

## Chapter #1

### EEE8013 – EEE3001

## Linear Controller Design and State Space Analysis

Ordinary Differential Equations .....	2
1. Introduction.....	2
2. First Order ODEs .....	2
3. Second Order ODEs.....	8
3.1 General Material .....	8
3.2 Solution of second order ODEs/IVPs. ....	12
3.2.1 Roots are real and unequal.....	13
3.2.2 Roots are Complex (and hence not equal) .....	14
3.2.4 Roots are real and equal.....	16
4. Tutorial Exercise I .....	21

# Ordinary Differential Equations

## 1. Introduction

To understand the properties (dynamics) of a system, we can model (represent) it using differential equations (DEs). The response/behaviour of the system is found by solving the DEs. In our cases, the DE is an Ordinary DE (ODE), i.e. not a partial derivative. The main purpose of this Chapter is to learn how to solve first and second order ODEs in the time domain. This will serve as a building block to model and study more complicated systems. Our ultimate goal is to control the system when it does not show a “satisfactory” behaviour. Effectively, this will be done by modifying the ODE.

Note for MSc students: There are footnotes throughout the notes containing commands/examples using the Symbolic Toolbox, this is **assessed material!**

## 2. First Order ODEs

The general form of a first order ODE is:

$$\frac{dx(t)}{dt} = f(x(t), t) \quad (1)$$

where<sup>1</sup>  $x, t \in \mathbb{R}$

**Analytical solution:** Explicit formula for  $x(t)$  (a solution which can be found using various methods) which satisfies  $\frac{dx}{dt} = f(x, t)$

---

<sup>1</sup> The proper notation is  $x(t)$  and not  $x$  but we drop the brackets in order to simplify the presentation.

**Example 1.1:** Prove that  $x = e^{-3t}$  and  $x = -10e^{-3t}$  are solutions of  $\frac{dx}{dt} = -3x$ .

$$\frac{dx}{dt} = -3x \Leftrightarrow \frac{d(e^{-3t})}{dt} = -3(e^{-3t}) \Leftrightarrow -3e^{-3t} = -3e^{-3t}$$

$$\frac{dx}{dt} = -3x \Leftrightarrow \frac{d(-10e^{-3t})}{dt} = -3(-10e^{-3t}) \Leftrightarrow 30e^{-3t} = 30e^{-3t} \quad \blacksquare^2$$

Obviously there are infinite solutions to an ODE and for that reason the found solution is called the **General Solution** of the ODE.

**First order Initial Value Problem :**  $\frac{dx}{dt} = f(x, t), \quad x(t_0) = x_0$

An initial value problem is an ODE with an initial condition, hence we do not find the general solution but the **Specific Solution** that passes through  $x_0$  at  $t=t_0$ .

**Analytical solution:** Explicit formula for  $x(t)$  which satisfies  $\frac{dx}{dt} = f(x, t)$  and

passes through  $x_0$  when  $t = t_0$ .

**Example 1.2:** Prove that  $x = e^{-3t}$  is a solution, while  $x = -10e^{-3t}$  is not a solution of  $\frac{dx}{dt} = -3x, x_0 = 1$

Both expressions ( $x = e^{-3t}$  and  $x = -10e^{-3t}$ ) satisfy the  $\frac{dx}{dt} = -3x$  but at  $t=0$

$$x(t) = e^{-3t} \Rightarrow x(0) = 1$$

$$x(t) = -10e^{-3t} \Rightarrow x(0) = -10 \neq 1 \quad \blacksquare^3$$

---

<sup>2</sup> `clc, clear all, syms t, x1=exp(-3*t); dx=diff(x1,t); isequal(dx,-3*x1)  
x2=-10*exp(-3*t); dx=diff(x2,t); isequal(dx,-3*x2)`

<sup>3</sup> `clc, clear all, syms t, x1=exp(-3*t); x2=-10*exp(-3*t); x0=1;  
x0_1=double(subs(x1,t,0)); x0_2=double(subs(x2,t,0)); isequal(x0,x0_1),  
isequal(x0,x0_2),`

For that reason some books use a different symbol for the specific solution:

$$\phi(t, t_0, x_0).$$

You must be clear about the difference between an ODE and the solution to an IVP! From now on we will just study IVP unless otherwise explicitly mentioned.

## Linear First Order ODEs

A linear 1<sup>st</sup> order ODE is given by:

$$\begin{cases} a(t)x' + b(t)x = c(t), a(t) \neq 0 & \text{Non autonomous} \\ ax' + bx = c, a \neq 0 & \text{Autonomous} \end{cases} \quad (2)$$

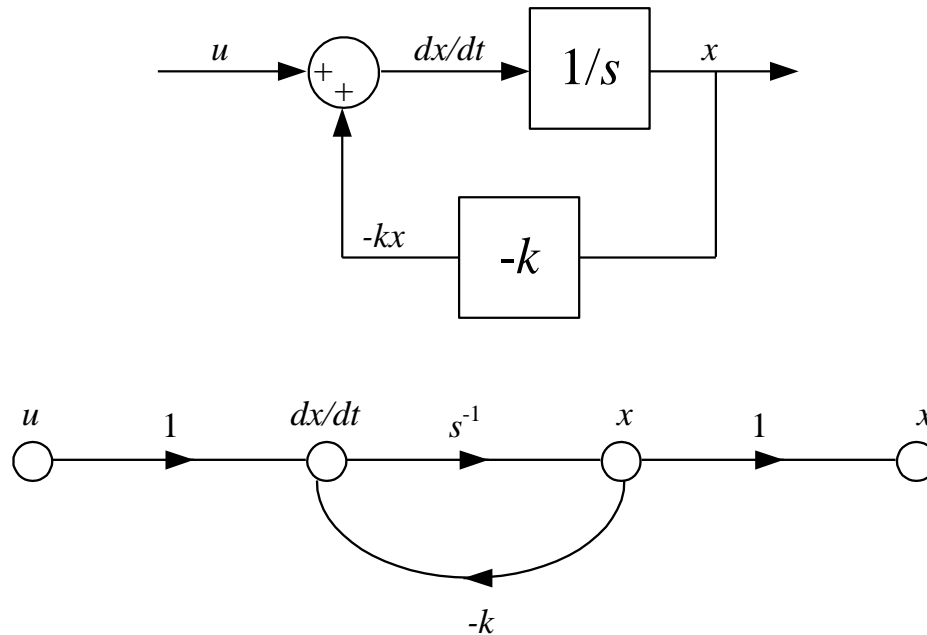
with  $a, b, c \in \mathbb{R}$  and  $a \neq 0$ .

