# Revision



$\dot{x} = A \cdot x$ , $x \in \mathbb{R}^{n \times 1}$ , $A \in \mathbb{R}^{n \times n}$

$x = e^{\lambda t} \cdot e$ , $\lambda \in \mathbb{R} = ?$ $e \in \mathbb{R}^{n \times 1} = ?$

$\lambda \cancel{e^{\lambda t}} \cdot e = A \cancel{e^{\lambda t}} \cdot e$

$\lambda e - Ae = 0$

$\lambda \cdot I \cdot e - Ae = 0$

$Ae - \lambda e = 0$

$(A - \lambda I) e = 0$

$|A - \lambda I| = \dots$

$(\lambda I - A) e = 0$ , if $\lambda$ = given

$n \times n$ homogeneous sys

Impose $|\lambda I - A| = 0 \Rightarrow \dots \lambda =$

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$n = 2$ • $\lambda_1 = \dots$ $\lambda_2 = \dots$

$\lambda_1 \neq \lambda_2$

$\to e_1$ $e_2$ $\Rightarrow$ 2 L. I. eigen vectors

$x = c_1 e^{\lambda_1 t} e_1 + c_2 e^{\lambda_2 t} \cdot e_2$

vectors

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \qquad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow -\lambda \cdot (-7-\lambda) + 6 = 0$$

$$\Rightarrow \lambda^2 + 7\lambda + 6 = 0$$

$$\Rightarrow \lambda_1 = -1$$

$$\lambda_2 = -6$$

$$(A - \lambda I) e = 0$$

$$\begin{bmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0$$

• $\lambda = -1$ $\quad \begin{bmatrix} +1 & 1 \\ -6 & -6 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0 \Rightarrow e_1 + e_2 = 0$

$$e_1 = 1 \Rightarrow e_2 = -1$$

$$\Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\cdot \lambda = -6$ $\quad \begin{bmatrix} 6 & 1 \\ -6 & -1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0$

$6 \cdot e_1 + e_2 = 0 \Rightarrow \begin{matrix} e_1 = 1 \\ e_2 = -6 \end{matrix}$

$\rightarrow \begin{bmatrix} 1 \\ -6 \end{bmatrix}$

$$x(t) = c_1 e^{-t} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$$x(0) = c_1 \cdot 1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \cdot 1 \cdot \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ -c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ -6 \cdot c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \ldots \ldots \begin{matrix} c_1 = \ldots \\ c_2 = \ldots \end{matrix}$$

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$\cdot \lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$

$\Rightarrow x(t) = c_1 \cdot (et + b) e^{\lambda t} + c_2 e^{\lambda t} \cdot e$

$e \rightarrow$ eigenvector.

$e = (A - \lambda I) \cdot b.$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \qquad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow (1-\lambda)\cdot(3-\lambda) + 1 = 0$$

$$\Rightarrow \lambda^2 - 4\cdot\lambda + 4 = 0$$

$$\Rightarrow (\lambda - 2)^2 = 0$$

$$\lambda = 2$$

$$(A - \lambda I)\cdot e = 0 \Rightarrow$$

$$\begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = 0$$

$$\bullet\ \lambda = 2 \Rightarrow \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0$$

$$e_1 + e_2 = 0 \Rightarrow e_1 = 1, \quad e_2 = -1.$$

$$\lambda = 2 \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(A-\lambda I)\cdot b = e$$

$$\cdot\ \lambda = 2 \Rightarrow \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$b_1 + b_2 = -1 \Rightarrow b_1 = 0,\quad b_2 = -1.$$

$$x(t) = c_1 (et + b) e^{\lambda t} + c_2 e \cdot e^{\lambda t}$$

$$= c_1 \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

$$x(0) = c_1 \cdot \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot 0 + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) \cdot 1 + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot 1.$$

$$c_1 \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$$

$$c_2 = 1$$
$$c_1 = -1$$

• $\lambda = a + bj$ , $a, b \in \mathbb{R}$. $j = i$

$$x = c_1 e \cdot e^{\lambda t} + c_2 \bar{e} e^{\bar{\lambda} t}$$

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow (1-\lambda)^2 + 4 = 0 \Rightarrow$$

$$\lambda_1 = 1 + 2i$$

$$\lambda_2 = 1 - 2i$$

• $\lambda = 1 + 2i \longrightarrow \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$$(A - \lambda I) \cdot e = 0$$

$$\begin{bmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0$$

$$\cancel{\begin{bmatrix} 1 - 1 \mp 2i \\ \end{bmatrix}}$$

$$\begin{bmatrix} 1-(1+2i) & 2 \\ -2 & 1-(1+2i) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0 \Rightarrow$$

$$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0 \Rightarrow$$

$$-\cancel{2}e_1 - \cancel{2}i\, e_2 = 0$$

$$-e_1 - i e_2 = 0 \qquad e_2 = 1 \Rightarrow e_1 = -i$$

$$x = c_1 e^{(1+2i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2 e^{(1-2i)t} \cdot \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$x_0 = c_1 \cdot \begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \dots \quad \boxed{c_1 = -\frac{1}{2i}, \quad c_2 = \frac{1}{2i}}$$

Cross check!!!!!!!!!!

