Summary of Lec 1.

11

$\dot{x} = f(x, t)$ 1st order ODE.

An soln $x(t) = \dots$ $0 = 0$

$\dot{x} = -3 \cdot x$ $\boxed{x_1 = e^{-3t}}$ is a soln

$(e^{-3t})' = -3 \cdot (e^{-3t}) \Leftrightarrow$

$-3 e^{-3t} = -3 e^{-3t}$ ✓

$\boxed{x_2 = \underline{10 e^{-3t}}}$

$\dot{x}_2 = -3 \cdot x_2$

$-30 e^{-3t} = -3 \cdot 10 e^{-3t}$ ✓

$x(t) = c \cdot e^{-3t} \rightarrow$ Gen soln.

I.V.P. $=$ ODE $+ x(t_0) = x_0$

$x(0) = x_0$

$\dot{x} = -3 \cdot x$, $\underline{x_0 = 1}$. $x_1 = e^{-3t} \rightarrow$ soln $\rightarrow$ sp. soln.

$x_1(0) = e^0 = 1$

~~$x_2 = 2$~~

$x_2 = 10 e^{-3t} \rightarrow$ soln

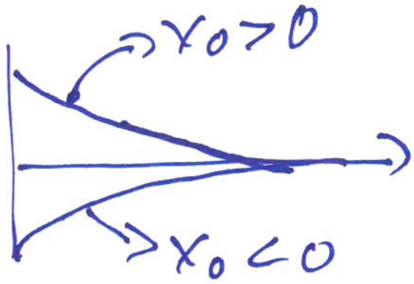
$x_2(0) = 10 \cdot e^0 = \underline{10}$

$\dot{x} + k \cdot x = u.$ $\quad u \to \boxed{\text{sys}} \to x$

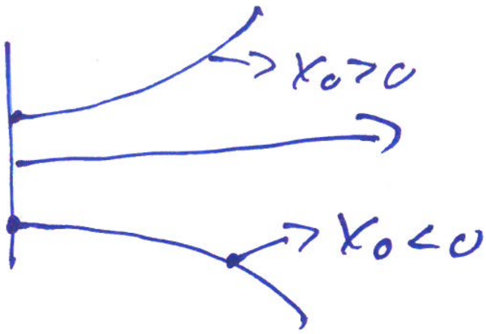
$\hookrightarrow x(t) = e^{-kt} \cdot x_0 + e^{-kt} \int_0^t e^{kt_1} \cdot u(t_1)\, dt_1$

- $u = 0 \quad x = e^{-kt} \cdot x_0$

  - $k > 0 \quad x \to 0$



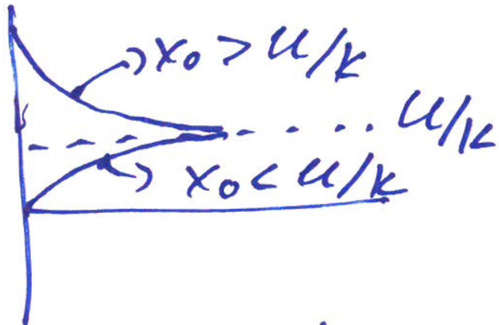
  - $k < 0 \quad x \to \pm\infty$



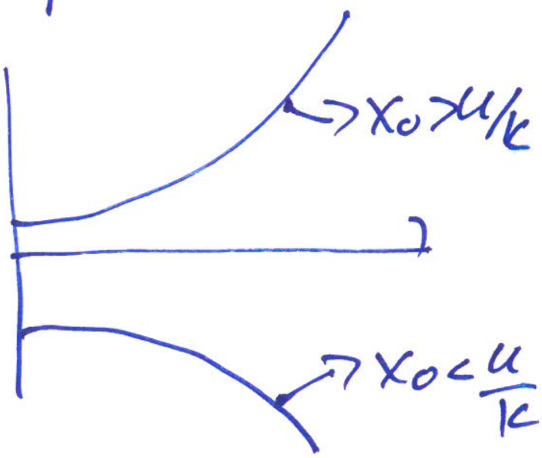
- $u \neq 0$ (but const).

