Revision

ODE / I.V.P.

$\dot{x} + kx = u$ $\qquad u = 0 \qquad x = e^{rt}$

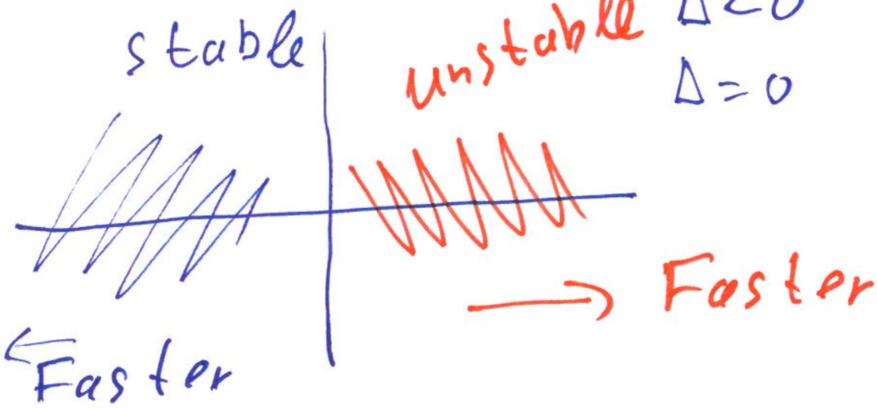
$\ddot{x} + A\dot{x} + Bx = u.$

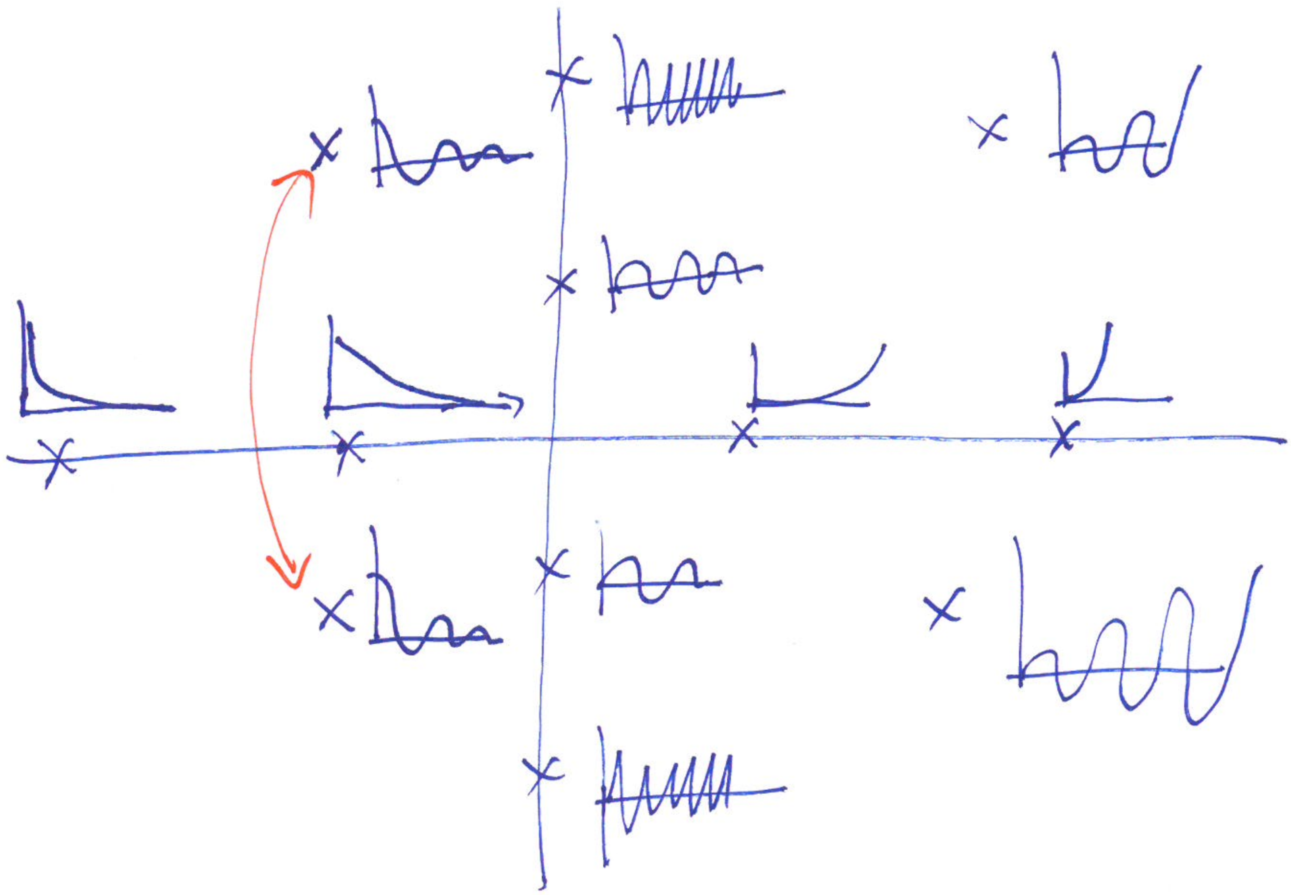
C.E. $\quad r + k = 0 \Rightarrow r = -k$

