Chapter #3

EEE8013

Linear Controller Design and State Space Analysis

Solution of State Space Models Using Matlab

1. Solution using Eigenvalues and Eigenvectors

1.1 Case 1: Real and unequal eigenvalues

Example: 3.1

Assume that $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

To find the eigenvalues using the analytical method:

$$|\lambda I - A| = 0 \Leftrightarrow \begin{vmatrix} \lambda+2 & -2 \\ -2 & \lambda+5 \end{vmatrix} = 0 \Leftrightarrow \lambda^2 + 7\lambda + 6 = 0$$

To find the eigenvalues we need to solve the characteristic equation:

$\lambda^2 + 7\lambda + 6 = 0 \Rightarrow \lambda = -6, -1.$

Using Matlab we can use the command eig(A)

```
>> A=[-2 2;2 -5];

>> eig(A)

ans =

    -6

    -1
```

To find the eigenvalues and the corresponding eigenvectors we use the Matlab command [e,v]=eig(A).

```
>>  A=[-2 2;2 -5];

>> [e,v]=eig(A)
```

e =

-0.4472 -0.8944

0.8944 -0.4472

$\mathbf{v}_1 = \begin{bmatrix} -0.4472 \\ 0.8944 \end{bmatrix}$

$\mathbf{v}_2 = \begin{bmatrix} -0.8944 \\ -0.4472 \end{bmatrix}$

v =

-6 0

0 -1

$\lambda_1 = -6$

$\lambda_2 = -1$

Thus we should say that the general solution of the state space system $\dot{x} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} x$ can be written as:

$$x = C_1 \times v_1 \times e^{-6t} + C_2 \times v_2 \times e^{-t}$$

$$x(t) = C_1 \begin{bmatrix} -0.4472 \\ 0.8944 \end{bmatrix} e^{-6t} + C_2 \begin{bmatrix} -0.8944 \\ -0.4472 \end{bmatrix} e^{-t}$$

To find the particular solution we can use the initial conditions to find C_1 and C_2.

Assume: $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$x(0) = C_1\begin{bmatrix}-0.4472\\0.8944\end{bmatrix} + C_2\begin{bmatrix}-0.8944\\-0.4472\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-0.4472C_1 - 0.8944C_2\\0.8944C_1 - 0.4472C_2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-0.4472 & -0.8944\\0.8944 & -0.4472\end{bmatrix}\begin{bmatrix}C_1\\C_2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \Leftrightarrow \begin{bmatrix}C_1\\C_2\end{bmatrix} = \begin{bmatrix}-0.4472 & -0.8944\\0.8944 & -0.4472\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix}$$

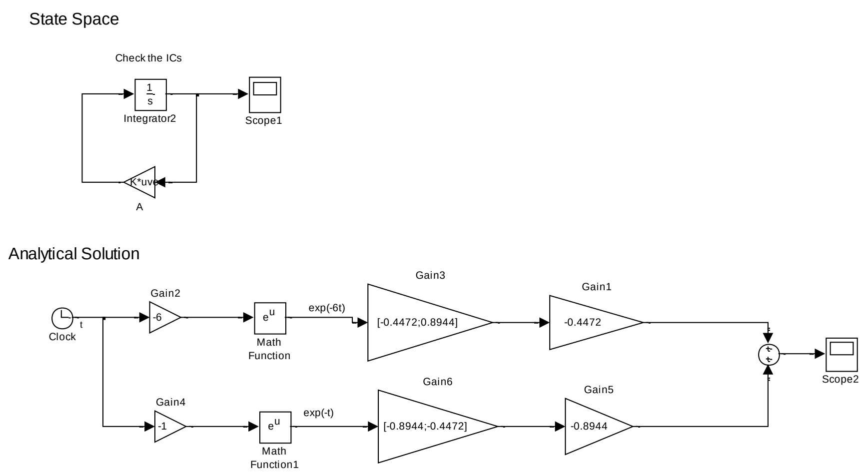
```
>> C=inv(e)*[1 0]'

C =

   -0.4472

   -0.8944
```

The particular solution is:

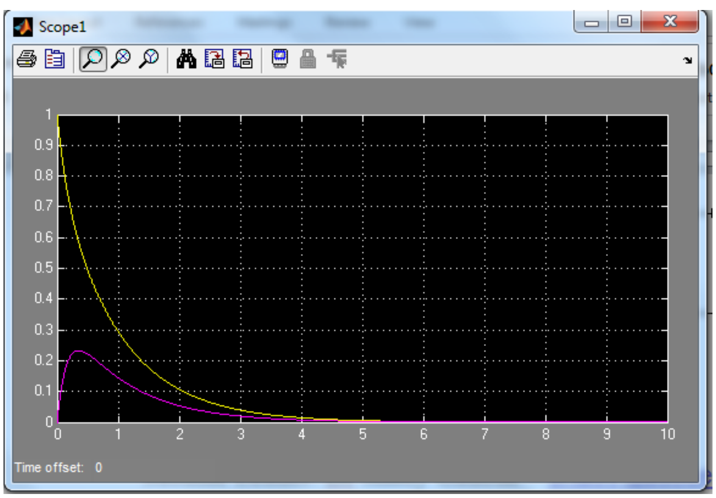
$$x(t) = -0.4472\begin{bmatrix}-0.4472\\0.8944\end{bmatrix}e^{-6t} - 0.8944\begin{bmatrix}-0.8944\\-0.4472\end{bmatrix}e^{-t}$$

We can now simulate the state space system and the analytical solution using eigenvectors.