  $x(t) = \frac{u}{k}(1 - e^{-kt})$

  - $k > 0 \quad x \to u/k$



  - $k < 0 \quad x \to \pm\infty$



So in Linear Sys:

1) I.C. → No stability
2) u → No stability
3) k → <u>YES</u> stability

$$x(t) = \frac{u}{k}\left(1 - e^{-kt}\right) + e^{-kt} \cdot x_0$$

• If $x_0 = u/k$

$$\Rightarrow \; = \frac{u}{k} - \cancel{\frac{u}{k} e^{-kt}} + \cancel{\frac{u}{k} e^{-kt}}$$

$$\Rightarrow \quad x(t) = u/k$$

$u/k \rightarrow$ EQ. point of $\dot{x} + kx = u$.

$$\dot{x} + 0 \cdot x = 5 \Rightarrow$$

$$\frac{dy}{dt} = 5 \Rightarrow \int \frac{dy}{dt}\, dt = \int 5 \, dt$$

$$\Rightarrow \quad x = 5t + c.$$

$$\ddot{x} = f(\dot{x}, x, t).$$

$$\ddot{x} + A\dot{x} + Bx = \cancel{u}. \to 0$$

$$\ddot{x} + A\dot{x} + Bx = 0$$

$$\ddot{x} - 2\dot{x} - 3x = 0$$

$x_1 = e^{3t}$ is a soln.

$x_2 = 3e^{3t}$ — " — . .

$x_3 = -\pi e^{3t}$ . . . . .

$x_4 = e^{-t}$ . . . . .

$x_5 = 10 \cdot e^{-t}$ . . . . .

$\vdots$

$x_6 = \sqrt{\pi} \cdot e^{3t} - \frac{1}{\sqrt{2}} e^{-t}$

$\vdots$

$x_n = C_1 \cdot x_1 + C_2 \cdot x_4$ . . . .

$x_n = C_1 x_2 + C_2 \cdot x_6$

so if x_A, x_B are 2 soln
then $C_1 \cdot x_A + C_2 \cdot x_B$ is
ALWAYS a soln

If I have 2 soln, then can I write <u>ANY</u> other soln as a L.C. of these 2.

$\varphi = \sqrt{3}\, e^{-t} + 3e^{3t}$

$x_7 = 3.33 \cdot x_4 - \frac{1}{\sqrt{\pi}} x_1$

$x_6 = 10 \cdot x_1 + 20 x_2$

$x_5 = 10 e^{-t}$

$x_3 = \frac{1}{3} e^{3t}$

$x_2 = 3 \cdot e^{3t}$

$x_4 = e^{-t}$

$x_1 = e^{3t}$

$$\boxed{\ddot{x} - 2\dot{x} - 3x = 0}$$

I pick x_1, x_2.

Can I find C_1, C_2 : $C_1 x_1 + C_2 x_2 = \varphi$

Ass $x_1 = 3e^{3t}$ $x_2 = -10 e^{-t} + \sqrt{5}\, e^{3t}$

$\varphi = \sqrt{3}\, e^{-t} + 3e^{3t} = C_1 x_1 + C_2 x_2$

$C_1 \cdot 3e^{3t} + C_2 (-10 \cdot e^{-t} + \sqrt{5}\, e^{3t}) =$

$\sqrt{3}\, e^{-t} + 3e^{3t} \Rightarrow$

. . . . .