$r^2 + A \cdot r + B = 0 \rightarrow \Delta > 0$

$\Delta < 0$

$\Delta = 0$





(72)

$$x = e^{-kt} \cdot x_0 + e^{-kt} \int_0^t e^{kt_1} u(t_1)\, dt_1$$

$\underline{\underline{u = \text{const}}}$ $\quad = \frac{u}{k}\left(1 - e^{-kt}\right)$

$\ddot{x} + A\dot{x} + Bx = 0$

- $r_1 \neq r_2 \in \mathbb{R}$ $\quad x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, $c_1 = \ldots$, $c_2 = \ldots$
- $r_1 = r_2 = r \in \mathbb{R}$ $\quad x = c_1 e^{rt} + c_2 \cdot t \cdot e^{rt}$, $c_1 = \ldots$, $c_2 = \ldots$
- $r = a \pm bi$ $\quad x = c_1 e^{rt} + c_2 e^{\bar{r}t}$, $c_1 = \ldots$, $c_2 = \ldots$

$$x = e^{at}\left(c_1 \cos bt + c_2 \sin bt\right)$$

S.S. $\quad \dot{x} = Ax + Bu$ | ODE $\to$ S.S. $\leftrightarrow$ T.F. ctrb, obsv

Solve $\quad \dot{x} = Ax$, $x \in \mathbb{R}^{n \times 1}$, $A \in \mathbb{R}^{n \times n}$

- eigenvalues $\quad |A - \lambda I| = 0$
- eigenvectors $\quad (A - \lambda I) \cdot e = 0$

$\lambda_1 \neq \lambda_2$ $\quad x = c_1 e^{\lambda_1 t} \cdot e_1 + c_2 e^{\lambda_2 t} \cdot e_2$

$\lambda_1 = \lambda_2 = \lambda$ $\quad x = c_1 (et + b) e^{\lambda t} + c_2 e^{\lambda t} \cdot e$

$\lambda = a \pm bi$ $\quad x = c_1 e^{\lambda t} e + c_2 e^{\bar{\lambda} t} \bar{e}$

Solution Matrix

$x_1, x_2 \to X = [x_1 \quad x_2]$

$$x(t) = X \cdot X^{-1}(0) \cdot x_0$$

- $\lambda_1 \neq \lambda_2 \quad X = [e_1 e^{\lambda_1 t} \quad e_2 e^{\lambda_2 t}]$
- $\lambda_1 = \lambda_2 = \lambda \quad X = [e e^{\lambda t} \quad (ct+b) e^{\lambda t}]$
- $\lambda = a + bi \quad X = [e e^{\lambda t} \quad \bar{e} e^{\bar{\lambda} t}]$

or

$[\mathrm{Re}(\ ) \quad \mathrm{Im}(\ )]$

or

$[\cos \quad \sin]$

$\dot{x} = ax \qquad x = e^{at} \cdot x_0, \quad x \in \mathbb{R}, \quad a \in \mathbb{R}$

$\dot{X} = A \cdot X \qquad X(t) = e^{At} \cdot x_0 \quad ???$

$\dot{x} = A \cdot x \rightarrow e_1, e_2 \quad \lambda_1, \lambda_2$

simple

$\rightarrow$ Soln Matrix

T.V.M.