(Simulink File ex 1)



To obtain the state response of the system double click on either Scope 1 or Scope 2.



State response

Example

[Simulink file ex 2]

A system is given by $\ddot{x}+7\dot{x}+6x=0, x(0)=1, \dot{x}(0)=0$

a) Find the particular solution

You can solve this part as in chapter 1.

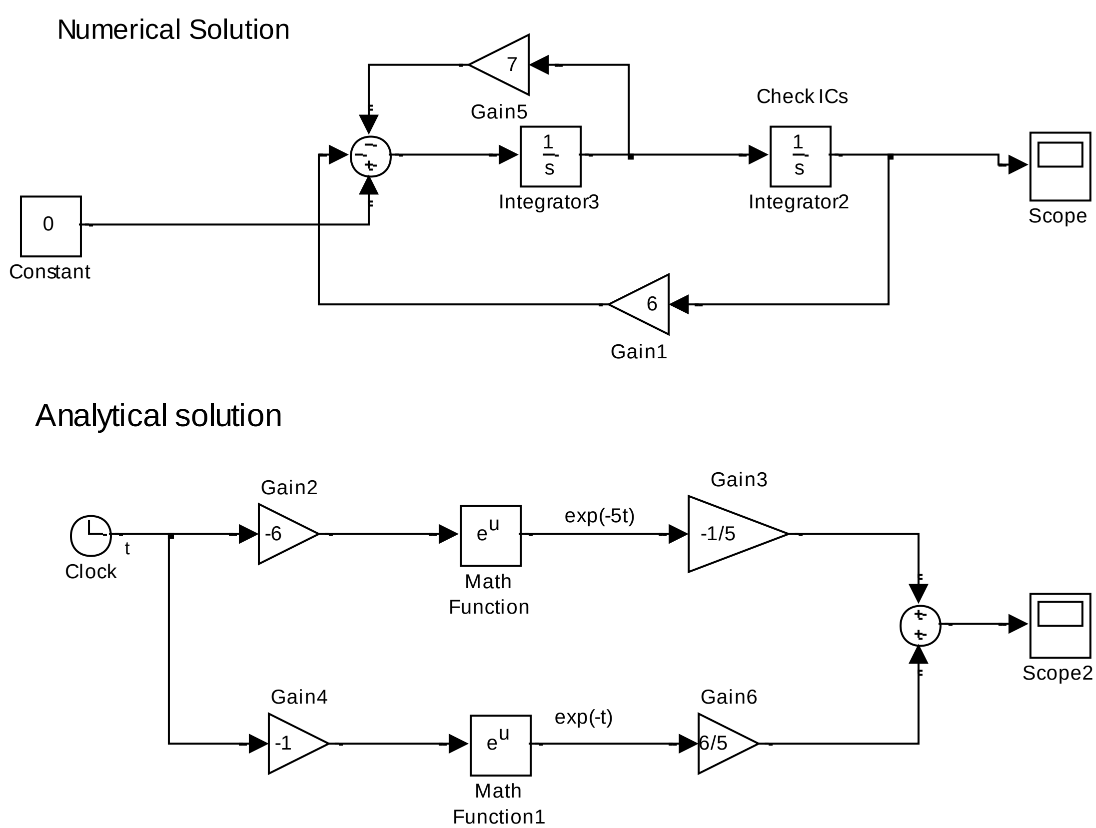
The general solution is $x=C_1e^{-t}+C_2e^{-6t}$.

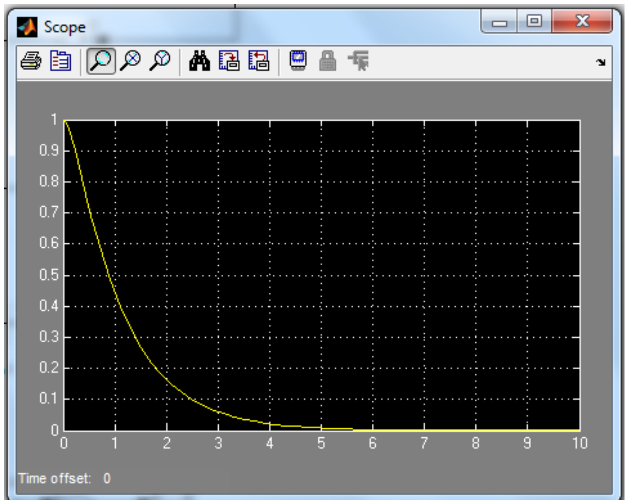
Using the initial conditions:

=> C_1=6/5, C_2=-1/5 and the particular solution is: $x=\frac{6}{5}e^{-t}-\frac{1}{5}e^{-6t}$

Note: since r_1 *and* r_2 are negative the response is stable and it converges exponentially to zero (homogeneous system) without oscillations.