$C_1 = 3 \cdot C_1 + 2 C_2 \sqrt{5}$ $= 3$

$C_2 = \ldots = -\frac{\sqrt{3}}{10}$

$$c_1 \cdot e^{3t} + c_2 \cdot 3e^{3t} = \sqrt{3}\, e^{-t} + 3e^{3t}$$

$$\nexists\, c_1, c_2 : \text{!}$$

$$x'' + A x' + B x = 0 \qquad x_1, x_2$$

<u>ALL</u> other soln $\quad c_1 x_1 + c_2 x_2$

<u>iff</u> $\begin{vmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix} \neq 0$

$$\begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix} \neq c \cdot \begin{bmatrix} x_2 \\ \dot{x}_2 \end{bmatrix}$$

$$\begin{vmatrix} 3e^{3t} & -10e^{-t} + \sqrt{5}\, e^{3t} \\ 9e^{3t} & 10e^{-t} + 3\sqrt{5}\, e^{3t} \end{vmatrix} =$$

<u>crosscheck</u> $\quad = 120\, e^{2t} \neq 0$

$$\ddot{x} + A\dot{x} + Bx = 0$$

$$\boxed{\begin{aligned}\dot{x} + kx &= 0\\ x &= e^{-kt}\cdot c\end{aligned}}$$

HOPE

TRY

$$x = e^{rt}$$

$$r^2 \cancel{e^{rt}} + rA\cancel{e^{rt}} + B\cancel{e^{rt}} = 0$$

$$r^2 + A\cdot r + B = 0 \longrightarrow \text{C.E.}, \; r = \text{ch. values of ODE}$$

$$r_{1,2} = \frac{-A \pm \sqrt{A^2 - 4B}}{2}.$$

- $A^2 - 4B > 0 \Rightarrow r_1 = \dots \Rightarrow x_1 = e^{r_1 t}$

  $r_2 = \dots \Rightarrow x_2 = e^{r_2 t}$

$$W(x_1, x_2) = \begin{vmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 \cdot e^{r_2 t} \end{vmatrix}$$

$$= e^{r_1 t}\cdot r_2 \cdot e^{r_2 t} - r_1 e^{r_1 t}\cdot e^{r_2 t}$$

$$= r_2 e^{(r_1+r_2)t} - r_1 e^{(r_1+r_2)t}$$

$|W| = 0$ ~~only~~ $r_1 = r_2$

All $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$

3. r_1 and $r_2 < 0 \quad x \to 0$

4. r_1 or $r_2 > 0 \quad x \to \pm\infty$

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ex $\quad x'' + 11\dot{x} + 30x = 0$

- Find C.E.

$x = e^{rt} \Rightarrow \dots \Rightarrow r^2 + 11r + 30 = 0$

- Determine stability

$r^2 + 11r + 30 = 0 \qquad r_{1,2} = \dfrac{-11 \pm \sqrt{11^2 - 4 \cdot 1 \cdot 30}}{2}$

