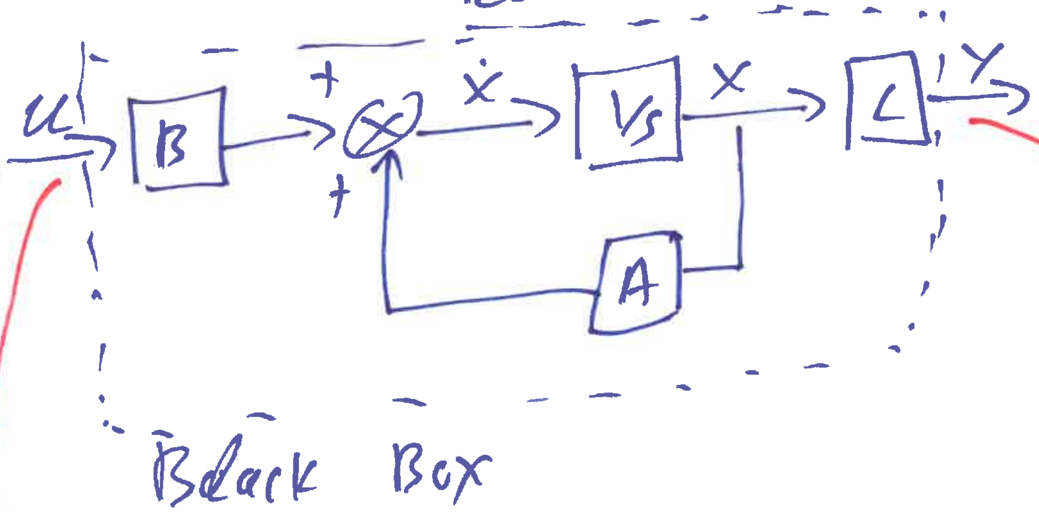
Revision



u can control ALL states x → CTRB

Y "observe" ALL states X → OBSV

ODE + SFD

$$M_c = [B \quad AB \quad A^2B \dots A^{n-1}\cdot B]$$

$$M_o = [C \quad AC \quad A^2C \dots A^{n-1}C]^T$$

$$\text{rank}(M) = n \begin{cases} \to \text{CTRB} \\ \to \text{OBSV} \end{cases}$$

$\dot{x} = ax$, $x \in \mathbb{R}$, $a \in \mathbb{R}$

$x = e^{rt} \rightarrow r = ?$

$r = a$ CE $\Rightarrow x = e^{at}$

$\ddot{x} + A\dot{x} + Bx = 0$

$x, A, B \in \mathbb{R}.$

$x = e^{rt} \rightarrow r = ?$

$r^2 + Ar + B = 0$ CE

$\Delta = \begin{cases} \rightarrow \Delta > 0 \\ \rightarrow \Delta = 0 \\ \rightarrow \Delta < 0 \end{cases}$

$\dot{x} = Ax$, $x \in \mathbb{R}^2$, $A \in \mathbb{R}^{2\times 2}$

$r \rightarrow$ eigenvalue
$e \rightarrow$ eigenvector

$\underbrace{x}_{\text{vector}} = \underbrace{e^{rt}}_{\text{scalar.}} \cdot \underbrace{e}_{\text{vector}}$

$r = ?$

$e = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$x = e^{rt} \cdot e \Rightarrow \dot{x} = r e^{rt} \cdot e$

$r \cancel{e^{rt}} \cdot e = A \cancel{e^{rt}} \cdot e$

$r \cdot e = A \cdot e$

$A = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \rightarrow$ Given.

$$r\cdot\begin{bmatrix}a_1\\a_2\end{bmatrix} = A\cdot\begin{bmatrix}a_1\\a_2\end{bmatrix}$$

$$r\begin{bmatrix}a_1\\a_2\end{bmatrix} = \begin{bmatrix}k_1 & k_2\\k_3 & k_4\end{bmatrix}\cdot\begin{bmatrix}a_1\\a_2\end{bmatrix}$$

$$\boxed{r}\boxed{a_1} = k_1\cdot\boxed{a_1} + k_2\cdot\boxed{a_2}$$
$$\boxed{r}\boxed{a_2} = k_3\cdot\boxed{a_1} + k_4\cdot\boxed{a_2}$$

$\square \to$ unknowns.

$\bigcirc \to$ knowns.

$\to$ NL 3×2 sys.

