Revision

$\dot{X} = A \cdot X$, $x = ?$

$X = e\, e^{\lambda t}$

$e \to$ eigenvector

$\lambda \to$ eigenvalue

$e = \ldots$

$(A - \lambda I) \cdot e = 0$

$\lambda \to$ Given $\to$ $|A - \lambda I| = 0 \Rightarrow \lambda = \ldots$

$n = 2$ $\quad X_1, X_2$ L.I.

$$\boxed{X = C_1 X_1 + C_2 \cdot X_2}$$

1. $\Delta > 0$ $\quad \lambda_1 \neq \lambda_2,\ \lambda_1, \lambda_2 \in \mathbb{R}$ $\Rightarrow$ $X_1 = e_1 e^{\lambda_1 t}$, $X_2 = e_2 \cdot e^{\lambda_2 t}$

   $\lambda_1 \downarrow e_1$, $\lambda_2 \downarrow e_2$

2. $\Delta = 0$ $\quad \lambda_1 = \lambda_2 \Rightarrow \in \mathbb{R}$ $\Rightarrow$ $X_1 = e_0 e^{\lambda_0 t}$, $X_2 = (et + b) e^{\lambda t}$

   $\downarrow e \to b$

3. $\Delta < 0$ $\quad \lambda_1 = a + bi,\ \lambda_2 = a - bi = \bar{\lambda}_1$ $\Rightarrow$ $X_1 = e_1 e^{\lambda_1 t}$, $X_2 = e_2 e^{\lambda_2 t}$

   $\downarrow e_1$ $\qquad \downarrow e_2 = \bar{e}_1$

   $\lambda \downarrow e$

   or

   $X_1 = Re(e e^{\lambda t})$

   $X_2 = Im(e e^{\lambda t})$

   or

   cos sin...

$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ $\lambda_0 = 1 + 2i$ $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 86

$e = [1 \quad i]^T$

$$x = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(1+2i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(1-2i)t}$$

$\cancel{c_1 = -\frac{1}{2i}}$ $\cancel{c_2 = \frac{1}{2i}}$

$c_1 = 1/2$ $c_2 = 1/2$

$$x = \frac{1}{2} \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(1+2i)t} + \frac{1}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(1-2i)t}$$

$$= \frac{1}{2} e^t \left(\begin{bmatrix} 1 \\ i \end{bmatrix} e^{2it} + \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-2it} \right)$$

$e^{a+b} = e^a e^b$

$e^{j\theta} = \cos\theta + j\sin\theta$

$e^{j\theta} + e^{-j\theta} = 2\cos\theta$

$e^{j\theta} - e^{-j\theta} = j2\sin\theta$

$$X = \frac{1}{2} e^{t} \cdot \begin{bmatrix} e^{2it} + e^{-2it} \\ i(e^{2it} - e^{-2it}) \end{bmatrix}$$

$$X = e^{t} \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix}$$

$$X_1 = \mathrm{Re}(e\, e^{\lambda t})$$
$$X_2 = \mathrm{Im}(e\, e^{\lambda t})$$

$$e = \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \lambda = 1 \cancel{+} 2i$$

$$e \cdot e^{\lambda t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(1+2i)t}$$
$$= e^{t} \begin{bmatrix} 1 \\ i \end{bmatrix} e^{2it}$$
$$= e^{t} \begin{bmatrix} 1 \\ i \end{bmatrix} (\cos 2t + i \sin 2t)$$

$$X_1 = e^{t} \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix} \qquad X_2 = e^{t} \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}$$

$$x = C_1 \cdot X_1 + C_2 X_2 \overset{t=0}{\Longrightarrow} X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} C_1 = 1 \\ C_2 = 0 \end{matrix}$$

$$X = e^{t} \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix}$$

$x = c_1 x_1 + c_2 x_2$ $\rightarrow$ $\lambda = \dots$ $e = \dots$ simple ~~DIV~~ special cases

$\downarrow$

$x(t) = X(t) \cdot C$ $\rightarrow$ F.S.M.