(b)





The response of x(t)

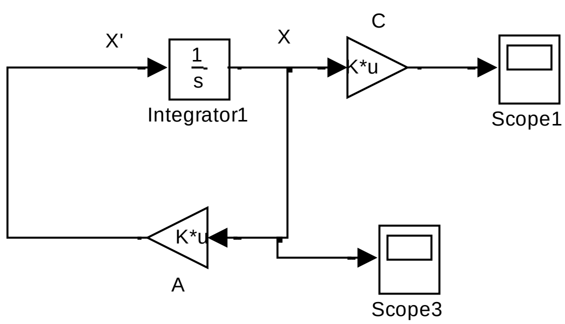
(c)Transform the system to state space form if y=x(t)

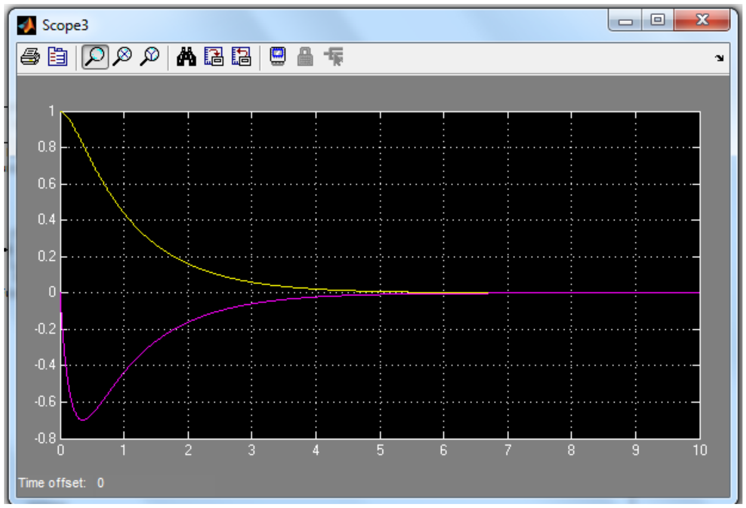
You can solve this part as in chapter 2. By defining $x_1 = x$, $x_2 = \dot{x}$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad y = x_1 \Leftrightarrow y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

(d)

State space





State response using state space representation

(e) Find the eigenvalues. Is the system stable? What will be the response type?

```
>> clear all

>> A=[0 1;-6 -7];

>> eig(A)

ans =

    -1

    -6
```

The eigenvalues of the system are: -1, -6 hence the system is stable with overdamped response.

To find the eigenvectors

```
>> [e,v]=eig(A)


e =
```

```
    0.7071   -0.1644
   -0.7071    0.9864


v =


    -1      0
     0     -6
```

(f) Find the general solution using the eigenvalues and eigenvectors

$$x(t) = C_1 \begin{bmatrix} 0.7071 \\ -0.7071 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -0.1644 \\ 0.9864 \end{bmatrix} e^{-6t}$$

To find the particular solution. (Compare your answer with i.)

$$x(0) = C_1 \begin{bmatrix} 0.7071 \\ -0.7071 \end{bmatrix} + C_2 \begin{bmatrix} -0.1644 \\ 0.9864 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

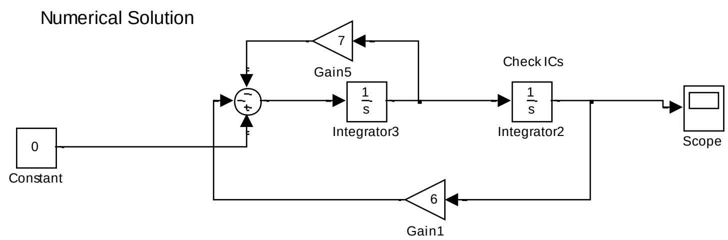
$$\Rightarrow \begin{bmatrix} 0.7071 & -0.1644 \\ -0.7071 & 0.9864 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0.7071 & -0.1644 \\ -0.7071 & 0.9864 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

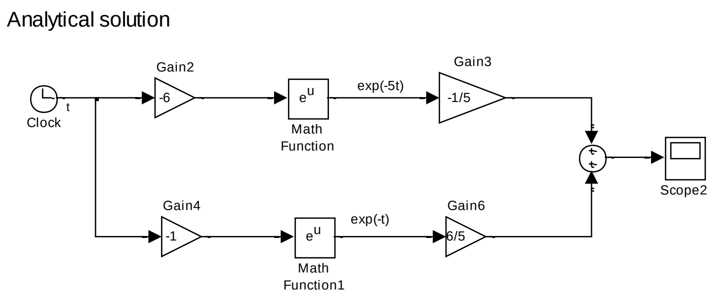
```
>> C=inv(e)*[1 0]'


C =

    1.6971

    1.2166
```

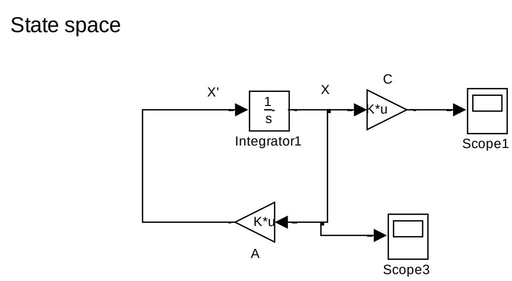
The particular solution is:

