Chapter #4

EEE8086-EEE8115

Robust and Adaptive Control Systems

Nonlinear Dynamics

1. Linear Combination

Until now we have seen only linear (autonomous) systems which unfortunately can only fully describe a small class of real systems. In general a nonlinear system is given by:

$$\dot{x}(t) = f\big(x(t), u(t)\big) \tag{1}$$

with $x \in \mathbb{R}^{n \times 1}, u \in \mathbb{R}^{q \times 1}, f \in \mathbb{R}^{n \times 1}$

The main property of a linear system (as we have seen) is that if you have 2 solutions x_1 and x_2 then also their linear combination is a solution.

For example, if we have: $\dot{x}(t) = -3x$ and $x_1 = e^{-3t}$ and $x_2 = 7e^{-3t}$ then $x_3 = 10x_1 - 2x_2 = 10e^{-3t} - 14e^{-3t}$ is also a solution as:

$$\frac{d\big(10e^{-3t} - 14e^{-3t}\big)}{dt} = -30e^{-3t} + 42e^{-3t} = 12e^{-3t}$$

$$-3\big(10e^{-3t} - 14e^{-3t}\big) = -30e^{-3t} + 42e^{-3t} = 12e^{-3t}$$

However this is not the case in nonlinear systems: $\dot{x}(t) = -x^2$ and a solution:

$$x = \frac{1}{t+C} \quad \text{as} \quad \dot{x} = -\frac{1}{(t+C)^2} \quad \text{and} \quad -x^2 = -\left(\frac{1}{t+C}\right)^2 = -\frac{1}{(t+C)^2} \quad \text{but}$$

$$x_A = 3\frac{1}{t+C} \quad \text{is not a solution:}$$

$$\dot{x} = -\frac{3}{(t+C)^2} \text{ and } -x^2 = -\left(\frac{3}{t+C}\right)^2 = -\frac{9}{(t+C)^2} \neq -\frac{3}{(t+C)^2} = \dot{x}$$

2. Equilibrium points

Before we see more differences between linear and nonlinear systems it is important to understand the concept of the equilibrium or rest or singular point which is defined as the point where

$$\dot{x}(t) = 0 \tag{2}$$

For first order linear systems we have that

$$\dot{x}(t) = 0 \Leftrightarrow A(t)x_{EP}(t) + B(t)u(t) = 0 \Leftrightarrow x_{EP}(t) = -A^{-1}(t)B(t)u(t) \tag{3}$$

For autonomous systems: $x_{EP}(t) = 0$ and for that reason in Chapter 3 all the state spaces were drawn with respect to the origin:
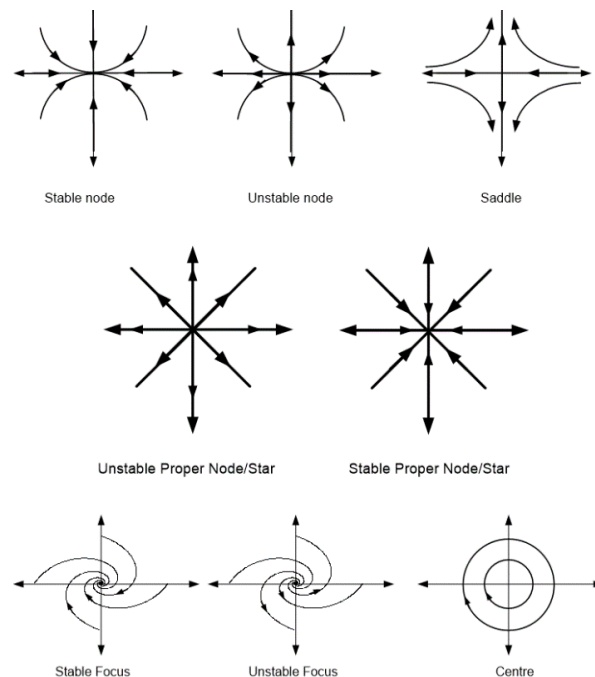
One of the biggest differences between linear and nonlinear systems is that a nonlinear system may have more than one EPs.