$X(t) \rightarrow [x_1 \quad x_2]$, $C \rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$X = [e_1 e^{\lambda_1 t} \quad e_2 e^{\lambda_2 t}]$$

$$X = [e e^{\lambda t} \quad (et + b) e^{\lambda t}]$$

$$X = [e e^{\lambda t} \quad \bar{e} e^{\bar{\lambda} t}] \text{ or } [\mathrm{Re}(e e^{\lambda t}) \quad \mathrm{Im}(e e^{\lambda t})]$$

$$x(0) = X(0) \cdot C \Rightarrow C = X^{-1}(0) \cdot x_0$$

$x(0) \rightarrow x_0$

$$x(t) = X(t) \cdot X^{-1}(0) \, x_0$$

$$x(t) = \Phi(t, 0) \cdot x_0$$

$\Phi \rightarrow$ S.T.M.; $0 \rightarrow t_0$

if Φ given, you know $x(t)$ $\forall t$

89

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \qquad \lambda = 1+2i \qquad e = [1 \quad i]^T \qquad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$X = \left[\begin{bmatrix} 1 \\ i \end{bmatrix} e^{(1+2i)t} \qquad \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(1-2i)t} \right]$$

$$X(0) = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \qquad \boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

$$|X(0)| = -2i$$

$$X^{-1}(0) = \frac{1}{-2i} \begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$$

$$X^{-1}(0) \cdot x_0 = -\frac{1}{2i} \begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= -\frac{1}{2i} \begin{bmatrix} -i \\ -i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} e^{(1+2i)t} & e^{(1-2i)t} \\ i e^{(1+2i)t} & -i e^{(1-2i)t} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} e^{(1+2i)t} + e^{(1-2i)t} \\ i\left(e^{(1+2i)t} - e^{(1-2i)t}\right) \end{bmatrix} \frac{1}{2}$$

90

$$= e^{t} \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}$$

$\therefore$ STM $\rightarrow$ LTV

$\dot{x} = a \cdot x$, $a, x \in \mathbb{R}$ $\rightarrow$ $x = e^{at} \cdot x_0$

$\dot{x} = A \cdot x$, $A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^{n \times 1}$

$$x(t) = \underbrace{e^{At}}_{\text{STM}} \cdot x_0$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$e^{A} \neq \begin{bmatrix} e^{a} & e^{b} \\ e^{c} & e^{d} \end{bmatrix}$$

expm(A).

exp(A)

Taylor series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!} \cdot (x-a)^3 \ldots\ldots$$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \ldots$$

$\dot{x} = A x$
- $\to e, \lambda$ simple
- $\to$ STM L.T.V.
- $\to e^{At}$ No S.C., H.O.S.

$e^{At} = ?$ simple case

$n = 2 \quad \lambda_1 \neq \lambda_2$

$A e_1 = \lambda_1 e_1 \qquad A \cdot e_2 = \lambda_2 e_2$

$$\underset{2\times 2}{A} \cdot \underbrace{[\underset{2\times 1}{e_1} \quad \underset{2\times 1}{e_2}]}_{2\times 2} = \underset{2\times 2}{[e_1 \quad e_2]} \cdot \underset{2\times 2}{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}$$

$A \cdot T = T \cdot \Lambda$ $\quad T \to$ eigen matrix

$A = T \cdot \Lambda \cdot T^{-1}$

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots$$

$$= I + (T \cdot \Lambda T^{-1}) \cdot t + \frac{(T \cdot \Lambda T^{-1})^2 \cdot t^2}{2} + \cdots$$

$$T \cdot \Lambda \cdot T^{-1}$$

$$(T \cdot \Lambda T^{-1})^2 = (T \cdot \Lambda \cancel{T^{-1}}) \cdot (\cancel{T} \cdot \Lambda \cdot T^{-1}) = T \Lambda^2 \cdot T^{-1}$$

$$(T \Lambda T^{-1})^3 = T \cdot \Lambda^3 \cdot T^{-1}$$

$$\vdots \qquad\qquad \vdots$$

$$I = T \cdot T^{-1}$$

$$e^{At} = T \cdot \boxed{T^{-1}} + T \cdot \Lambda \cdot \boxed{T^{-1}} \cdot t + \frac{T \cdot \Lambda^2 \cdot T^{-1}\, t^2}{2} + \dots$$

$$= T \cdot (\underbrace{I + \Lambda t + \frac{\Lambda^2 \cdot t^2}{2} + \dots}_{e^{\Lambda t}}) \cdot T^{-1}$$