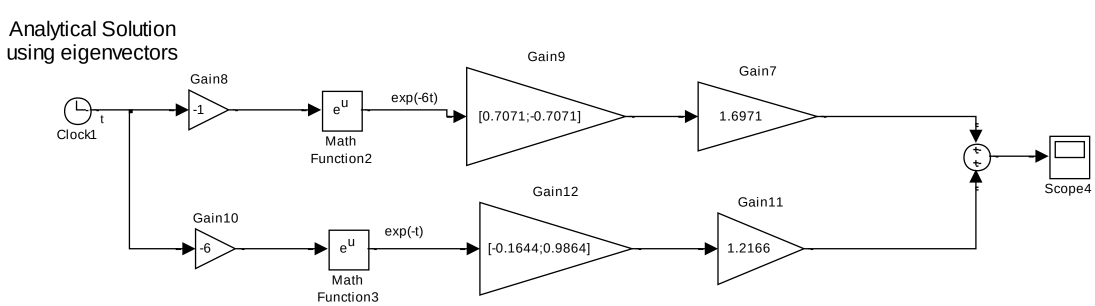
$$x(t) = 1.6971\begin{bmatrix} 0.7071 \\ -0.7071 \end{bmatrix} e^{-t} + 1.2166\begin{bmatrix} -0.1644 \\ 0.9864 \end{bmatrix} e^{-6t}$$

You can now simulate this solution.









In the lecture notes we found a solution for the same problem using eigenvectors as $x(t)=6/5\begin{bmatrix}1\\-1\end{bmatrix}e^{-t}-1/5\begin{bmatrix}1\\-6\end{bmatrix}e^{-6t}$. Simulate this solution and check that it gives the same response as other solutions.

Tutorial problem 3: (ex 3 simulink file)

A homogeneous system is given by $\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}2&1&5\\7&3&6\\1&5&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

(a) Use Matlab to find the system eigenvalues and eigenvectors. Is the system stable? Justify your answer.

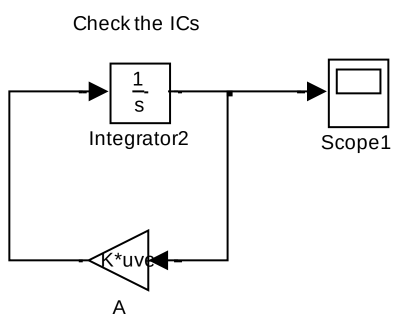
```
>> A=[2 1 5;7 3 6;1 5 2];

>> [e,v]=eig(A)

e =

   0.3893         -0.4221 - 0.4075i   -0.4221 + 0.4075i

   0.7716         -0.3641 + 0.4002i   -0.3641 - 0.4002i

   0.5031          0.6025              0.6025

v =

  10.4432               0                  0

        0         -1.7216 + 2.6447i        0

        0               0            -1.7216 - 2.6447i
```

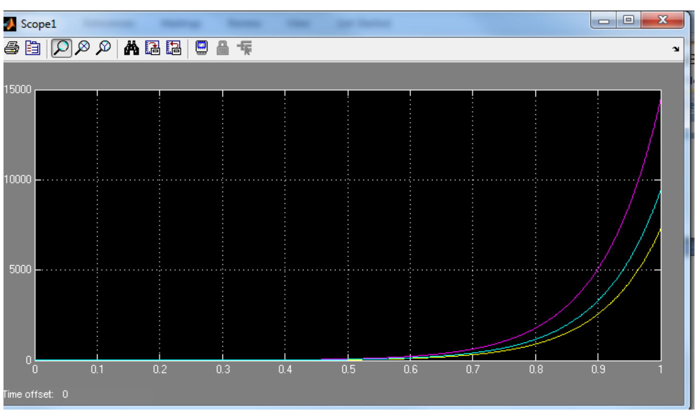
$\mathbf{v}_1$ $\mathbf{v}_2$ $\mathbf{v}_3$

Since one eigenvalue is 10.4432 (positive real) the system is unstable.

(b) Simulate the system in sate space form using $\mathbf{x}(0)=\begin{bmatrix}1\\0\\0\end{bmatrix}$ and plot the state response for $t\in[0\quad 1]$.

State Space





State response

Tutorial problem 4:

A homogeneous system is given by $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}2 & -1 & 1\\ -7 & 3 & 2\\ -1 & 5 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

(a) Use Matlab to find the system eigenvalues and eigenvectors. Is the system stable? Justify your answer.

```
>> A=[2 -1 1;-7 3 2;-1 5 0];

>> [e,v]=eig(A)

e =

   -0.0000    0.2171   -0.2604

    0.7071    0.5524   -0.5527

    0.7071    0.8048    0.7916

v =

    5.0000         0         0

         0    3.1623         0

         0         0   -3.1623
```