$\rightarrow e^{At} \rightarrow$ ~~no~~ S.S.

No S.S.

H.O.S.

$e^{At}, \quad \lambda_1 \neq \lambda_2$

$Ae_1 = e_1 \lambda_1 \qquad Ae_2 = e_2 \lambda_2$

$$\underset{2\times2}{A} \cdot \underbrace{[\underset{2\times1}{e_1} \quad \underset{2\times1}{e_2}]}_{2\times2} = \underset{2\times2}{\boxed{[e_1 \quad e_2]}} \cdot \underset{2\times2}{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}$$

$A \cdot T = T \cdot \Lambda$

$A = T \cdot \Lambda T^{-1}$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$$

$$= I + T \cdot \Lambda T^{-1} \cdot t + \frac{(T \Lambda T^{-1} t)^2}{2} + \frac{(T \cdot \Lambda T^{-1} \cdot t)^3}{6} + \dots$$

$$(T \Lambda T^{-1})^2 = (T \cdot \Lambda T^{-1}) \cdot (T \cdot \Lambda \cdot T^{-1})$$

$$= T \cdot \Lambda \cdot T^{-1} \cdot T \cdot \Lambda \cdot T^{-1}$$

$$= T \cdot \Lambda \cdot I \; \Lambda \cdot T^{-1}$$

$$= T \cdot \Lambda^2 \cdot T^{-1}$$

$$(T \cdot \Lambda T^{-1})^3 = (T \cdot \Lambda T^{-1})^2 \cdot (T \cdot \Lambda T^{-1})$$

$$= T \cdot \Lambda^2 \cdot T^{-1} \cdot T \cdot \Lambda \cdot T^{-1}$$

$$= T \cdot \Lambda^3 T^{-1}$$

$$e^{At} = I + \frac{T \cdot \Lambda^2 T^{-1} \cdot t^2}{2} + \frac{T \cdot \Lambda^3 T^{-1} \cdot t^3}{6} + \dots$$

$$+ T \cdot \Lambda T^{-1} \cdot t$$

$$= \boxed{T \cdot T^{-1}} + T \Lambda T^{-1} t + \frac{1}{2} T \Lambda^2 T^{-1} \cdot t^2 + \dots$$

$$= T \cdot I \cdot T^{-1} + T \Lambda T^{-1} \cdot t + \frac{1}{2} T \Lambda^2 T^{-1} t^2 + \dots$$

$$= T \cdot \underbrace{(I + \Lambda t + \tfrac{1}{2} \Lambda^2 t^2 + \dots)}_{?} T^{-1}$$

$$= T e^{\Lambda t} T^{-1} \qquad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$x = c_1 e_1 e^{\lambda_1 t} + c_2 e_2 e^{\lambda_2 t} \longleftrightarrow \cancel{x} = e^{At} \cdot x_0 .$$

$$x = e^{At} \cdot x_0 = T \cdot e^{\Lambda t} \cdot T^{-1} \cdot x_0$$

$$= [e_1 \;\; e_2]\, e^{\Lambda t} \cdot [e_1 \;\; e_2]^{-1} \cdot x_0$$

$$= [e_1 \;\; e_2]\, e^{\Lambda t} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \cdot x_0$$

$$= [e_1 \;\; e_2] \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \cdot x_0$$

$$= \Big(\underbrace{\underset{2\times 1}{e_1}\, e^{\lambda_1 t}\, \underset{1\times 2}{w_1}}_{2\times 2} + e_2 e^{\lambda_2 t} w_2 \Big) \cdot x_0$$

$$= \Big(e^{\lambda_1 t} \cdot \underbrace{e_1 w_1}_{2\times 2} + e^{\lambda_2 t} e_2 w_2 \Big) \underset{2\times 1}{x_0}$$

$$= e^{\lambda_1 t} \cdot e_1 \cdot \underbrace{w_1 \cdot x_0} + e^{\lambda_2 t} e_2 w_2 \cdot x_0$$