In engineering books the most common form of (2) is (since  $a \neq 0$ ):

$$\begin{aligned} x' + k(t)x &= u(t) & (3) \\ \text{with } k, u &\in \mathbb{R} \end{aligned}$$

Note: We say that  $u$  is the input to our system that is represented by (3)

The block diagram and signal flow representation of this ODE is<sup>4</sup>:



The solution of (3) (using the integrating factor) is given by:

$$x(t) = e^{-kt} x(t_0) + e^{-kt} \int_{t_0}^t e^{kt_1} u(t_1) dt_1$$

The term  $e^{-kt} x(t_0)$  is called transient response, while  $e^{-kt} \int_{t_0}^t e^{kt_1} u(t_1) dt_1$  comes

from the input signal  $u$ .

If we assume that  $u$  is constant:

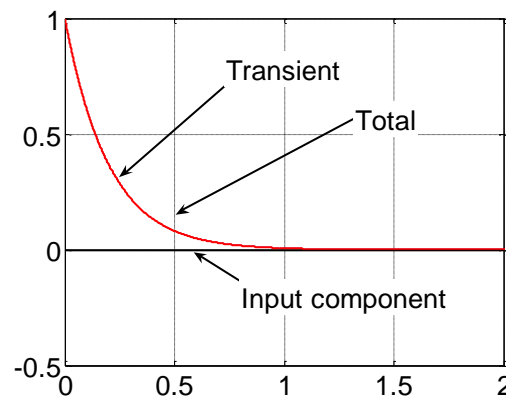
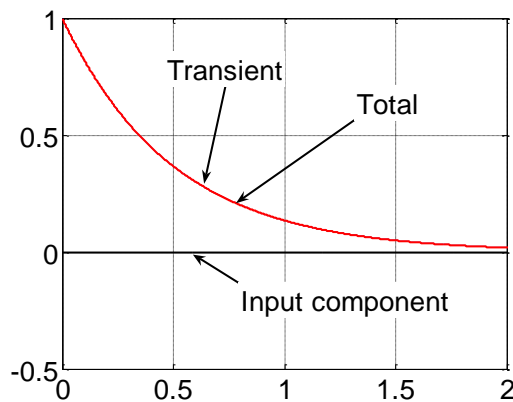
$$x(t) = e^{-kt} x(t_0) + e^{-kt} \int_{t_0}^t e^{kt_1} u dt_1 \Leftrightarrow x(t) = e^{-kt} x(t_0) + u \frac{1}{k} (1 - e^{-k(t-t_0)})$$

<sup>4</sup> The symbols  $1/s$  and  $s^{-1}$ , denote integration and have nothing to do with the Laplace Transform.

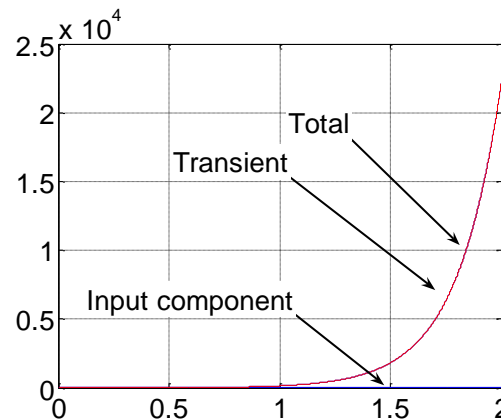
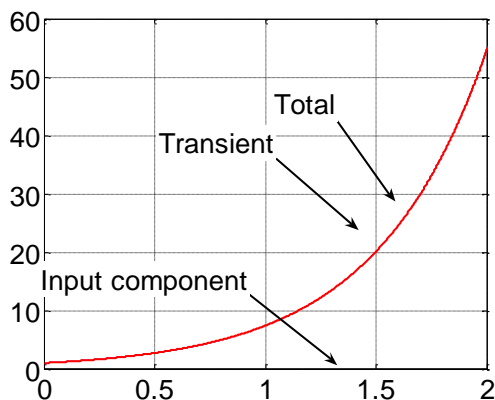
$$\text{Hence: } \lim_{t \rightarrow \infty} x(t) = \begin{cases} 0 + u \frac{1}{k} (1 - 0) = u/k, & k > 0 \\ \pm\infty, & k < 0 \end{cases}$$

Thus we say that if  $k > 0$  the system is stable (and the solution converges exponentially at  $u/k$ ) while if  $k < 0$  the system is unstable (and the solution diverges exponentially to  $\pm\infty$ ).