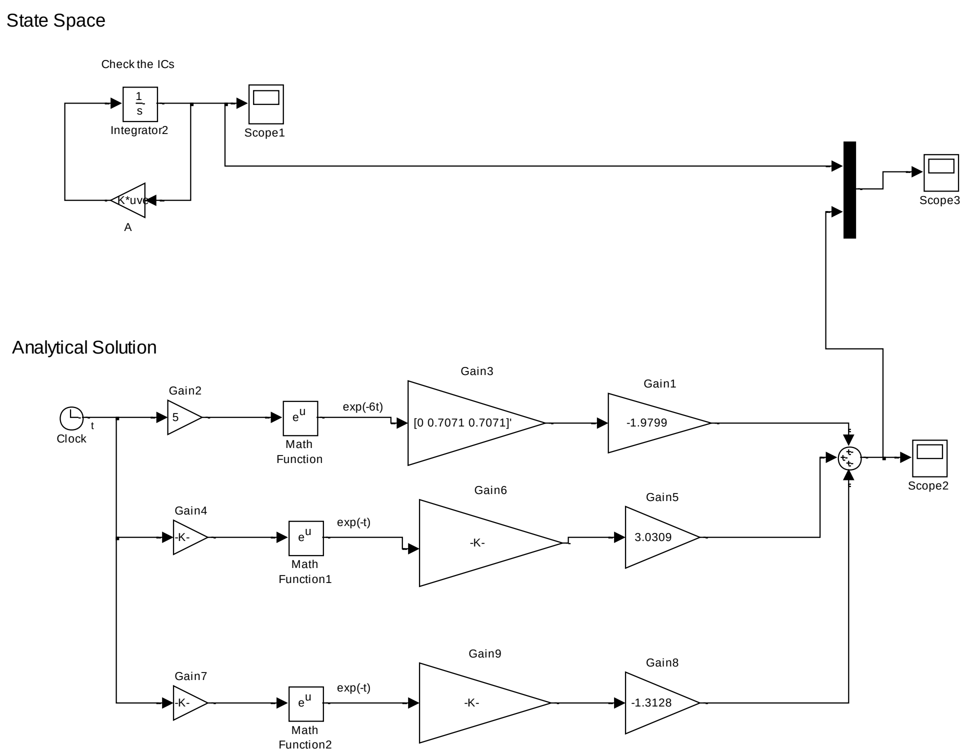
The system is unstable with overdamped response.

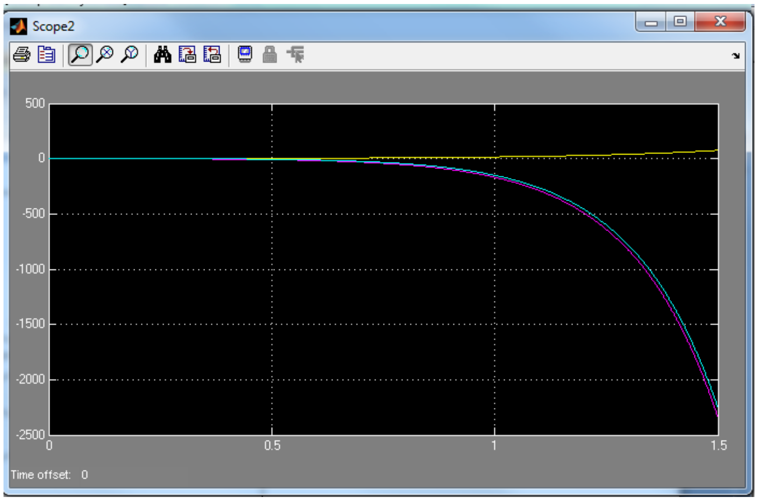
(b) Write down the general solution using the previously found eigenvectors?

$$x(t)=C_1\begin{bmatrix}0\\0.7071\\0.7071\end{bmatrix}e^{5t}+C_2\begin{bmatrix}0.2171\\0.5524\\0.8048\end{bmatrix}e^{3.1623t}+C_3\begin{bmatrix}-0.2604\\-0.5527\\0.7916\end{bmatrix}e^{-3.1623t}$$

(c) For $\mathbf{x}(0)=\begin{bmatrix}1\\1\\0\end{bmatrix}$ find the particular solution then simulate both state space form and particular solution and plot the state response for $t\in[0 \quad 1.5]$.

Could we still use *inv(e)*X(0)* to find the 3 constants??





State response

1.2 Case 2: Repeated Eigenvalues

A homogeneous system is given by $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Analytically find the Eigen values, Eigenvectors and general solution.

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow \lambda_{1,2} = 2 \Rightarrow \mathbf{v}_{1,2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

In that case I have that $(\lambda I - A)^2 b = 0$ (b is called the generalised eigenvector of A), which can be written as $(\lambda I - A)(\lambda I - A)b = 0$. Now I substitute $v = (\lambda I - A)b$ and I have $(\lambda I - A)v = 0$, i.e. v is one eigenvector of A for the eigenvalue λ. Now it can be proved that the general solution of this system can be written as: $x(t) = C_1(vt - b)e^{\lambda t} + C_2 v e^{\lambda t}$.

So in that case

$$v = (\lambda I - A)b \Leftrightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (2I - A)b \Leftrightarrow$$
$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} b_1 + b_2 \\ -b_1 - b_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence the solution is: $x(t) = C_1 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{2t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$

If we are given the initial conditions $[1\ 0]^T$:

$$x(0) = \begin{bmatrix} 0 \\ -C_1 \end{bmatrix} + \begin{bmatrix} C_2 \\ -C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} C_2 = 1 \\ C_1 = -1 \end{cases}$$

Now simulate both state space form and particular solution (analytical) and plot the state response for $t \in [0 \quad 1.5]$.

