})^2 \cdot t^2}{2} + \frac{(T\Lambda T^{-1})^3 \cdot t^3}{3!} + \dots$$

$$(T \cdot \Lambda T^{-1})^2 = (T \cdot \Lambda \cdot T^{-1}) \cdot (T \cdot \Lambda \cdot T^{-1})$$

$$T \cdot \Lambda \cdot \cancel{T^{-1}} \cdot \cancel{T} \cdot \Lambda \cdot T^{-1}$$

$$= T \cdot \Lambda^2 \cdot T^{-1}$$

$$(T \cdot \Lambda \cdot T^{-1})^3 = T \cdot \Lambda^3 \, T^{-1}$$

$$e^{At} = I + (T \cdot \Lambda \cdot T^{-1}) t + \frac{(T \cdot \Lambda^2 \cdot T^{-1}) \cdot t^2}{2} + \frac{(T \cdot \Lambda^3 T^{-1}) \cdot t^3}{3!} + \dots$$

$$T T^{-1} \downarrow T \cdot I T^{-1}$$

$$= T \cdot I \cdot T^{-1} + T \cdot \Lambda T^{-1} t + \frac{T \cdot \Lambda^2 T^{-1} t^2}{} + \dots$$

$$T \cdot \Big(\underbrace{I + \Lambda t + \frac{\Lambda^2 t^2}{2} + \frac{\Lambda^3 t^3}{3!} + \dots}_{e^{\Lambda t}} \Big) T^{-1}$$

$$e^{At} = T e^{\Lambda t} \cdot T^{-1}$$

$$e^{\Lambda t} \rightarrow \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

Homework.

$$I + \Lambda t + \frac{(\Lambda t)^2}{2} + \frac{(\Lambda t)^3}{3!} + \dots$$

$$\Lambda^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$$

$$\Lambda^3 = \dots = \begin{bmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1^2 t^2 & 0 \\ 0 & \lambda_2^2 t^2 \end{bmatrix} \frac{1}{2} + \begin{bmatrix} \lambda_1^3 t^3 & 0 \\ 0 & \lambda_2^3 t^3 \end{bmatrix} \frac{1}{3!}$$

element 1,1: $\underbrace{1 + \lambda_1^2 t^2 \cdot \frac{1}{2} + \frac{\lambda_1^3 t^3}{3!} + \dots}$

Taylor series expansion of $e^{\lambda t}$

so $I + \Lambda t + \frac{\Lambda^2 t^2}{2} + \frac{\Lambda^3 t^3}{3!} = \dots$

$$\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$A \rightarrow \begin{matrix} \lambda_1 & \lambda_2 \\ \downarrow & \downarrow \\ e_1 & e_2 \end{matrix}$$

$$x = c_1 e_1 e^{\lambda_1 t} + c_2 e_2 e^{\lambda_2 t}$$

$$x = e^{At} \cdot x_0$$

$$x = e^{At} \cdot x_0 = T \cdot e^{\Lambda t} T^{-1} \cdot x_0$$

$$= \underbrace{[e_1 \quad e_2]}_{2\times 2} \underbrace{e^{\Lambda t}}_{2\times 2} \cdot \underbrace{[e_1 \quad e_2]^{-1}}_{2\times 2} \underbrace{x_0}_{2\times 1}$$

$$= [\underbrace{e_1}_{2\times 1} \quad e_2] \underset{2\times 2}{e^{\Lambda t}} \cdot \begin{bmatrix} w_1 \\ \underbrace{w_2}_{1\times 2} \end{bmatrix} \cdot x_0$$

$$= [e_1 \quad e_2] \cdot \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \cdot x_0$$

$$= (\underbrace{e_1}_{2\times 1} e^{\lambda_1 t} \cdot \underbrace{w_1}_{1\times 2} + e_2 e^{\lambda_2 t} \cdot w_2) x_0$$

$$(e^{\lambda_1 t} \cdot e_1 \cdot \underbrace{w_1}_{1\times 2} + e^{\lambda_2 t} \cdot e_2 \cdot \underbrace{w_2}_{1\times 2}) \underbrace{x_0}_{2\times 1}$$

$$\rightarrow e^{\lambda_1 t} \cdot e_1 \cdot w_1 x_0 + e^{\lambda_2 t} \cdot e_2 w_2 \cdot x_0$$

$$x = e^{At} \cdot x_0 \xrightarrow{?} x = c_1 \cdot e_1 \, e^{\lambda_1 t} + c_2 \cdot e_2 \cdot e^{\lambda_2 t}$$

$$\Big\downarrow$$

$$x = e^{\lambda_1 t} \cdot e_1 \underbrace{w_1 \cdot x_0}_{c_1} + e^{\lambda_2 t} e_2 \underbrace{w_2 \, x_0}_{c_2}$$

