# Revision

$\ddot{x} + A\dot{x} + Bx = 0$

- If $x_1, x_2$ are sol$^n$s $\longrightarrow$ $y = c_1 x_1 + c_2 x_2$ is also a sol$^n$
  i.e. ANY L.C. $\longrightarrow$
- if $x_1, x_2$ are L.I. sol$^n$s $\longrightarrow$ ALL other sol$^n$s
  $\ast$ $y = c_1 x_1 + c_2 x_2$

$\downarrow$

I want to find 2 L.I. sol$^n$s

Try $x = e^{rt} \longrightarrow r^2 + Ar + B = 0$ C.E.

$r_1 \quad r_2$ eigenvalues

$$r_{1,2} = \frac{-A \pm \sqrt{\Delta}}{2} \qquad \Delta = A^2 - 4 \cdot B$$

■ $\Delta > 0$ $\quad r_1 = \dfrac{-A + \sqrt{\Delta}}{2}$ $\quad r_1 \neq r_2$

overdamped. $\quad r_2 = \dfrac{-A - \sqrt{\Delta}}{2}$ $\quad r_1, r_2 \in \mathbb{R}$

L.I. $\quad x_1 = e^{r_1 t} \quad x_2 = e^{r_2 t}$ sol'ns

$c_1, c_2 \in \mathbb{R} \leftarrow x = c_1 \cdot e^{r_1 t} + c_2 e^{r_2 t}$

$x_0 = \ldots \quad \dot{x}_0 = \ldots\ldots \Big\} \Rightarrow c_1 = \ldots,\ c_2 = \ldots$

■ $\Delta < 0$ $\quad r_1 = \dfrac{-A + i\sqrt{-\Delta}}{2}$ $\quad r_1 = \bar{r}_2$

underdamped $\quad r_2 = \dfrac{-A - i\sqrt{-\Delta}}{2}$ $\quad r_1, r_2 \in \mathbb{C}$

L.I. $\quad x_1 = e^{r_1 t} \quad x_2 = e^{r_2 t}$ sol'ns

$x = c_1 x_1 + c_2 x_2 \quad c_1, c_2 \in \mathbb{C}$

Try $\quad y_1 = \frac{1}{2}(x_1 + x_2)$, $\quad y_2 = \frac{1}{2i}(x_1 - x_2)$

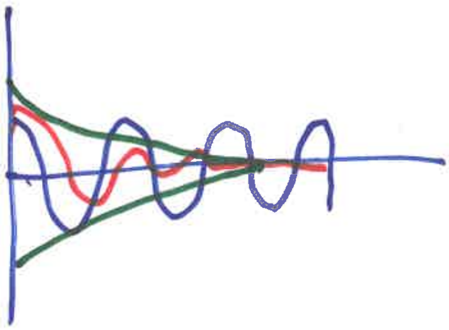
L.I.

$y_1 = e^{at} \cdot \cos bt$, $\quad y_2 = e^{at} \sin bt$

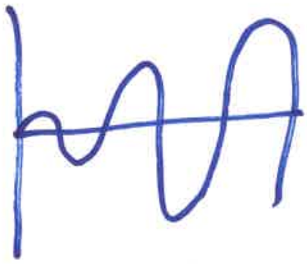
$a = -A/2$, $\quad b = \dfrac{\sqrt{-\Delta}}{2}$

$x = e^{at}(c_1 \cdot \cos bt + c_2 \sin bt) \quad c_1, c_2 \in \mathbb{R}$

$a<0$



$a>0$



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$\Delta = 0$ $\qquad r_1 = r_2 = r = -A/2 \in \mathbb{R}.$

$$\Downarrow$$

$$x_1 = e^{rt}$$

$$x_2 = t e^{rt}$$

$$x = e^{rt} c_1 + c_2 t e^{rt}, \quad c_1, c_2 \in \mathbb{R}.$$

e.g.

$r = -1$ $\qquad x_1 = e^{-t} \longrightarrow$ conv.