Exercise: Repeat the previous example for the system $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

1.3 Case 3: Complex Eigenvalues

If I have complex eigenvalues then $\lambda_1 = \overline{\lambda_2}$ and the corresponding eigenvectors are $\mathbf{v}_1 = \overline{\mathbf{v}_2}$. In that case the general solution is given by:

$$x(t) = A_1 v_1 e^{\lambda_1 t} + A_2 v_2 e^{\lambda_2 t}$$

Exercise:

Find the numerical and analytical solution of a homogeneous system (u=0) with a state matrix: $A=\begin{bmatrix}-1/2 & 1\\ -1 & -1/2\end{bmatrix}$, assuming $\mathbf{x}(0)=\begin{bmatrix}1\\0\end{bmatrix}$

2. Solution using State Transition Matrix

This is a general approach to solve state space models. As described in Chapter 1 for a scalar homogeneous DE: $\dot{x}=ax$ the solution was $x(t)=e^{at}x(0)$ (no special cases) so can we do the same with $\dot{x}=Ax$, i.e. $x=e^{At}x(0)$?

The solution of the homogeneous system can then be written as:

$$x(t)=\varphi(t)x(0)=e^{\mathbf{A}t}x(0)$$

The solution of the non-homogeneous system can be written as:

$$x(t)=\varphi(t)x(0)+\int_0^t \varphi(t-\tau)Bud\tau$$

Where the first term is the response to initial conditions and the second term is the response due to external input u(t).

In Matlab we use the command **expm** (not exp!!) to calculate $e^{\mathbf{A}t}$

Example:

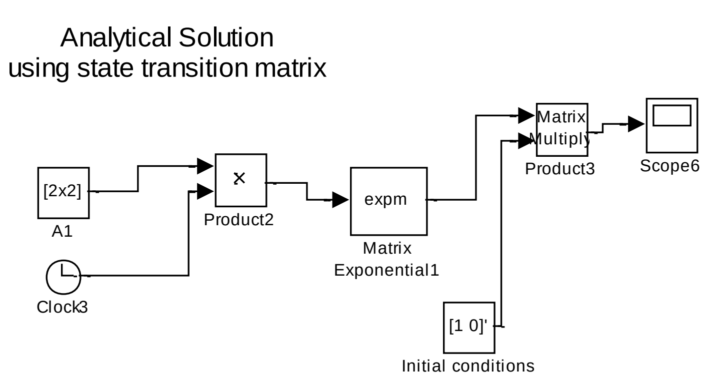
For the system $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-2 & 2\\ 2 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$, $A=\begin{bmatrix}-2 & 2\\ 2 & -5\end{bmatrix}$ assuming that $\boldsymbol{x}(0)$=[1 2] what is the value of $x(5)$?

```
>> clear all

>> A=[-2 2;2 -5];

>> X_5=expm(A*5)*[1 2]'

X_5 =

    0.0108

    0.0054
```

Example:

For the system described in Tutorial problem 2 $\Rightarrow\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -6 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

Find the solution using state transition matrix method $x(t)=e^{At}x(0)=\varphi(t)x(0)$



Expm: DSP system toolbox/Math functions/Matrices and linear algebra/Matrix operations

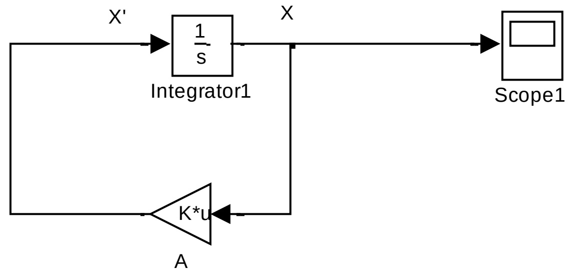
Check that you get the same answer as in ex 2!

Tutorial problem 5:

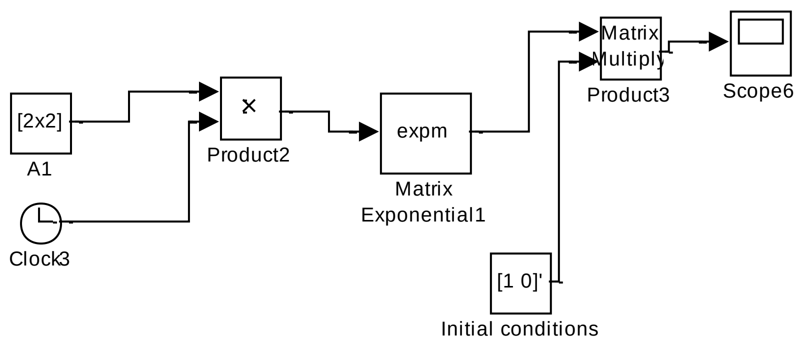
For the following system $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

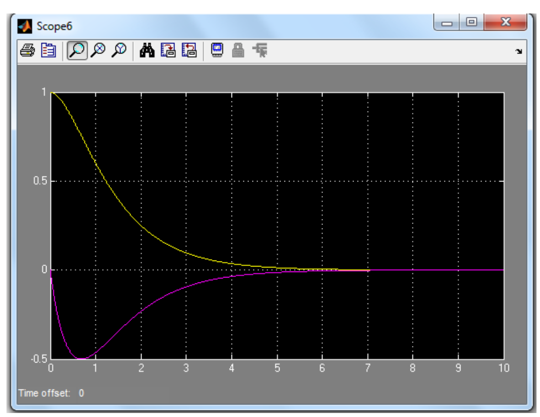
a) Simulate the state space system and plot the state response. $x_1(0)=1$, $x_2(0)=0$.
b) Find the state response using the state transition matrix method.