**Example 1.3:**  $u=0$  and  $k=2$  &  $5$ ,  $x_0=1$ ,  
 $x(t) = e^{-2t} \cdot 1 + 0$ ,  $\lim_{t \rightarrow \infty} x(t) = 0$ , as  $2 > 0$

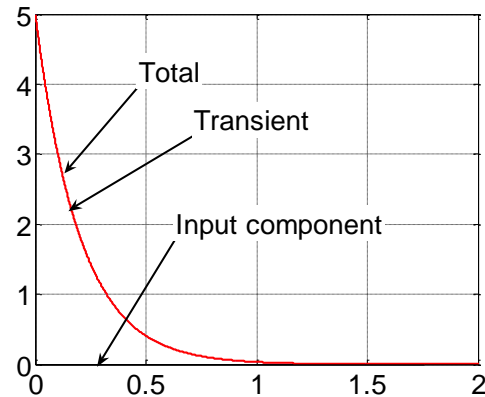
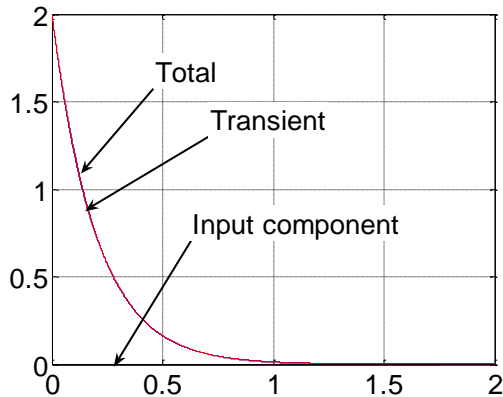
■<sup>5</sup>

**Example 1.4:**  $u=0$  and  $k=-2$  &  $5$ ,  $x_0=1$

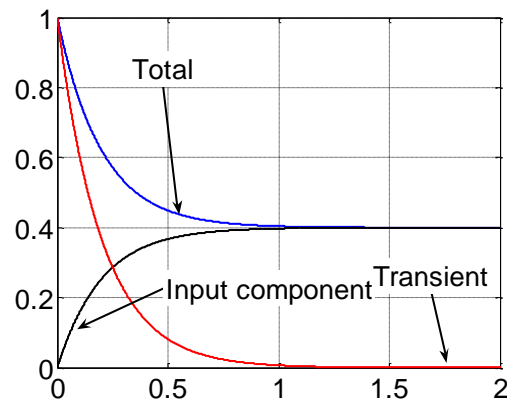
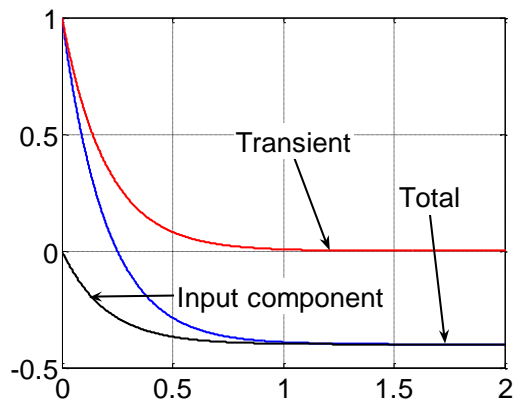


■

<sup>5</sup> `clc, clear all, syms x(t), dx=diff(x); dsolve(dx+2*x, x(0)==1)`

**Example 1.5:**  $u=0$  and  $k=5$ ,  $x_0=1$  &  $5$ 

■

**Example 1.6:**  $u=-2$  &  $2$  and  $k=5$ ,  $x_0=1$ ■<sup>6</sup>**Comments:**

- In real systems we cannot have a state (say the speed of a mass-spring system) that becomes infinite, obviously the system will be destroyed when  $x$  gets to a high value.
- For the dynamics (settling time, stability...) of the system we should only focus on the homogenous ODE:  $x' + k(t)x = 0$

---

```
6 clc, clear all, close all, syms t t1, x0=1; k=5; t0=0; u=2;
t2=0:0.01:2; x_x0=exp(-k*t)*x0; x_u=exp(-k*t)*int(exp(k*t1)*u,t0,t);
x_x0_t=double(subs(x_x0,t,t2)); x_u_t=double(subs(x_u,t,t2));
hold on, plot(t2,x_x0_t), plot(t2,x_u_t), plot(t2,x_u_t+x_x0_t)
```

### 3. Second Order ODEs

#### 3.1 General Material

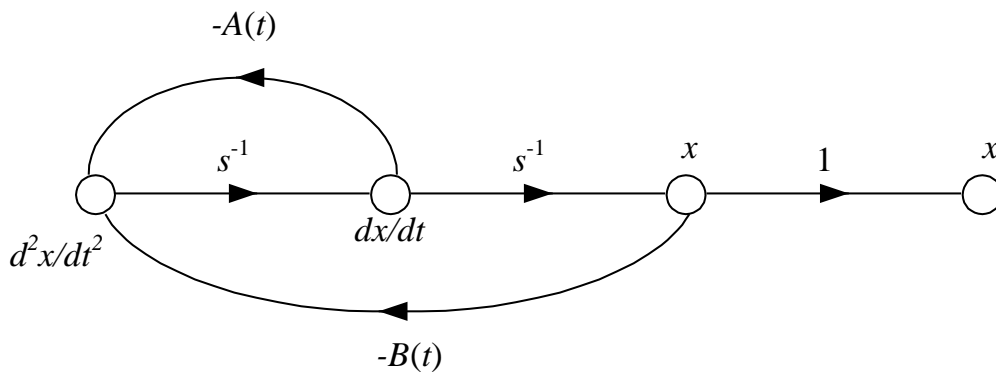
A second order ODE has as a general form:

$$\frac{d^2x(t)}{dt^2} = f(x'(t), x(t), t) \quad (4)$$

A linear 2<sup>nd</sup> order ODE is given by:

$$\begin{cases} x''(t) + A(t)x'(t) + B(t)x(t) = u(t), & \text{Non autonomous} \\ x''(t) + Ax'(t) + Bx(t) = u(t), & \text{Autonomous} \end{cases} \quad (5)$$

Its signal flow diagram is:



And again we focus on autonomous homogeneous systems:

$$x''(t) + A(t)x'(t) + B(t)x(t) = 0 \quad (6)$$

Again we define as an analytical solution of (6) an expression that satisfies it.