- $3x+y=0$
  $6x+y=0$
  - $x=0, y=0$ (Trivial)
    $(x,y)=(0,0)$.
  - $x\neq 0$ $\left.\begin{array}{l} y=-3x \\ y=-6\cdot x\end{array}\right\}$ $3=6$
    No more soln.

- $3x+y=0$
  $6x+2y=0$
  $(x,y)=(0,0)$
  $\left.\begin{array}{l} y=-3x \\ 2y=-6x\end{array}\right\}$ $\left.\begin{array}{l} y=-3x \\ y=-3x\end{array}\right\}$ $y=-3x$
  assume $x=k \Rightarrow y=-3k$
  $(x,y)=(k,-3k)$
  inf. soln.

$$\begin{bmatrix} 3 & 1 \\ 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{vmatrix} 3 & 1 \\ 6 & 1 \end{vmatrix} = 3-6 = -3} \rightarrow \text{inv } \exists$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 6 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix} = 6-6 = 0} \rightarrow \text{in } \nexists$$

$$\left.\begin{array}{l} k_1 x + k_2 y = 0 \\ k_3 x + k_4 y = 0 \end{array}\right\}$$

- $(x,y) = (0,0)$ ← 1 sol
- if $\begin{vmatrix} k_1 & k_2 \\ k_3 & k_4 \end{vmatrix} \neq 0$
- if $\begin{vmatrix} k_1 & k_2 \\ k_3 & k_4 \end{vmatrix} = 0 \rightarrow$ inf. soln.

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$$r \cdot a_1 = k_1 a_1 + k_2 a_2$$

$$r \cdot a_2 = k_3 a_1 + k_4 \cdot a_2$$

$r \to$ Given $\to$ L.S. 2×2

$$(k_1 - r) \cdot a_1 + k_2 \cdot a_2 = 0$$

$$k_3 \cdot a_1 + (k_4 - r) \cdot a_2 = 0$$

$$\begin{bmatrix} k_1 - r & k_2 \\ k_3 & k_4 - r \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Trivial: $a_1 = a_2 = 0 \to X = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{rt} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- Inf soln:

$$\begin{vmatrix} k_1 - r & k_2 \\ k_3 & k_4 - r \end{vmatrix} = 0$$

$$\left| \underbrace{\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}}_{A} - \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}_{r \cdot I} \right| = 0$$

$$|A - rI| = 0 \to \text{C.E.}$$

$$\begin{vmatrix} k_1 - r & k_2 \\ k_3 & k_4 - r \end{vmatrix} = 0$$

$$(k_1 - r)(k_4 - r) - k_2 k_3 = 0$$

$$\Delta = \ldots \begin{cases} \Delta > 0 \\ \Delta < 0 \\ \Delta = 0 \end{cases} \quad r_1 \quad r_2$$

$$\downarrow \qquad \downarrow$$

$$e_1 \qquad e_2$$

Gen. $e, x \in \mathbb{R}^{n \times 1}$, $A \in \mathbb{R}^{n \times n}$

$$\dot{x} = Ax, \quad x = e \cdot e^{rt}$$

$$r \cancel{e^{rt}} \cdot e = A e \cancel{e^{rt}}$$

$$\underbrace{re - Ae}_{\downarrow} = 0 \qquad \text{N.L.} \quad \begin{matrix} n+1 \text{ unknowns} \\ n \text{ eqns} \end{matrix}$$

$$(rI - A)e = 0$$

$$|rI - A| = 0$$

$$|A - rI| = 0 \Rightarrow \begin{matrix} r_1 = \cdot & \rightarrow & e_1 \\ r_2 = \ldots & & e_2 \\ r_3 = \ldots & & \vdots \end{matrix}$$

$$\dot{x} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \cdot x \qquad soln = ?$$

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- stability
- Gen soln
- spec. soln $\quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$x = \ell \, e^{rt} \qquad \ell = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad x = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{rt}$$

C.E

- $|A - rI|$

$$\left| \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} - \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -2-r & 2 \\ 2 & -5-r \end{vmatrix} = 0$$

$$(-2-r)(-5-r) - 2 \cdot 2 = 0$$

$$(r+2)(r+5) - 4 = 0$$

$$r^2 + 7r + 6 = 0 \qquad \text{C.E.}$$

$r_1 = -1 \quad r_2 = -6 \rightarrow$ sys is stable

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$$r_1 = -1 \rightarrow x_1 = e_1 e^{-t}$$
$$r_2 = -6 \rightarrow x_2 = e_2 e^{-6t}$$
$\Big\} \Rightarrow$ L.I. solns