State space



Analytical solution using
state transtion matrix



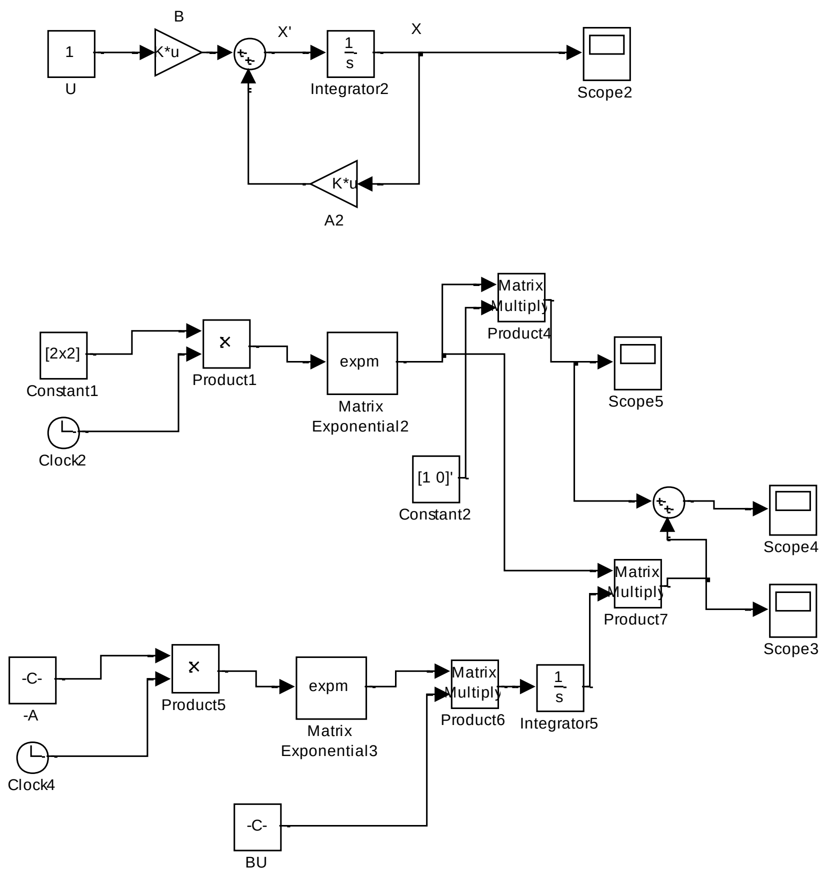


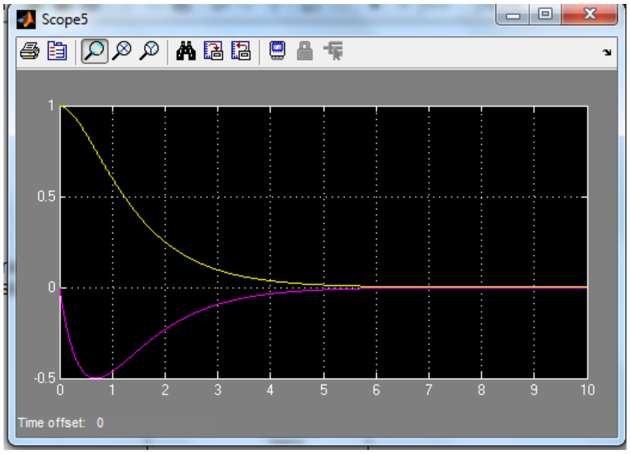
State response

Now try to find the same solution using eigenvalues and eigenvectors and crosscheck your answer?

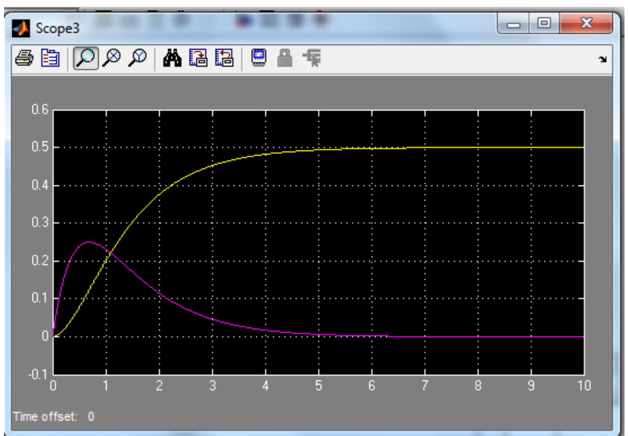
c) Find the state response for a unit step input assuming same initial conditions and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ using state space representation and using $x(t) = e^{\mathbf{A}t}x(0) + \int_0^t e^{\mathbf{A}(t-\tau)}Bud\tau$. Show each component of the response (i.e. the part that depends on the initial conditions and the part that depends on the input signal).