**Example 1.7:** Given  $x'' - 2x' - 3x = 0$  prove that  $x = e^{3t}$  and  $x = e^{-t}$  are two solutions:

$$(e^{3t})'' - 2(e^{3t})' - 3(e^{3t}) = 0 \Leftrightarrow$$

$$9e^{3t} - 6e^{3t} - 3e^{3t} = 0 \Leftrightarrow$$

$$0 = 0$$

$$(e^{-t})'' - 2(e^{-t})' - 3(e^{-t}) = 0 \Leftrightarrow$$

$$e^{-t} + e^{-t} - 3e^{-t} = 0 \Leftrightarrow$$

■<sup>7</sup>

$$0 = 0$$

Assume that you have 2 solutions for a 2<sup>nd</sup> order ODE  $x_1$  and  $x_2$  (we will see later how to get these two solutions), then:

$$\left. \begin{aligned} x_1''(t) + A(t)x_1'(t) + B(t)x_1(t) &= 0 \\ x_2''(t) + A(t)x_2'(t) + B(t)x_2(t) &= 0 \end{aligned} \right\}$$

obviously I can multiply these two equations with arbitrary constants:

$$\left. \begin{aligned} C_1x_1''(t) + C_1A(t)x_1'(t) + C_1B(t)x_1(t) &= 0 \\ C_2x_2''(t) + C_2A(t)x_2'(t) + C_2B(t)x_2(t) &= 0 \end{aligned} \right\}$$

and now I can add them and collect similar terms:

$$\underbrace{(C_1x_1(t) + C_2x_2(t))}'' + A(t)\underbrace{(C_1x_1(t) + C_2x_2(t))}' + B(t)\underbrace{(C_1x_1(t) + C_2x_2(t))} = 0$$

which means that  $C_1x_1(t) + C_2x_2(t)$  (i.e. the linear combination of  $x_1$  and  $x_2$ )

is also a solution of the ODE.

---

<sup>7</sup> `clc, clear all, close all, syms x(t) t`  
`Dx=diff(x); D2x=diff(x,2); ODE=D2x-2*Dx-3*x;`  
`subs(ODE, x, exp(-t)), subs(ODE, x, exp(3*t))`

**Example 1.8:** Given  $x'' - 2x' - 3x = 0$  prove that  $x = e^{3t} + 2e^{-t}$  is a solution:

$$(e^{3t} + 2e^{-t})'' - 2(e^{3t} + 2e^{-t})' - 3(e^{3t} + 2e^{-t}) = 0 \Leftrightarrow$$

$$9e^{3t} + 2e^{-t} - 2(3e^{3t} - 2e^{-t}) - 3e^{3t} - 6e^{-t} = 0 \Leftrightarrow$$

$$9e^{3t} + 2e^{-t} - 6e^{3t} + 4e^{-t} - 3e^{3t} - 6e^{-t} = 0 \Leftrightarrow$$

$$9e^{3t} - 6e^{3t} - 3e^{3t} + 2e^{-t} + 4e^{-t} - 6e^{-t} = 0 \Leftrightarrow$$

$$0 = 0$$

■<sup>8</sup>

Now, the question is, if we have  $x_1$  and  $x_2$ , can ALL other solutions of the ODE, be expressed as a linear combination of  $x_1$  and  $x_2$ ? So assume a third solution  $\varphi(t)$ :

$$\varphi''(t) + A(t)\varphi'(t) + B(t)\varphi(t) = 0$$

Now, the question can be written as, can we find constants  $C_1$  and  $C_2$  such as:

$$\begin{cases} \varphi(t) = C_1x_1(t) + C_2x_2(t) \\ \varphi'(t) = C_1x_1'(t) + C_2x_2'(t) \end{cases}$$

This equation can be seen as a 2by2 system with unknowns  $C_1$  and  $C_2$  as:

$$\begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \varphi(t) \\ \varphi'(t) \end{bmatrix}$$

From linear algebra this system of equations has a unique solution if:

---

<sup>8</sup> `clc, clear all, close all, syms x(t) t; Dx=diff(x); D2x=diff(x,2); ODE=D2x-2*Dx-3*x; subs(ODE, x, exp(3*t)+exp(-t))`

$$\begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = x_1(t)x_2'(t) - x_2(t)x_1'(t) \neq 0$$

Note: The matrix  $W(x_1(t), x_2(t)) = \begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix}$  is called the Wronskian<sup>9</sup> of the ODE.

We also know from linear algebra that the determinant is not zero if:

$$\begin{bmatrix} x_1(t) \\ x_1'(t) \end{bmatrix} \neq C \begin{bmatrix} x_2(t) \\ x_2'(t) \end{bmatrix}$$

So if the two solutions  $x_1$  and  $x_2$  are linear independent (LI) then ANY other solution can be described by the linear combination of  $x_1$  and  $x_2$ . So now we have to look for two LI solutions for the 2<sup>nd</sup> order ODE.