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$$A = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \to \left.\begin{matrix} \lambda_1 = -1, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \lambda_2 = -6, \begin{bmatrix} 1 \\ -6 \end{bmatrix} \end{matrix}\right\} \Rightarrow$$

$$x_1 = e^{-t} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad x_2 = e^{-6t} \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$$X = \left[ e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad e^{-6t} \begin{bmatrix} 1 \\ -6 \end{bmatrix} \right]$$

$$X = \begin{bmatrix} e^{-t} & e^{-6t} \\ -e^{-t} & -6\,e^{-6t} \end{bmatrix}$$

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$$X = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \qquad x = c_1 \cdot x_1 + c_2 \cdot x_2$$

$$x(t) = X(t) \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

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$x'' + 2 \cdot x' + x = 0 \qquad x_0 = 1, \; x_0' = 0$

1) Find Spec. soln. ✓

2) S.S. model ✓

3) Solve S.S. model using eig. ✓

4) Find S.T.M.

5) Solve S.S. model using S.T.M.

1) $r^2 + 2 \cdot r + 1 = 0 \qquad \Delta = 4 - 4 \cdot 1 = 0$

$r_1 = r_2 = r = -1.$

$x = c_1 e^{-t} + c_2 t e^{-t}$

$\dot{x} = -c_1 e^{-t} + c_2 e^{-t} + c_2 t (-1) e^{-t}$

$$\left.\begin{aligned} x_0 &= c_1 \cdot 1 + c_2 \cdot 0 \cdot 1 = c_1 = 1. \\ \dot{x}_0 &= -c_1 \cdot 1 + c_2 \cdot 1 + 0 = 0 \end{aligned}\right\} \Rightarrow$$

$c_2 = 1.$

$x(t) = e^{-t} + t e^{-t}$

2) $\ddot{x} = -2\dot{x} - x$

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}\right\} \Rightarrow \begin{aligned} &\dot{x}_1 = \dot{x} = x_2 \Rightarrow \dot{x}_1 = x_2 \\ &\dot{x}_2 = \ddot{x} = -2\dot{x} - x = -2 \cdot x_2 - x_1 \end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow -\lambda(-2-\lambda) + 1 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda + 1)^2 = 0 \Rightarrow \lambda = -1$$

$$(A - \lambda I) \cdot e = 0$$

$$\begin{pmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{pmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0 \Rightarrow$$

$$e_1 + e_2 = 0 \Rightarrow e_1 = 1 \Rightarrow e_2 = -1$$

$$\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1&1\\-1&-1\end{bmatrix}\cdot\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

$$1=b_1+b_2 \Rightarrow b_2=0$$
$$b_1=1$$

$\boxed{\text{Homework: } b_1=0,\ b_2=1 \text{ Find } c_1, c_2}$

$$x(t)=c_1\,(et+b)e^{\lambda t}+c_2e^{\lambda t}e$$

$$=c_1\left(\begin{bmatrix}1\\-1\end{bmatrix}t+\begin{bmatrix}1\\0\end{bmatrix}\right)e^{-t}+c_2e^{-t}\cdot\begin{bmatrix}1\\-1\end{bmatrix}$$

$$x(0)=c_1\cdot\left(\begin{bmatrix}1\\-1\end{bmatrix}\cdot 0+\begin{bmatrix}1\\0\end{bmatrix}\right)\cdot 1+c_2\cdot 1\cdot\begin{bmatrix}1\\-1\end{bmatrix}$$

$$=c_1\begin{bmatrix}1\\0\end{bmatrix}+c_2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}c_1+c_2\\-c_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

$$\left.\begin{array}{l}c_1+c_2=1\\-c_2=0\end{array}\right\}\Rightarrow c_1=1$$

$$x=\left(\begin{bmatrix}1\\-1\end{bmatrix}t+\begin{bmatrix}1\\0\end{bmatrix}\right)e^{-t}$$

$$=\begin{bmatrix}(t+1)e^{-t}\\-te^{-t}\end{bmatrix}\Rightarrow x_1=(t+1)e^{-t}\ -$$

4) F.S.M

$$X = \left[ e\, e^{\lambda t} \quad (et + b) e^{\lambda t} \right]$$

$$X = \left[ \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \quad \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{-t} \right].$$

$$X_0 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \quad X_0^{-1} = \frac{1}{1} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

S.T.M. $\Phi(t, t_0)$

$$X \cdot X^{-1}(0) =$$

$$\begin{bmatrix} e^{-t} & (t+1) e^{-t} \\ -e^{-t} & -t e^{-t} \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$x(t) = X \cdot X^{-1}(0) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} e^{-t} & (t+1) e^{-t} \\ -e^{-t} & -t e^{-t} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (t+1) e^{-t} & -e^{-t} + (t+1) e^{-t} \\ -t e^{-t} & e^{-t} - t e^{t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (t+1)e^{-t} + 0 \\ -te^{-t} + 0 \end{bmatrix} \to (t+1)\cdot e^{-t} \quad -$$

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