(76)

$$\dot{X} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \cdot X, \quad X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -2-\lambda & 2 \\ 2 & -5-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$(\lambda+2)(\lambda+5) - 4 = 0$$

$$\lambda^2 + 7\cdot\lambda + 6 = 0$$

$$\Delta = 49 - 4\cdot 6 = 25$$

$$\lambda_{1,2} = \frac{-7 \pm 5}{2} \quad \begin{matrix} \to \lambda_1 = -1 \\ \to \lambda_2 = -6 \end{matrix}$$

$$(A - \lambda I)\cdot e$$

$$\begin{bmatrix} -2-\lambda & 2 \\ 2 & -5-\lambda \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0$$

$$\bullet\ \lambda = -1 \quad \begin{bmatrix} -2+1 & 2 \\ 2 & -5+1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0$$

$$e_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$-V_1 + 2V_2 = 0, \quad V_2 = 1 \Rightarrow V_1 = 2$$

• $\lambda = -6$

$$e_2 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2+6 & 2 \\ 2 & -5+6 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$2 \cdot v_1 + v_2 = 0, \quad v_2 = 1$$

$$v_1 = -0.5$$

Gen. soln:

$$x(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \cdot \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} e^{-6t}$$

$$x(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot 1 + c_2 \cdot \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} \cdot 1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 2 \cdot c_1 - c_2 \cdot 0.5 &= 1 \\ c_1 + c_2 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} c_2 &= -0.4 \\ c_1 &= 0.4 \end{aligned}$$

spec. soln

$$x(t) = 0.4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} - 0.4 \cdot \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} e^{-6t}$$

$$= \begin{bmatrix} 0.8 e^{-t} + 0.2 e^{-6t} \\ 0.4 e^{-t} - 0.4 e^{-6t} \end{bmatrix}$$

~~F.S.M.S~~

F.S.M

$$\Phi(t) = \left[\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} \quad \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} e^{-6t} \right]$$

$\rightarrow$ F.S.M.

$$X(0) = \begin{bmatrix} 2 & -0.5 \\ 1 & 1 \end{bmatrix}$$

$$|X(0)| = 2 \times 1 - (-0.5) = 2.5$$

$$X^{-1}(0) = \begin{bmatrix} 1 & 0.5 \\ -1 & 2 \end{bmatrix} \cdot \frac{1}{2.5}$$

$$x(t) = \Phi(t) \cdot \Phi^{-1}(0) \cdot x_0 =$$

$$\cancel{\begin{bmatrix} 1 & 0.5 \\ -1 & 2 \end{bmatrix} \cdot \frac{1}{}}$$

$$\frac{1}{2.5} \begin{bmatrix} 2e^{-t} & -0.5e^{-6t} \\ e^{-t} & e^{-6t} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0.5 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$= \ldots\ldots =$ Homework

$$\begin{bmatrix} 1 & 0.5 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2e^{-t} & -0.5e^{-6t} \\ e^{-t} & e^{-6t} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2e^{-t} + 0.5e^{-6t} \\ e^{-t} - e^{-6t} \end{bmatrix}$$

$$\frac{1}{2.5} \begin{bmatrix} 2e^{-t} + 0.5e^{-6t} \\ e^{-t} - e^{-6t} \end{bmatrix} =$$

$$\begin{bmatrix} 0.8 \cdot e^{-t} + 0.2e^{-6t} \\ 0.4e^{-t} - 0.4e^{-6t} \end{bmatrix}$$ as in p. 77

$$x = e^{At} \cdot x_0$$

$$= T\, e^{\Lambda t} \cdot T^{-1} \cdot x_0$$

$$T = \begin{bmatrix} 2 & -0.5 \\ 1 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & 0.5 \\ -1 & 2 \end{bmatrix} \frac{1}{2.5}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{-6t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

$$T = \begin{bmatrix} -0.5 & 2 \\ 1 & 1 \end{bmatrix}$$

79b

$$X = \begin{bmatrix} 2 & -0.5 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ -1 & 2 \end{bmatrix} \frac{1}{2.5} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{1}{2.5} \begin{bmatrix} 2e^{-t} & -0.5e^{-6t} \\ e^{-t} & e^{-6t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

$$\frac{1}{2.5} \begin{bmatrix} 2e^{-t} + 0.5e^{-6t} \\ e^{-t} - e^{-t6} \end{bmatrix} =$$

$$\begin{bmatrix} 0.8e^{-t} + 0.2e^{-6t} \\ 0.4e^{-t} - 0.4e^{-6t} \end{bmatrix}$$