**Example 1.9:** Prove that two solutions of  $x'' - 2x' - 3x = 0$ ,  $x_1 = e^{3t}$  and  $x_2 = e^{-t}$  are linear independent.

$$W(x_1(t), x_2(t)) = \begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix} = \begin{bmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{bmatrix} \Rightarrow |W| = \begin{vmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{vmatrix} \Rightarrow$$

$$|W| = e^{3t}(-e^{-t}) - 3e^{3t}e^{-t} = -e^{2t} - 3e^{2t} = -4e^{2t} \quad \blacksquare^{10}$$

<sup>9</sup> From the Polish mathematician Józef Maria Hoëne-Wroński

<sup>10</sup> `clc, clear all, close all, syms t, x1=exp(3*t); x2=exp(-t); Dx1=diff(x1); Dx2=diff(x2); W=[x1, x2; Dx1, Dx2], det(W)`

**Example 1.10:** Prove that two solutions of  $x'' - 2x' - 3x = 0$ ,  $x_1 = e^{3t}$  and  $x_2 = 2e^{3t}$  are NOT linear independent.

$$W(x_1(t), x_2(t)) = \begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix} = \begin{bmatrix} e^{3t} & 2e^{3t} \\ 3e^{3t} & 6e^{3t} \end{bmatrix} \Rightarrow$$

$$|W| = \begin{vmatrix} e^{3t} & 2e^{3t} \\ 3e^{3t} & 6e^{3t} \end{vmatrix} = 6e^{6t} - 6e^{6t} = 0 \quad \blacksquare$$

**Example 1.11:** For the ODE  $x'' - 2x' - 3x = 0$  prove that the solution  $x = -e^{3t} + 2e^t$  cannot be written as any combination of  $x_1 = e^{3t}$  and  $x_2 = 2e^{3t}$ .  
 $x = C_1x_1 + C_2x_2 \Leftrightarrow -e^{3t} + 2e^t = C_1e^{3t} + C_2e^{3t} = (C_1 + C_2)e^{3t}$   
 From this expression we have that  $C_1 + C_2 = -1$  (and hence we have the term  $-e^{3t}$ ) but there is no term  $e^t$  for  $2e^t$ .  $\blacksquare$

### 3.2 Solution of second order ODEs/IVPs.

But how can we find two LI solutions? For homogeneous 1<sup>st</sup> order ODEs with  $u=0$  the solution was:  $x(t) = e^{-kt}C$  so we will try a similar approach for 2<sup>nd</sup> order ODEs:

$$x'' + Ax' + Bx = 0, \text{ assume}^{11} x = e^{rt} \Rightarrow x' = re^{rt} \ \& \ x'' = r^2e^{rt} \Rightarrow$$

$$x'' + Ax' + Bx = 0 \Leftrightarrow r^2e^{rt} + Are^{rt} + Be^{rt} = 0 \Leftrightarrow$$

$$r^2 + Ar + B = 0 \tag{7}$$

---

<sup>11</sup> Notice that we do NOT know what is the value of  $r$ .

This is called the Characteristic Equation (CE) and we have to check its roots:

$$r = \frac{-A \pm \sqrt{A^2 - 4B}}{2}, \text{ these are the Characteristic values or Eigenvalues.}$$

### 3.2.1 Roots are real and unequal

If  $A^2 > 4B$  the system is called **Overdamped** and the two roots are  $r_1$  and  $r_2$  with  $r_1 \neq r_2$ ,  $r_1, r_2 \in \mathbb{R}$ . Then  $x_1 = e^{r_1 t}$  and  $x_2 = e^{r_2 t}$  are two linear independent solutions as:

$$\begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = e^{r_1 t} r_2 e^{r_2 t} - e^{r_2 t} r_1 e^{r_1 t} \neq 0$$

hence the general solution is  $x = C_1 x_1 + C_2 x_2 = C_1 e^{r_1 t} + C_2 e^{r_2 t}$  (8)

If  $r_1$  and  $r_2 < 0$  then  $x \rightarrow 0$  and the system is stable.

If  $r_1$  or  $r_2 > 0$  then  $x \rightarrow \pm\infty$  and the system is unstable.

**Example 1.12:** The CE of  $x'' + 11x' + 30x = 0$  is  $r^2 + 11r + 30 = 0$  which means

$$\text{that the two roots are: } r_{1,2} = \frac{-11 \pm \sqrt{11^2 - 4 \cdot 1 \cdot 30}}{2} = \frac{-11 \pm 1}{2} \Rightarrow \begin{cases} r_1 = -5 \\ r_2 = -6 \end{cases} \quad 12$$

$$\text{and hence the 2 LI solutions are } \begin{cases} x_1 = e^{r_1 t} = e^{-5t} \\ x_2 = e^{r_2 t} = e^{-6t} \end{cases}$$

This means that the general solution is  $x = C_1 e^{-5t} + C_2 e^{-6t}$  and hence the ODE is stable<sup>13</sup>. The Wronskian is

<sup>12</sup> roots([1 11 30])

<sup>13</sup> clc, clear all, close all, syms x(t) Dx=diff(x,1); D2x=diff(x,2); ODE=D2x+11\*Dx+30\*x; dsolve(ODE)

$$\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} e^{-5t} & e^{-6t} \\ -5e^{-5t} & -6e^{-6t} \end{vmatrix} = -6e^{-5t}e^{-6t} + 5e^{-6t}e^{-5t} = -e^{-11t} \neq 0$$

If the initial condition is  $x(0)=1, x'(0)=0$  then:

$$\left. \begin{array}{l} C_1 + C_2 = 1 \\ -5C_1 - 6C_2 = 0 \end{array} \right\} \Rightarrow \left. \begin{array}{l} C_1 = 6 \\ C_2 = -5 \end{array} \right\} \Rightarrow x = 6e^{-5t} - 5e^{-6t} \quad \blacksquare^{14}$$

### 3.2.2 Roots are Complex (and hence not equal)

If  $A^2 < 4B$  then the system is called **Underdamped** and the two roots are

$r_1 = a + bj$  and  $r_2 = \bar{r}_1 = a - bj$  with  $r_1 \neq r_2, r_1, r_2 \in \mathbb{C}$ . Then  $x_1 = e^{r_1 t} = e^{(a+bj)t}$

and  $x_2 = e^{r_2 t} = e^{(a-bj)t}$  are two linear independent solutions as

$$\begin{vmatrix} e^{(a+bj)t} & e^{(a-bj)t} \\ (a+bj)e^{(a+bj)t} & (a-bj)e^{(a-bj)t} \end{vmatrix} = e^{(a+bj)t}(a-bj)e^{(a-bj)t} - e^{(a-bj)t}(a+bj)e^{(a+bj)t} = (a-bj)e^{2at} - (a+bj)e^{2at} = e^{2at}(a-bj-a-bj) = -2e^{2at}bj \neq 0$$

Hence the general solution is

$$x = C_1 x_1 + C_2 x_2 = C_1 e^{rt} + C_2 e^{\bar{r}t} \quad (9)$$

but remember that  $C_1$  and  $C_2$  are complex now variables such as  $x \in \mathbb{R}$ .

