Revision

$\dot{x} = A \cdot x \;,\; x \in \mathbb{R}^{n\times 1} \;,\; A \in \mathbb{R}^{n\times n}$

Try $\underbrace{x}_{n\times 1} = \underbrace{e^{\lambda t}}_{1\times 1} \cdot \underbrace{e}_{n\times 1}$

$\lambda \to$ eigenvalue
$e \to$ eigenvector.

$\lambda \cdot \cancel{e^{\lambda t}} \cdot e = A \cdot \cancel{e^{\lambda t}} \cdot e$

$(\lambda I - A) e = 0$ or $(A - \lambda I) \cdot e = 0$

$n+1$ unknowns.
n eqns.
N.L.S.

$\lambda \to$ Given
$\downarrow$
$n\times n$ L.S.

$e = 0$, More solns (inf. solns)

$\downarrow$ $\quad |A - \lambda I| = 0 \Rightarrow \lambda = \ldots$

$x = 0$ $\quad$ C.E.

$n = 2$. $\quad \lambda_1 = \ldots \quad \lambda_2 = \ldots \to e, b$

$x_1 = . \quad x_2 = \quad \to$ L.I.

$x = c_1 \cdot x_1 + c_2 \cdot x_2$

$\downarrow t = 0$

$x_0 = [\;\;] \leftarrow c_1 x_1(0) + c_2 x_2(0)$

$|A-\lambda I|=0$ 2nd order P.E.

- $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$

$\downarrow e_1 \quad \downarrow e_2 \to$ L. I.

$x_1 = e_1 e^{\lambda_1 t}$ $\quad x_2 = e_2 e^{\lambda_2 t}$

- $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$

$\downarrow e, \; b \; : \; e = (A - \lambda I) \cdot b$

$x_1 = e\, e^{\lambda t}$, $x_2 = (et + b) e^{\lambda t}$

- $\lambda = a \pm bi$, $a, b \in \mathbb{R}$

$\downarrow e, \bar{e}$ $\quad x_1 = e\, e^{\lambda t} \quad x_2 = \bar{e}\, e^{\bar{\lambda} t}$

or $\quad x_1 = \mathrm{Re}\,(e\, e^{\lambda t})$

$\quad x_2 = \mathrm{Im}\,(e\, e^{\lambda t})$

or $\quad \cos\ldots \quad \sin\ldots$

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$A = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}, x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

C.E. $|A - \lambda I| = 0$

$\begin{vmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{vmatrix} = 0 \quad (-\lambda)(-7-\lambda) - (-6)\cdot 1 = 0$

$\lambda^2 + 7\cdot\lambda + 6 = 0$

$\lambda_1 = -1 \, , \, \lambda_2 = -6$

$(A - \lambda I)\cdot e = 0 \qquad \left(e = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right)$

$\begin{bmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

• $\lambda_1 = -1$: $\begin{bmatrix} 1 & 1 \\ -6 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$v_1 + v_2 = 0$

$v_1 = 1 \Rightarrow v_2 = -1$

$\boxed{v_1 = 100 \Rightarrow v_2 = -100}$

$e_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

• $\lambda = -6$ $\begin{bmatrix} 6 & 1 \\ -6 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$6 \cdot v_1 + v_2 = 0$$

$$\cancel{v_2 = 1 \Rightarrow v_1 = -1/6}$$

$$v_1 = 1 \Rightarrow v_2 = -6$$

$$e_2 = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

Gen. $\cancel{x = c_1 e_1 e^{t}}$

$$x = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot e^{-t} + c_2 \cdot \begin{bmatrix} 1 \\ -6 \end{bmatrix} e^{-6t}$$

sp. soln.

$$x_0 = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot 1 + c_2 \begin{bmatrix} 1 \\ -6 \end{bmatrix} \cdot 1.$$

$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ -c_1 - 6c_2 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= \dots \\ c_2 &= \dots \end{aligned}$$

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$$A=\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \qquad x_0=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

C.D. $|A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix}=0$$

$$(1-\lambda)(3-\lambda)+1=0$$

$$\lambda^2-4\cdot\lambda+4=0$$

$$(\lambda-2)^2=0 \Rightarrow \lambda=2$$

• $\lambda=2$ $\quad (A-\lambda I)\cdot e=0$

$$\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix}\cdot\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_1+v_2=0$, $\quad v_1=1$

$v_2=-1$

$$e=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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$(A-\lambda I)\cdot b = e$

$\lambda = 2 \quad \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\Rightarrow b = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$b_1 + b_2 = -1 \qquad b_1 = 0 \qquad b_2 = -1.$

$X = c_1 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$

$\underbrace{}_{0}$ at $t=0$

$X_0 = c_1 \cdot \left(0 + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) \cdot 1 + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot 1.$

$= \ldots \qquad c_1 = -1 \qquad c_2 = 1.$

$X = -1 \cdot \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) e^{2t} + 1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$

$-1 \left(\begin{bmatrix} t \\ -t \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) e^{2t} + \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$

$- \begin{bmatrix} t+0 \\ -t-1 \end{bmatrix} e^{2t} + \begin{bmatrix} e^{2t} \\ -e^{+2t} \end{bmatrix}$

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$$\begin{bmatrix} -t \cdot e^{2t} \\ (t+1) e^{2t} \end{bmatrix} + \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} =$$

$$= \begin{bmatrix} -t \cdot e^{2t} + e^{2t} \\ t\, e^{2t} \end{bmatrix} = \cancel{\begin{bmatrix} 1-t \\ t \end{bmatrix} e^{2t}}$$

$\downarrow$ for $b_1 = 0$, $b_2 = -1$.