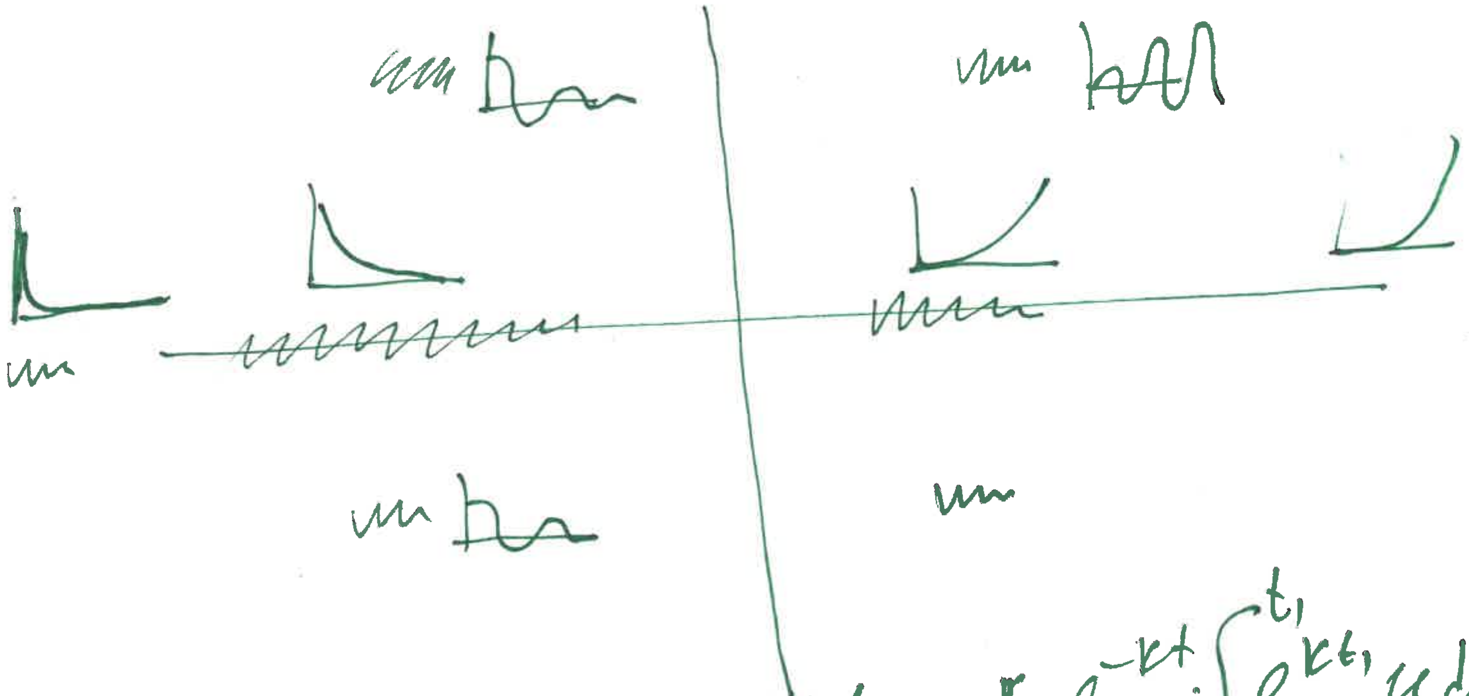
same as p. 77 and 78b

$\dot{x} + kx = u.$ $\quad u = 0 \quad e^{rt}$

$\ddot{x} + A\dot{x} + Bx = 0 \quad e^{rt}$

C.E. $\quad r + k = 0 \Rightarrow r = -k$

$r^2 + Ar + B = 0 \rightarrow \Delta \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$



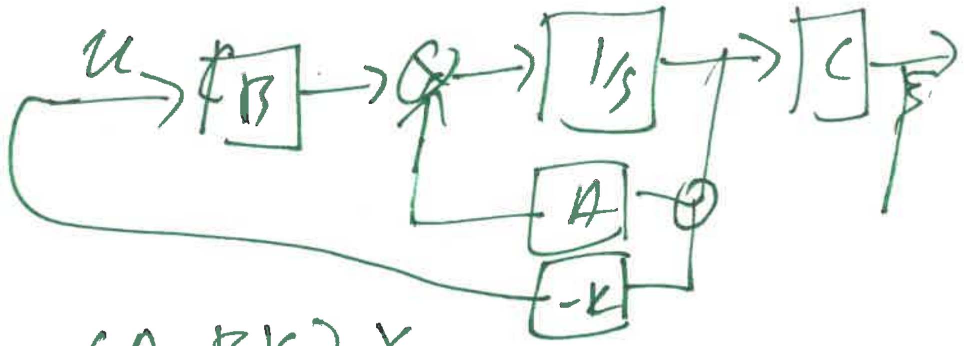
$\dot{x} + kx = u \rightarrow x(t) = e^{-kt} x_0 + e^{-kt} \int_0^{t_1} e^{kt_1} u \, dt_1$

$\downarrow$

$\dot{x} = Ax + Bu \qquad e^{At}.$

$$\dot{x} = Ax + B \cdot u$$

$$u = -K \cdot x$$



$$\dot{x} = Ax - BKx = (A - BK) \cdot x$$