**Example 1.13:** The CE of  $x'' + 2x' + 5x = 0$  is  $r^2 + 2r + 5 = 0$  which means that

$$\text{the two roots are: } r_{1,2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4j}{2} = -1 \pm 2j \Rightarrow \begin{cases} r_1 = -1 + 2j \\ r_2 = -1 - 2j \end{cases}$$

---

<sup>14</sup> `clc, clear all, close all, syms x(t) Dx=diff(x,1); D2x=diff(x,2); ODE=D2x+11*Dx+30*x; dsolve(ODE, x(0)==1, Dx(0)==0)`

and hence the 2 LI solutions are  $\begin{cases} x_1 = e^{rt} = e^{(-1+2j)t} \\ x_2 = e^{r_2t} = e^{(-1-2j)t} \end{cases}$

This means that the general solution is  $x = C_1 e^{(-1+2j)t} + C_2 e^{(-1-2j)t}$  and hence the ODE is stable. The Wronskian is

$$\begin{aligned} \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} &= \begin{vmatrix} e^{(-1+2j)t} & e^{(-1-2j)t} \\ (-1+2j)e^{(-1+2j)t} & (-1-2j)e^{(-1-2j)t} \end{vmatrix} = \\ &(-1-2j)e^{(-1+2j)t}e^{(-1-2j)t} - (-1+2j)e^{(-1+2j)t}e^{(-1-2j)t} = \\ &(-1-2j)e^{-2t} - (-1+2j)e^{-2t} = (-1-2j+1-2j)e^{-2t} = \\ &-4je^{-2t} \neq 0 \end{aligned}$$

If the initial condition is  $x(0)=1, x'(0)=0$  then:

$$\left. \begin{aligned} C_1 + C_2 &= 1 \\ (-1+2j)C_1 + (-1-2j)C_2 &= 0 \end{aligned} \right\} \Rightarrow \left. \begin{aligned} C_1 &= \frac{1}{2} + \frac{1}{4}j \\ C_2 &= \frac{1}{2} - \frac{1}{4}j \end{aligned} \right\} \Rightarrow$$

$$x = \left(\frac{1}{2} + \frac{1}{4}j\right)e^{(-1+2j)t} + \left(\frac{1}{2} - \frac{1}{4}j\right)e^{(-1-2j)t} \quad \blacksquare$$

An alternative approach is not to use  $x_1$  &  $x_2$  but a linear combination of them:

$$y_1 = e^{rt} + e^{\bar{r}t}, y_2 = e^{rt} - e^{\bar{r}t}$$

Note that $\begin{vmatrix} e^{rt} + e^{\bar{r}t} & e^{rt} - e^{\bar{r}t} \\ re^{rt} + \bar{r}e^{\bar{r}t} & re^{rt} - \bar{r}e^{\bar{r}t} \end{vmatrix} \neq 0$
---

Using Euler's formula:  $e^{(a+bj)t} = e^{at}(\cos bt + j \sin bt)$  and hence:

$$y_1 = e^{(a+bj)t} + e^{(a-bj)t} = e^{at}(\cos bt + j \sin bt + \cos bt - j \sin bt) = 2e^{at} \cos bt$$

$$y_2 = e^{(a+bj)t} - e^{(a-bj)t} = e^{at}(\cos bt + j \sin bt - \cos bt + j \sin bt) = j2e^{at} \sin bt$$

As  $y_1$  and  $y_2$  are solutions so do  $y_1 \times \frac{1}{2}$ ,  $y_2 \times \frac{1}{2j}$ . So the general solution when

we have complex roots is:

$$x(t) = e^{at} (C_1 \cos bt + C_2 \sin bt), C_1, C_2 \in \mathbb{R} \quad (10)$$

**Example 1.14:** The CE of  $x'' + 2x' + 5x = 0$  is  $r^2 + 2r + 5 = 0$  which means that

$$\text{the two roots are: } r_{1,2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4j}{2} = -1 \pm 2j \Rightarrow \begin{cases} r_1 = -1 + 2j \\ r_2 = -1 - 2j \end{cases}$$

$$\text{and hence the 2 LI solutions are } \begin{cases} x_1 = e^{-t} \cos(2t) \\ x_2 = e^{-t} \sin(2t) \end{cases}$$

This means that the general solution is  $x = e^{-t} (C_1 \cos 2t + C_2 \sin 2t)$  and hence the ODE is stable. The Wronskian is

$$\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} e^{-t} \cos(2t) & e^{-t} \sin(2t) \\ -e^{-t} \cos(2t) - 2e^{-t} \sin(2t) & -e^{-t} \sin(2t) + 2e^{-t} \cos(2t) \end{vmatrix} \neq 2e^{-2t}$$

If the initial condition is  $x(0) = 1, x'(0) = 0$  then:

$$\left. \begin{matrix} C_1 = 1 \\ -C_1 + 2C_2 = 0 \end{matrix} \right\} \Rightarrow \left. \begin{matrix} C_1 = 1 \\ C_2 = 0.5 \end{matrix} \right\} \Rightarrow$$

$$x = e^{-t} (\cos 2t + 0.5 \sin 2t) \quad \blacksquare$$

### 3.2.4 Roots are real and equal

If  $A^2 = 4B$  then the system is called **Critically damped** and the two roots are