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$$

$$I + \Lambda t + \frac{\Lambda^2 \cdot t^2}{2} + \dots$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{bmatrix} + \frac{\begin{bmatrix} \lambda_1^2 \cdot t^2 & 0 \\ 0 & \lambda_2^2 t^2 \end{bmatrix}}{2} + \dots$$

1st element: $1 + \lambda_1 t + \frac{\lambda_1^2 t^2}{2!} + \frac{\lambda_1^3 t^3}{3!} + \dots = e^{\lambda_1 t}$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$A = T \cdot \Lambda \cdot T^{-1}$$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \dots$$

$$\boxed{e^{At} = T\, e^{\Lambda t}\, T^{-1}}$$

$$A = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \qquad x(t) = \dots \begin{cases} e, x \\ \Phi \\ e^{At} \end{cases}$$

$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_1 = -1 \qquad \lambda_2 = -6$$

$$e_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad e_2 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} \quad \Big\} \Rightarrow \quad \begin{matrix} c_1 = 0.4 \\ c_2 = -0.4. \end{matrix}$$

$$x = 0.4 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} - 0.4 \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} e^{-6t}$$

$$= \begin{bmatrix} 0.8\, e^{-t} + 0.2\, e^{-6t} \\ 0.4 e^{-t} - 0.4\, e^{-6t} \end{bmatrix}$$

94

$$\mathbf{X} = \left[\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} \quad \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} e^{-6t} \right]$$

$$\mathbf{X}(0) = \begin{bmatrix} 2 & -0.5 \\ 1 & 1 \end{bmatrix}$$

$$|X(0)| = 2 + 0.5 = 2.5$$

$$\mathbf{X}^{-1}(0) = \begin{bmatrix} 1 & 0.5 \\ -1 & 2 \end{bmatrix} \cdot \frac{1}{2.5}$$

$$\mathbf{X}^{-1}(0) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2.5} \begin{bmatrix} 1 & 0.5 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2.5} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 2e^{-t} & -0.5e^{-6t} \\ e^{-t} & e^{-6t} \end{bmatrix} \frac{1}{2.5} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{2.5} \begin{bmatrix} 2e^{-t} + 0.5e^{-6t} \\ e^{-t} - e^{-6t} \end{bmatrix}$$

$$\frac{2}{2.5} = 0.8$$

$$\frac{0.5}{2.5} = 0.2$$

$$\frac{1}{2.5} = 0.4$$

$$= \begin{bmatrix} 0.8 \cdot e^{-t} + 0.2e^{-6t} \\ 0.4e^{-t} - 0.4e^{-6t} \end{bmatrix}$$

$$\cancel{e^{At} = T \cdot e^{A}}$$

$$e^{At} = T \cdot e^{\Lambda t} \cdot T^{-1}$$

$$= \begin{bmatrix} 2 & -0.5 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{bmatrix} \cdot \begin{bmatrix} 2 & -0.5 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & -0.5 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{bmatrix} \frac{1}{2.5} \begin{bmatrix} 1 & 0.5 \\ -1 & 2 \end{bmatrix}$$

$$x = e^{At} \cdot x_0$$

$$\begin{bmatrix} 2 & -0.5 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{bmatrix} \frac{1}{2.5} \begin{bmatrix} 1 & 0.5 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -0.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{bmatrix} \frac{1}{2.5} \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

$$= \dots = \begin{bmatrix} 0.8 \cdot e^{-t} + 0.2 e^{-6t} \\ 0.4 e^{-t} - 0.4 e^{-6t} \end{bmatrix}$$