$\Rightarrow r_1 = -5 \quad , r_2 = -6$

$e^{-5t} \to 0$ AND $e^{-6t} \to 0$

$x = C_1 e^{-5t} + C_2 e^{-6t} \to 0$

- Find spec. soln. : $x_0 = 1, \; x_0' = 0$

$\Rightarrow x(0) = \boxed{C_1 + C_2 = 1}$

$\dot{x} = -5C_1 e^{-5t} - 6 \cdot C_2 e^{-6t}$

$\dot{x}(0) = \boxed{-5C_1 - 6 \cdot C_2 = 0}$

$\dots \Rightarrow C_1 = 6, \quad C_2 = -5$

$\dot{x} + kx = 0$ $\quad$ $x = e^{rt}$

$r + k = 0 \Rightarrow r = -k.$

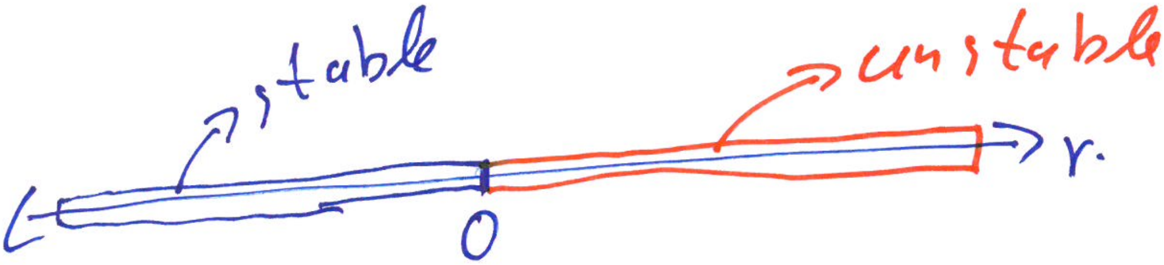
$k > 0 \rightarrow$ stable

$k < 0 \rightarrow$ unstable

$\rightarrow$ $r < 0$ stable

$r > 0$ unstable

Real.



$x = e^{-10t}$ $\qquad$ $\times$ $x = e^{-1000t}$

$\Downarrow$

Faster

$x = c_1 e^{t} + c_2 e^{-t}$

If I choose $x_0, \dot{x}_0$: $c_1 = 0$, $c_2 \neq 0$

$x = \underbrace{c_1 e^{-\cancel{10}t}}_{\text{Important term}} + c_2 e^{-1000t}$

- $A^2 - 4B < 0$

$$r_{1,2} = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$$

$$= \frac{-A \pm j\sqrt{4B - A^2}}{2}$$

$$r = a \pm bj$$

$$a = -A/2$$

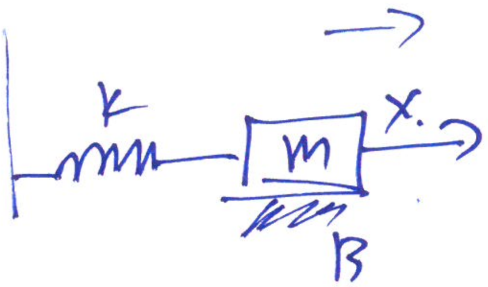
$$b = \frac{\sqrt{4B - A^2}}{2}$$

$\Rightarrow x_1 = e^{(a+bi)t} \quad x_2 = e^{(a-bi)t}$

2)

$$|W| = \begin{vmatrix} e^{(a+bi)t} & e^{(a-bi)t} \\ (a+bi)\cdot e^{(a+bi)t} & (a-bi)e^{(a-bi)t} \end{vmatrix}$$

$$= \dots = -2e^{2at}\, bj \neq 0$$



$$m\ddot{x} = F - kx - B\cdot\dot{x}$$

A $\underline{\text{REAL}}$ SYS $\to x_1 = e^{rt}$, $x_2 = e^{\bar{r}t}$

$$x = c_1\cdot e^{rt} + c_2 e^{\bar{r}t}$$

$$c_1, c_2 \in \mathbb{C}$$

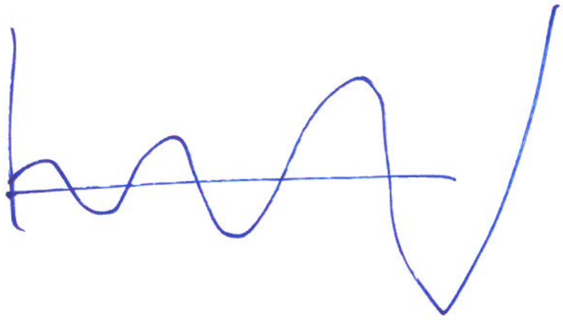
$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$x_1 = e^{rt} = e^{(a+bi)t}$$

$$= e^{at} \cdot e^{bit}$$

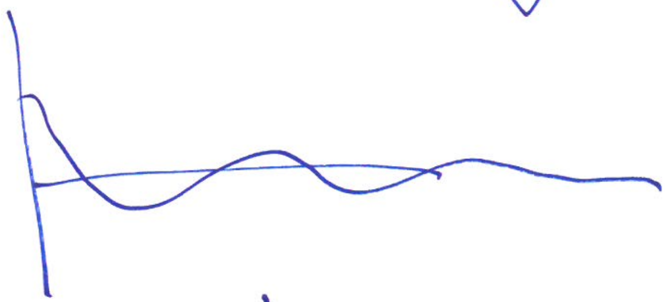
$$= e^{at} \cdot \left(\cos(bt) + \sin(bt) \right)$$