$\qquad x_2 = t \cdot e^{-t} \longrightarrow$ conv.

critically damped system

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$\dot{x} + kx = 0 \leftarrow x = e^{rt}$ $\qquad r + k = 0 \Rightarrow r = -k$

$$x = e^{-kt}$$

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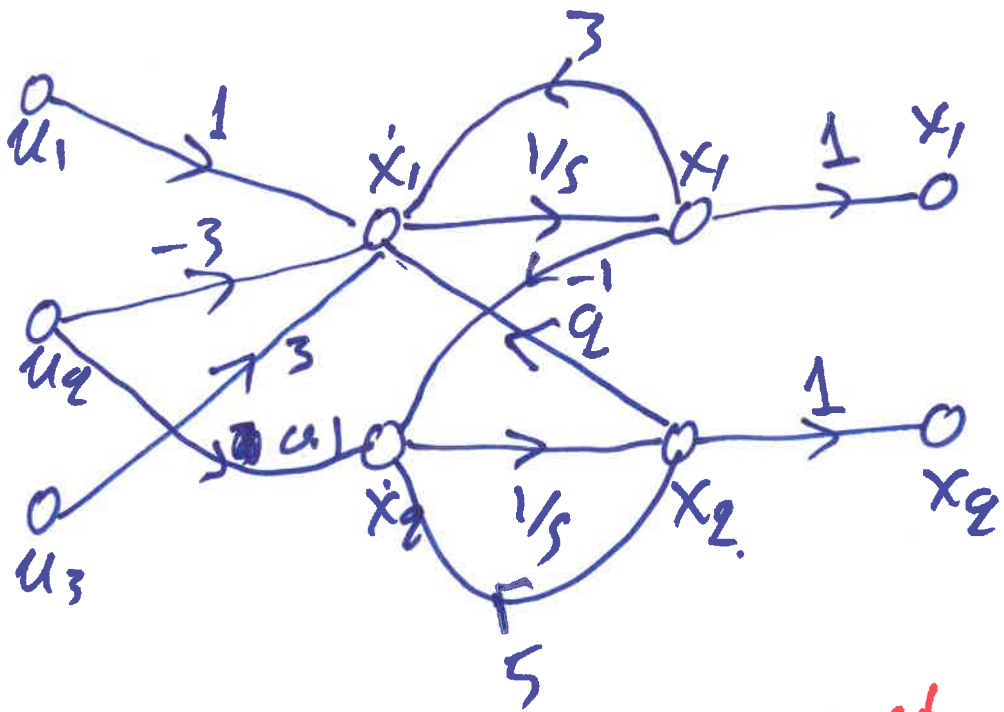
$\ddot{x} + A\dot{x} + Bx = u$ $\qquad$ stable $\qquad x_{ss} = \text{const.}$

$u = \text{const.}$ $\qquad \cancel{\ddot{x}_{ss}}^{\to 0} + A\cancel{\dot{x}_{ss}}^{\to 0} + B \cdot x_{ss} = u$

$$x_{ss} = u/B$$

$\dot{x}_1 = 3x_1 + 2x_2 + u_1 - 3u_2 + 3u_3$

$\dot{x}_2 = -x_1 + 5x_2 + 0.1u_2$




Linear Algebra.

Must be 2x1 — check

$$\underbrace{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix}3 & 2\\ -1 & 5\end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{2\times 1} + \underbrace{\begin{bmatrix}1 & -3 & 3\\ 0 & 0.1 & 0\end{bmatrix}}_{2\times 3}\underbrace{\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}}_{3\times 1}$$

$$\underbrace{\dot{x}}_{n\times 1} = \underbrace{A}_{n\times n}\cdot \underbrace{X}_{n\times 1} + \underbrace{B}_{n\times m}\cdot \underbrace{u}_{m\times 1}$$