$r = r_1 = r_2$  with  $r \in \mathbb{R}$ . One solution is  $x_1 = e^{rt}$  but how about  $x_2$ ? We can

use  $x_2 = te^{rt}$  and the general solution:

$$x = C_1 x_1 + C_2 x_2 = C_1 e^{rt} + C_2 t e^{rt} \quad (11)$$



The Wronskian is:

$$\begin{vmatrix} e^{r_1 t} & te^{r_1 t} \\ r_1 e^{r_1 t} & r_1 te^{r_1 t} + e^{r_1 t} \end{vmatrix} = e^{r_1 t} (r_1 te^{r_1 t} + e^{r_1 t}) - r_1 e^{r_1 t} te^{r_1 t} = r_1 te^{2r_1 t} + e^{2r_1 t} - r_1 te^{2r_1 t} = e^{2r_1 t} \neq 0$$

**Example 1.15:** The CE of  $x'' + 2x' + x = 0$  is  $r^2 + 2r + 1 = 0$  which means that

the two roots are:  $r_{1,2} = \frac{-2 \pm \sqrt{0}}{2} \Rightarrow \begin{cases} r_1 = -1 \\ r_2 = -1 \end{cases}$

and hence the 2 LI solutions are  $\begin{cases} x_1 = e^{-t} \\ x_2 = te^{-t} \end{cases}$

This means that the general solution is  $x = C_1 e^{-t} + C_2 t e^{-t}$  and hence the ODE is stable. The Wronskian is

$$\begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t} + e^{-t} \end{vmatrix} = e^{2t} \neq 0$$

If the initial condition is  $x(0) = 1, x'(0) = 0$  then:

$$\begin{cases} C_1 = 1 \\ -C_1 + C_2 = 0 \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = 1 \end{cases}$$

$$x = e^{-t} + te^{-t}$$

■

### Not assessed material

To see how  $x_2 = te^{r_1 t}$  was found go to the ODE and place  $x = e^{rt}$ :

$$(e^{rt})'' + A(e^{rt})' + Bx = e^{rt}(r^2 + Ar + B)$$

Since  $r_1$  is a double root of the CE:  $r^2 + Ar + B = a(r - r_1)^2$  for some constant

$a$ . So:  $(e^{rt})'' + A(e^{rt})' + Bx = e^{rt} a(r - r_1)^2$

Taking the time derivative wrt  $r$ :

$$\frac{d\left((e^{rt})''\right)}{dr} + A \frac{d\left((e^{rt})'\right)}{dr} + B \frac{d(e^{rt})}{dr} = \frac{d\left(e^{rt} a(r-r_1)^2\right)}{dr}$$

And as we can change the sequence of the differentiation:

$$\left(\frac{d(e^{rt})}{dr}\right)' + A \left(\frac{d(e^{rt})}{dr}\right) + B \frac{d(e^{rt})}{dr} = \frac{d\left(e^{rt} a(r-r_1)^2\right)}{dr}$$

By using simple calculus:

$$(e^{rt})'' + A(e^{rt})' + Be^{rt} = \frac{d(e^{rt})}{dr} a(r-r_1)^2 + e^{rt} \frac{d(a(r-r_1)^2)}{dr} \Leftrightarrow$$

$$(e^{rt})'' + A(e^{rt})' + Be^{rt} = e^{rt} t a(r-r_1)^2 + e^{rt} 2a(r-r_1)$$

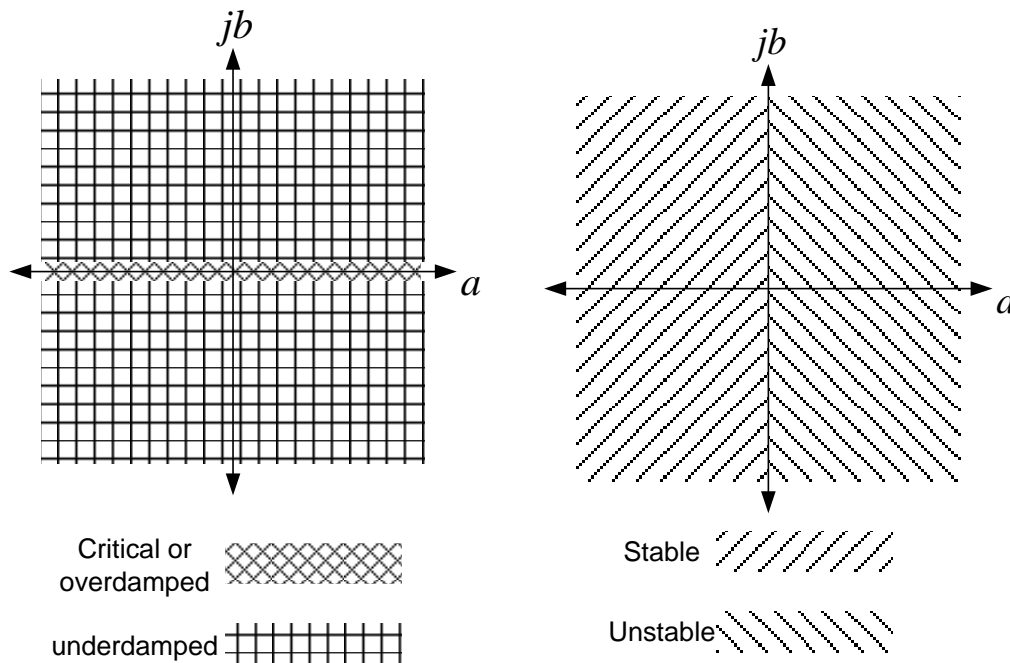
$$\text{By placing now where } r=r_1: (e^{rt})'' + A(e^{rt})' + Be^{rt} = 0$$

Which means that  $e^{rt}$  must be a solution of my ODE and:

$$\begin{vmatrix} e^{rt} & te^{rt} \\ r_1 e^{rt} & tr_1 e^{rt} + e^{rt} \end{vmatrix} = e^{rt} \cdot (tr_1 e^{rt} + e^{rt}) - te^{rt} \cdot r_1 e^{rt} = tr_1 e^{2rt} + e^{2rt} - tr_1 e^{2rt} = e^{2rt} \neq 0$$

And hence  $x_2(t) = e^{rt}$  is my second solution.