$$e^{\lambda_1 t} e_1 \cdot c_1 + e^{\lambda_2 t} e_2 c_2$$

$$x(t) = \underbrace{[x_1 \quad x_2] \cdot [x_1(0) \quad x_2(0)]^{-1}}_{e^{At}} \cdot x_0$$

$$\downarrow$$

$$e^{At} \cdot x_0$$

$\lambda_1 \neq \lambda_2$

$$x(t) = [e_1 e^{\lambda_1 t} \quad e_2 e^{\lambda_2 t}] \cdot [e_1 \quad e_2]^{-1} \cdot x_0$$

$$= \underbrace{[e_1 \quad e_2]}_{T} \cdot \underbrace{\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}}_{e^{\Lambda t}} \underbrace{[e_1 \quad e_2]^{-1}}_{T^{-1}} \cdot x_0$$

$$= \underbrace{T e^{\Lambda t} T^{-1}}_{\downarrow} \cdot x_0$$

$$= e^{At} \cdot x_0$$

$\dot{x} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \cdot x \quad , \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 2 \\ 2 & -5-\lambda \end{vmatrix} = 0$

$(\lambda+2)(\lambda+5) - 4 = 0$

$\lambda^2 + 7\lambda + 10 - 4 = 0 \rightarrow 6.$

$\Delta = 49 - 4 \cdot 6 = 25$

$\lambda_{1,2} = \dfrac{-7 \pm 5}{2} \rightarrow \lambda_1 = -1$, $\lambda_2 = -6$

$(A - \lambda I)e = 0$

$\begin{pmatrix} -2-\lambda & 2 \\ 2 & -5-\lambda \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

- $\lambda = -1$

$\begin{pmatrix} -2+1 & 2 \\ 2 & -5+1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$-v_1 + 2 \cdot v_2 = 0$

$v_2 = 1 \Rightarrow v_1 = 2$

$\lambda = -6 \qquad \begin{pmatrix} -2+6 & 2 \\ 2 & -5+6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\downarrow$

$\begin{bmatrix} -0.5 \\ 1 \end{bmatrix} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$2 \cdot v_1 + v_2 = 0$

$v_2 = 1 \Rightarrow v_1 = -0.5$

$$x(t) = c_1 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-6t} \cdot \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 2 \cdot c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} -0.5 \cdot c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left.\begin{aligned} 2c_1 - 0.5c_2 &= 1 \\ c_1 + c_2 &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} c_2 &= 0.4 \\ c_1 &= 0.6 \end{aligned}$$

$$x(t) = 0.6 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + 0.4 \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} e^{-6t}$$

$$= \begin{bmatrix} 1.2e^{-t} + (-0.2)e^{-6t} \\ 0.6e^{-t} + 0.4e^{-6t} \end{bmatrix}.$$

80

$$X = \left[\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} \quad \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} e^{-6t} \right]$$

$$X(0) = \begin{bmatrix} 2 & -0.5 \\ 1 & 1 \end{bmatrix}$$

$$|X(0)| = 2 + 0.5 = 2.5$$

$$X^{-1}(0) = \frac{1}{2.5} \begin{bmatrix} 1 & 0.5 \\ -1 & 2 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 2e^{-t} & -0.5e^{-6t} \\ e^{-t} & e^{-6t} \end{bmatrix} \frac{1}{2.5} \begin{bmatrix} 1 & 0.5 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \frac{1}{2.5} \cdot \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.2e^{-t} - 0.2e^{-t} \\ 0.6e^{-t} + 0.4e^{-6t} \end{bmatrix}$$

$$x = e^{At} \cdot x_0$$

$$= T \cdot e^{\Lambda t} \cdot T^{-1} \cdot x_0$$

$$T = \begin{bmatrix} 2 & -0.5 \\ 1 & 1 \end{bmatrix}, \quad T^{-1} = \frac{1}{2.5} \begin{bmatrix} 1 & +0.5 \\ -1 & 2 \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{-6t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

$$T = \begin{bmatrix} -0.5 & 2 \\ 1 & 1 \end{bmatrix}$$