ALL $\quad x = c_1 x_1 + c_2 x_2$

$$= c_1 e_1 e^{-t} + c_2 e_2 e^{-6t}$$

$r_1 = -1 \quad (A - \cancel{r} I)\cdot e_1 = 0$

$$(A + I)\cdot e_1 = 0$$

$$\left(\begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) e = 0$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

$$-a_1 + 2\cdot a_2 = 0$$
$$2a_1 - 4\cdot a_2 = 0 \Rightarrow -a_1 + 2a_2 = 0$$
$$a_1 = 2a_2$$

- $a_1 = 1 \Rightarrow a_2 = 1/2 \qquad a_1 = \sqrt{3}/\pi \qquad a_2 = \dfrac{\sqrt{3}}{2\pi}$
- $a_1 = 5 \Rightarrow a_2 = 5/2 \qquad \boxed{a_2 = 1 \qquad a_1 = 2}$

$$a_1 = 2 \qquad a_2 = 1 \qquad e_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$$

$$\lambda_2 = -6 \qquad (A - r_2 I) \cdot e_2 = 0$$

$$(A + 6 \cdot I) \cdot e_2 = 0$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2 \cdot a_3 + a_4 = 0$$

$$a_4 = -2 a_3$$

$$a_3 = 1 \Rightarrow a_4 = -2.$$

$$x_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-6t}$$

$$x = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-6t}$$

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$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} 2 \cdot c_1 + c_2 = 1 \\ c_1 - 2c_2 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 0.4 \\ c_2 = 0.2 \end{array}$$

$$\boxed{x = 0.4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + 0.2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-6t}}$$

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

C.E. $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{vmatrix} = 0$$

$$(-\lambda) \cdot (-7-\lambda) - (-6) \cdot 1 = 0$$

$$\lambda^2 + 7 \cdot \lambda + 6 = 0 \rightarrow \begin{array}{l} \lambda_1 = -1 \\ \lambda_2 = -6 \end{array}$$

Since both eigs are negative $\rightarrow$ sys is stable

• $\lambda_1 = -1$ $\quad (A - \lambda_1 I) \cdot e_1 = 0$

$$(A + I) \cdot e_1 = 0$$

$$\left(\begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \cdot e_1 = 0$$

$$\begin{bmatrix} 1 & 1 \\ -6 & -6 \end{bmatrix} e_1 = 0$$

$e_1 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $\quad a_1 + a_2 = 0$

$a_1 = 1 \Rightarrow a_2 = -1$

$e_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

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• $\lambda_2 = -6$ $\quad (A+6I)\cdot e_2 = 0$

$$\left(\begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\right) e_2 = 0$$

..... $e_2 = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$ Do it home

Gen. soln.

$$X = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 1 \\ -6 \end{bmatrix} \cdot e^{-6t}$$

$$C_1 = 6/5 \,,$$

$$C_2 = -1/5.$$

$\dot{X} = A \cdot X \quad n = 2$

C.E. $|A - \lambda I| = 0$

$\downarrow$

2nd order eqn.

$\lambda_1 = \ldots \to e_1 = \ldots$

$\lambda_2 = \ldots \to e_2 = \ldots$

$\downarrow$

$X = c_1 e_1 e^{\lambda_1 t} + c_2 e_2 e^{\lambda_2 t}$

$\Delta = 0$

.... $\lambda_1 = \lambda_2 = \lambda \to e \to X_1 = e\, e^{\lambda t}$

$X_2 = ?\ t \cdot \cancel{e\, e^{\lambda t}}$

$X_2 = (et + b)\, e^{\lambda t}$

b = Gen eigenvector

$e = (A - \lambda I) \cdot b$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$1 = -b_1 - b_2 \qquad b_2 = -1 - b_1$

- $b_1 = 2 \longrightarrow b_2 = -3$
- $b_1 = 1 \longrightarrow b_2 = -2$
- $b_1 = 0 \longrightarrow b_2 = -1.$

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$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \qquad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) + 1 = 0$$ unεA.

$$(\lambda - 2)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda = +2$$

$$(A - \lambda I) \cdot e = 0$$

$$(A - 2I) e = 0 \Rightarrow e_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$e = (A - \lambda I) \cdot b.$$

$$A - 2I = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$