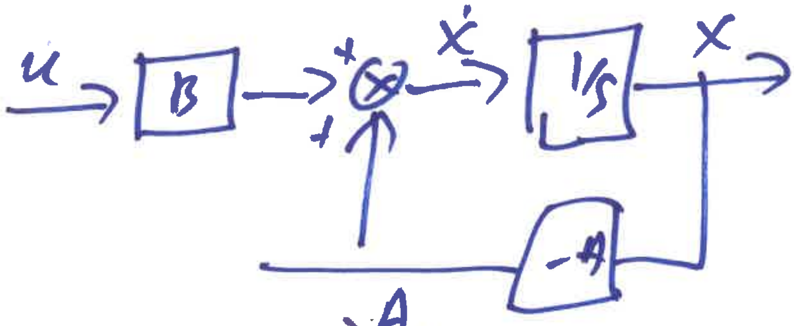
X → STATE VECTOR

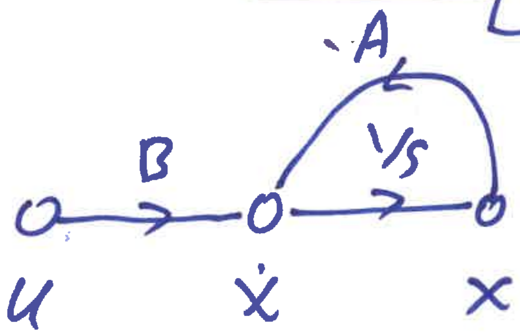
u → Input vector

A → state matrix

B → input ~~vector~~ matrix

$\dot{x} = Ax + B \cdot u.$

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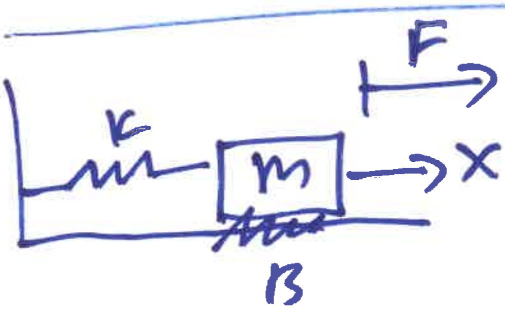


$x_1 = x = \text{disp}$

$x_2 = \dot{x} = \text{vel}$

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -K/m & -B/m \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{B} \underbrace{F}_{u.}$$

Monitor MSS ?

sensor $x_1$

sensor $x_2$

$y_1 = x_1 = \text{disp.}$

$y_2 = x_2 = \text{vel}$

case 1

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Y = [1 \quad 0] \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$ Case 2

$$Y = [0 \quad 1] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$ Case 3

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$$\underbrace{Y}_{p\times 1} = \underbrace{C}_{p\times n} \cdot \underbrace{X}_{n\times 1} + \underbrace{\cancel{D}}_{p\times m} \cdot \underbrace{U}_{m\times 1} \rightarrow 0$$

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$$\dot{X} = AX + B \cdot U \rightarrow \text{state}$$

$$Y = C \cdot X \rightarrow \text{output}$$

# ODEs → S.S.

$\ddot{x} + 3\dot{x} + 2x = u$ , $y = 4 \cdot x$

① $\ddot{x} = -3\dot{x} - 2x + u.$

② $\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}\right\} \overset{③}{\Longrightarrow} \left.\begin{aligned} \dot{x}_1 &= \dot{x} = x_2 \\ \dot{x}_2 &= \ddot{x} = -3\cdot\dot{x} - 2x + u. \\ &= -3x_2 - 2x_1 + u. \end{aligned}\right\} \Rightarrow$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\underset{2\times 1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \cdot \underset{1\times 1}{u}.}_{2\times 1}$$

$$y = \begin{bmatrix} 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dddot{x} - 3\ddot{x} - 2\dot{x} + 2x = u_1 - 6u_2$$

$$y_1 = \ddot{x} + u_2 = x_3 + u_2$$

$$y_2 = \ddot{x} + 3x + 5u_1 = x_3 + 3x_1 + 5u_1$$

$$y_3 = -3\ddot{x} + x + 5 \cdot u_2 = -3x_3 + x_1 + 5u_2$$

$$\dddot{x} = 3\ddot{x} + 2\dot{x} - 2x + u_1 - 6u_2$$

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= \ddot{x} \end{aligned}\right\} \overset{d/dt}{\Longrightarrow} \begin{aligned} \dot{x}_1 &= \dot{x} = x_2 \\ \dot{x}_2 &= \ddot{x} = x_3 \\ \dot{x}_3 &= \dddot{x} = 3\ddot{x} + 2\dot{x} - 2x + u_1 - 6u_2 \\ \dot{x}_3 &= 3 \cdot x_3 + 2x_2 - 2x_1 + u_1 - 6u_2 \end{aligned}$$

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}}_{3\times 1} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 2 & 3 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{3\times 1}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -6 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{2\times 1}$$

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$$\underset{3\times1}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} = \underset{3\times3}{\begin{bmatrix} 0 & 0 & 1 \\ 3 & 0 & 1 \\ 1 & 0 & -3 \end{bmatrix}} \cdot \underset{3\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} + \underset{3\times2}{\begin{bmatrix} 0 & 1 \\ 5 & 0 \\ 0 & 5 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}$$