- $a > 0$

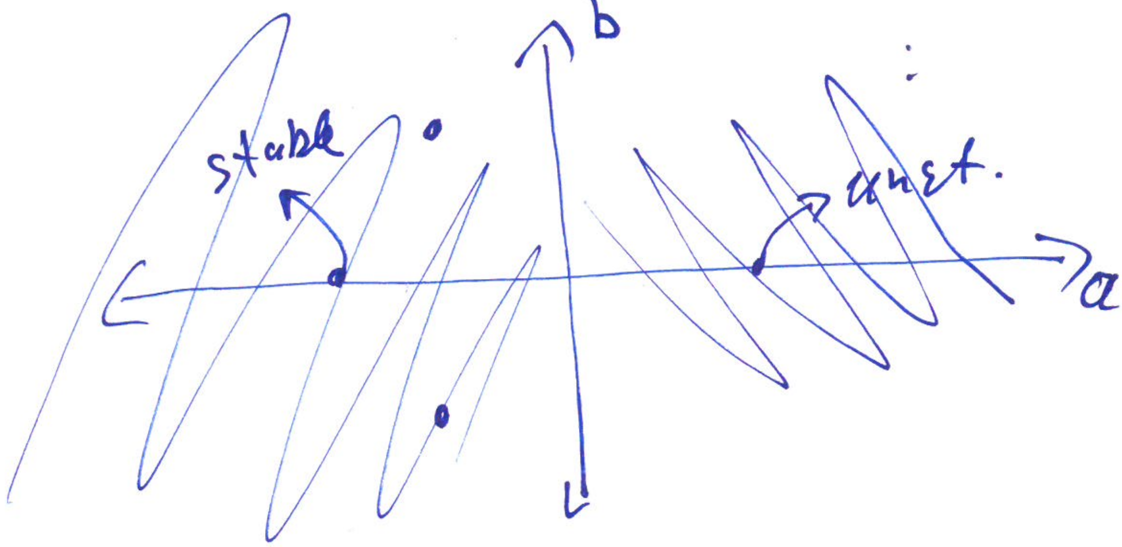


unstable

- $a < 0$



stable



$x_1 = e^{rt} = e^{at}\cdot(\cos bt + j\sin bt)$

$x_2 = e^{\bar{r}t} = e^{at}(\cos bt - j\sin bt)$

$y_1 = x_1 + x_2 = e^{at}(2\cdot\cos bt) = \underline{e^{at}\cdot 2\cdot\cos bt}$

$y_2 = x_1 - x_2 = \ldots\ldots = 2e^{at}\sin bt$

$x = c_1\cdot y_1 + c_2\cdot y_2$

$= c_1 2 \underline{e}^{at}\cos bt + c_2 2\cdot \underline{e}^{at}\sin bt$

$= e^{at}(c_1\cdot 2\cdot\cos bt + c_2\cdot 2\cdot\sin bt)$

$= e^{at}(c_A\cos bt + c_B\cdot\sin bt)$

- $A^2 = 4B$ $\qquad r_{1,2} = \dfrac{-A \pm 0}{2}$

$x_1 = e^{r_1 t} \qquad x_2 = e^{r_2 t}$

ARE NOT LI.

$x_3 = te^{rt}$

$x = c_1 x_1 + c_2 x_3$

$= c_1 e^{rt} + c_2 t e^{rt}$