$$b = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$X = c_1 \cdot \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

$$\Rightarrow \dots \Rightarrow \quad c_1 = -1 \quad c_2 = 0$$

$$X = -1 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{2t}$$

$$= \cancel{\begin{bmatrix} -t - e^{2t} \\ t \cdot e^{2t} \end{bmatrix}}$$

$$= -\begin{bmatrix} t - 1 \\ -1 \end{bmatrix} \cdot e^{2t} = \begin{bmatrix} -t + 1 \\ 1 \end{bmatrix} e^{2t}$$

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$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \qquad X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

C.E. $\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)^2 + 4 = 0 \Rightarrow \dots \quad \lambda_1 = 1+2i$$
$$\lambda_2 = 1-2i$$

$(A - \lambda I) \cdot e$

$$\begin{bmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

• $\lambda = 1+2i$

$$\begin{bmatrix} 1-(1+2i) & 2 \\ -2 & 1-(1+2i) \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-1-2i & 2 \\ -2 & 1-1-2i \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2i\cdot v_1 + 2v_2 = 0$$

$$-2\cdot v_1 - 2i v_2 = 0$$

$$-v_1 i + v_2 = 0 \Rightarrow \quad v_2 = 1 \Rightarrow v_1 = -i$$

$$\Rightarrow \boxed{e_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}} \quad e_2 = \bar{e_1} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Gen sol:

$$X = C_1 e^{(1+2i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix} + C_2 e^{(1-2i)t} \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$X_0 = C_1 \cdot \begin{bmatrix} -i \\ 1 \end{bmatrix} + C_2 \cdot \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left.\begin{matrix} -C_1 \cdot i + C_2 \cdot i = 1 \\ C_1 + C_2 = 0 \end{matrix}\right\} \Rightarrow \dots \quad C_1 = -\frac{1}{2i} \quad C_2 = \frac{1}{2i}$$

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Cross check

$$X_1 = \mathrm{Re}(e\, e^{\lambda t})$$

$$X_2 = \mathrm{Im}(e\, e^{\lambda t})$$

cos sin

$\dot{X} = AX \qquad n=2$

$X = c_1 \cdot X_1 + c_2 X_2.$

$2\times1 \qquad 1\times1$

$x_1 = e_1 e^{\lambda_1 t}$

$x_2 = e_2 e^{\lambda_2 t}$

$x_1 = e\, e^{\lambda t}$

$x_2 = (et+b)\, e^{\lambda t}$

$x_1 = e\, e^{\lambda t}$

$x_2 = \bar{e}\, e^{\bar{\lambda} t}$

or

$x_1 = Re(e\, e^{\lambda t})$

$x_2 = Im(e\, e^{\lambda t})$

$$X(t) = \underbrace{[\underbrace{x_1}_{2\times1} \quad \underbrace{x_2}_{2\times1}]}_{2\times2} \cdot \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{2\times1}$$

2×1

$$x(t) = \mathbb{X}(t)\cdot C, \qquad C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\mathbb{X} = [x_1 \quad x_2] \rightarrow \text{Fundamental soln. Matrix}$$

$t=0$

$$x(0) = \mathbb{X}(0)\cdot C \Rightarrow \boxed{C = \mathbb{X}^{-1}(0)\cdot X_0}$$

$$x(t) = \underbrace{\mathbb{X}(t)\cdot\mathbb{X}^{-1}(0)}_{\text{state transition Matrix}}\cdot X_0$$

- $\lambda_1 \neq \lambda_2$ $\quad x_1 = e_1 e^{\lambda_1 t}$ $\quad x_2 = e_2 e^{\lambda_2 t}$

$$X = [e_1 e^{\lambda_1 t} \quad e_2 e^{\lambda_2 t}]$$

$$X(0) = [e_1 \quad e_2]$$

- $\lambda_1 = \lambda_2 = \lambda$ $\quad x_1 = e e^{\lambda t}$

$\quad x_2 = (et + b) e^{\lambda t}$

$$X = [e e^{\lambda t} \quad (et + b) e^{\lambda t}]$$

$$X(0) = [e \quad b]$$

- $\lambda = a \pm bi$ $\quad x_1 = Re(e e^{\lambda t})$, $x_2 = In(e e^{\lambda t})$

$$X(t) = [Re(e e^{\lambda t}) \quad In(e e^{\lambda t})]$$

$$X(0) = [Re(e) \quad In(e)]$$

$\ddot{x} + 2\dot{x} + x = 0$, $x_0 = 1$

$\dot{x}_0 = 0$

1) Find the sp. soln.

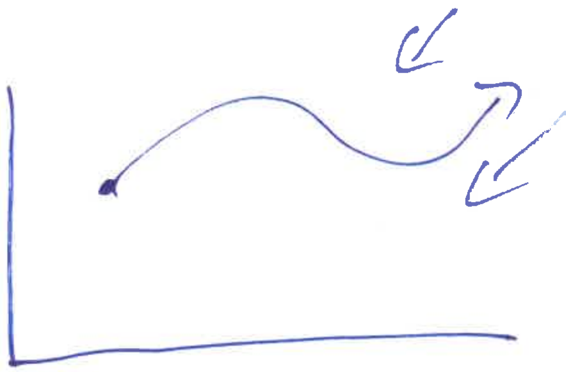
Homework.

2) ODE $\rightarrow$ S.S.
3) solve S.S. model, using eigs
4) Find the S.T.M.
5) Solve S.S. using

I

$$x(t) = X(t) \cdot X^{-1}(0) \cdot X_0$$

$$x(t) = \Phi(t, t_0) \cdot X_0$$



at $t=10$ $\quad STM = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$