State space I

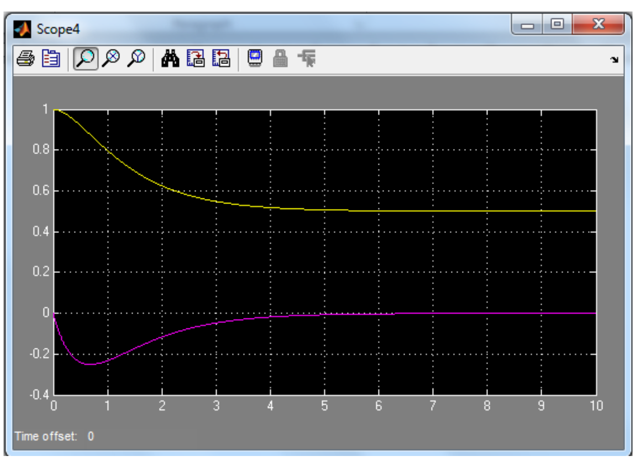




Response to initial conditions



Response to external input u(t)



Overall response

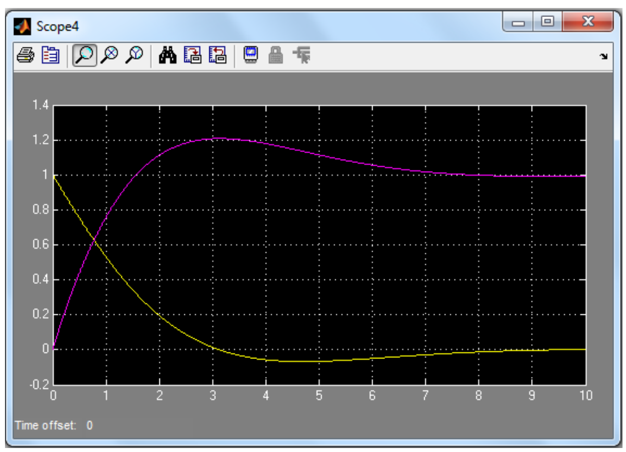
Tutorial problem 6:

Find the state response x(t) of the state space system described by:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1 & -0.5\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0.5\\ 0\end{bmatrix}u,\ y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Where u(t) is unit step and the system initial conditions are $[1\ 0]^T$. Simulate the state space form and the solution using state transition matrix.

This is a non-homogeneous system with non-zero initial conditions.



Overall response

Linear Controller Design and State Space Analysis
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Tutorial Exercise III

1. A system is given by $\dot{x} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} x$

   (a) Use Matlab to find the eigenvalues and eigenvectors of this system.

   (b) Find the general solution using the previously found eigenvectors.

   (c) Find the particular solution if $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   (d) Simulate the particular solution obtained in (c) and Crosscheck you answer by simulating the system in state space form.

   (e) Plot the state response of the system.

2. A system is given by $\ddot{x} + 7\dot{x} + 6x = 0, x(0) = 1, \dot{x}(0) = 0$

   (a) Find the particular solution of the differential equation.

   (b) Simulate both numerical and analytical solution obtained in (a) and plot the response *x*(t).

   (c) Transform the system to state space form if *y* = *x*(t).

   (d) Simulate the state space form and plot the state response.

   (e) Using Matlab find the eigenvalues and eigenvectors of the system. What is the response type?

   (f) Find the general solution using the eigenvectors then find the particular solution using the given initial conditions.

   (g) Simulate the particular solution obtained in (f) and crosscheck your answer.

3. A homogeneous system is given by $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 5 \\ 7 & 3 & 6 \\ 1 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(a) Use Matlab to find the eigenvalues and eigenvectors. Is the system stable?

(b) Simulate the state space form of the system using $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and plot the state response for $t \in [0 \quad 1]$.

4. A homogeneous system is given by $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -7 & 3 & 2 \\ -1 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(a) Use Matlab to find the eigenvalues and eigenvectors. Is the system stable?

(b) Write down the general solution using eigenvectors.

(c) If $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ find the particular solution.

(d) Simulate both state space and analytical solution in (c) and plot the state response for $t \in [0 \quad 1.5]$.

5. For the following system $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(a) Simulate the state space system and plot the state response. $x_1(0)=1$, $x_2(0)=0$.

(b) Find the state response using the state transition matrix. Simulate this solution using the initial conditions given in (a). Plot the state response.

(c) Find the state response for a unit step input assuming same initial conditions and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ using state space representation and using state transition matrix method $x(t) = e^{\mathbf{A}t}x(0) + \int_0^t e^{\mathbf{A}(t-\tau)}Bu\,d\tau$. Show each component of the response (i.e. the part that depends on the initial conditions and the part that depends on the input signal).

6. Find the output response y(t) of the state space system described by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u, \; y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where u(t) is unit step and the system initial conditions are $[1 \; 0]^T$. Simulate the state space form and the solution using state transition matrix. Crosscheck your answer.