## Root Space



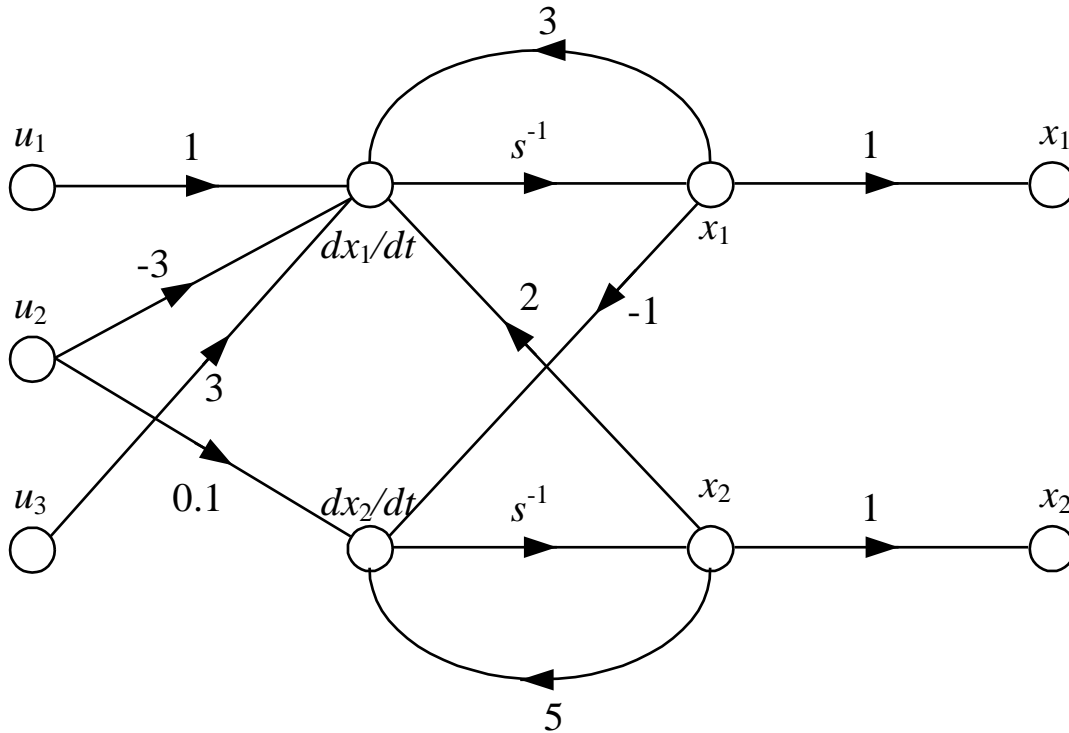
Name	Oscillations?	Components of solution
Overdamped	No	Two exponentials: $e^{k_1 t}, e^{k_2 t}, k_1, k_2 < 0$
Critically damped	No	Two exponentials: $e^{kt}, te^{kt}, k < 0$
Underdamped	Yes	One exponential and one cosine $e^{kt}, \cos(\omega t), k < 0$
Undamped	Yes	one cosine $\cos(\omega t)$

In the next chapter we will investigate more complicated 2<sup>nd</sup> order ODEs

like the following:

$$\frac{dx_1}{dt} = 3x_1 + 2x_2 + u_1 - 3u_2 + 3u_3$$

$$\frac{dx_2}{dt} = -x_1 + 5x_2 + 0.1u_2$$



#### 4. Tutorial Exercise I<sup>15 16</sup>

1. By using the general form of the analytic solution try to predict the response of the following systems. Your answer must describe the system as stable/unstable, convergent to zero/nonzero value. Crosscheck your answer by solving the DE:

i.  $5 \frac{dx}{dt} + 6x = 0, \quad x(0) = 0, x(0) = 1, x(0) = -1$

ii.  $5 \frac{dx}{dt} - 6x = 0, \quad x(0) = 0, x(0) = 1, x(0) = -1$

iii.  $5 \frac{dx}{dt} + 6x = 1, \quad x(0) = 0, x(0) = 1, x(0) = -1$

iv.  $5 \frac{dx}{dt} + 6x = -1, \quad x(0) = 0, x(0) = 1, x(0) = -1$

v.  $\frac{dx}{dt} - 3 = 0, \quad x(0) = 0, x(0) = 1, x(0) = -1$

2. Find the solution of  $\ddot{x} + 6\dot{x} + 5x = 0, x(0) = 2, \dot{x}(0) = 3$ . Briefly describe how the solution behaves for these initial conditions. Draw a sketch of the response.
3. Find the solution of  $\ddot{x} + 2\dot{x} + 6x = 0, x(0) = 1, \dot{x}(0) = 0$ . Briefly describe how the solution behaves for these initial conditions. Draw a sketch of the response.
4. Find the solution of  $\ddot{x} + \dot{x} + 0.25x = 0, x(0) = 2, \dot{x}(0) = 1/3$ . Briefly describe how the solution behaves for these initial conditions. Draw a sketch of the response.
5. Find the Wronskian matrices of the solutions of Q2-5.

---

<sup>15</sup> Please come to see me if you want exercises on how .

<sup>16</sup> It will be essential for the MSc students to solve these questions using (AND) Matlab where possible.