Example 1.1: Find the EPs of $\begin{aligned} x_1{}' &= x_1 - x_2 \\ x_2{}' &= x_1^2 + x_2^2 - 2 \end{aligned}$

The EPs are:

$$\begin{bmatrix} x_1{}' \\ x_2{}' \end{bmatrix} = 0 \Leftrightarrow \begin{bmatrix} x_1 - x_2 \\ x_1^2 + x_2^2 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} x_1 = x_2 \\ x_1^2 + x_1^2 = 2 \end{cases} \Leftrightarrow \begin{cases} x_1 = x_2 \\ 2x_1^2 = 2 \end{cases}$$

$$\Rightarrow \begin{cases} (x_1, x_2) = (1,1) \\ (x_1, x_2) = (-1,-1) \end{cases}$$

So we have 2 Eps at (1,1) and at (-1,-1).   ■[1]

Example 1.2: Find the EPs of $\begin{aligned} x_1{}' &= x_1^2 x_2 + 3x_1 x_2 - 10x_2 \\ x_2{}' &= x_1^2 x_2 - 4x_1 \end{aligned}$

The EPs are:

$$\left. \begin{aligned} x_1^2 x_2 + 3x_1 x_2 - 10x_2 &= 0 \\ x_1{}^2 x_2 - 4x_1 &= 0 \end{aligned} \right\}$$

From the 2nd expression we have that $x_1 = 0$ or $x_2 = 4$

From the 1st expression we have for $x_1 = 0$ $x_2 = 0$ i.e. $(x_1, x_2) = (0,0)$ and for $x_1 x_2 = 4$ we have that either $x_2 = 0$ which implies that $x_1 = 0$ (as before) or $x_1^2 + 3x_1 - 10 = 0$ which is a 2nd order polynomial and can easily be solved as:

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[1] syms x1 x2, f1=x1-x2, f2=x1^2+x2^2-2, [x1s x2s]=solve(f1,f2)

$$\Delta = 49 \Rightarrow x_{1_{1,2}} = \frac{-3 \pm 7}{2} \Rightarrow \begin{cases} x_{1_1} = -5 \Rightarrow (x_1, x_2) = (-5, 4) \\ x_{1_2} = 2 \Rightarrow (x_1, x_2) = (2, 4) \end{cases}$$

So the system has the following EPs:

$$(x_1, x_2) = (0,0), (x_1, x_2) = (-5, 4), (x_1, x_2) = (2, 4) \qquad \blacksquare^2$$

In linear systems we were using the term stable/unstable system but effectively we will referring to the equilibrium point. Now that we have multiple equilibriums we need to determine the stability of each EP. It has to be stated here that for nonlinear systems it does not make sense to talk about stable/unstable systems, but for stable/unstable EPs as it is possible to have systems with multiple EPs and some of them to be stable and other to be unstable. Also, the stability of linear systems was determined by the eigenvalues of the state matrix. Hence, now we need a state matrix for each fixed point.

3. Linearisation

In order to determine if each of the EPs of a nonlinear system is stable or not we have to take a "local" picture of each EP. This local picture is called a linearization. I.e. we will describe each EP in a neighbourhood around it. The main tool for that is the Taylor series. Remember that the Taylor series around a point x_0 is defined as:

$$f(x) = f(x_0) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0} \frac{(x-x_0)}{1!} + \left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_0} \frac{(x-x_0)^2}{2!} + \left.\frac{\partial^3 f(x)}{\partial x^3}\right|_{x=x_0} \frac{(x-x_0)^3}{3!} \ldots$$
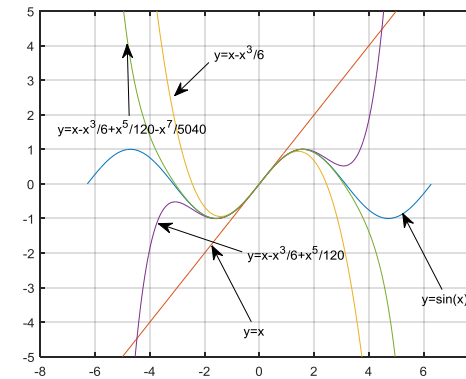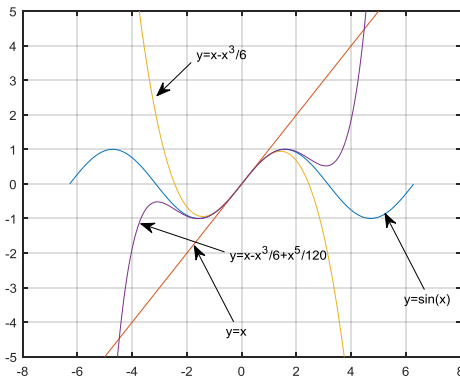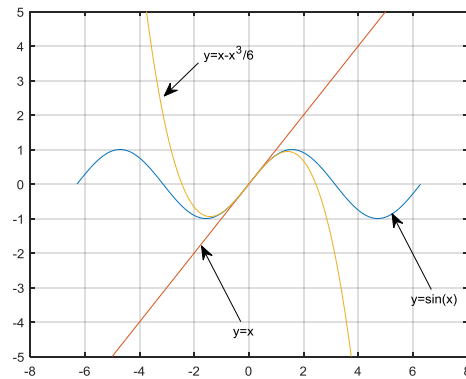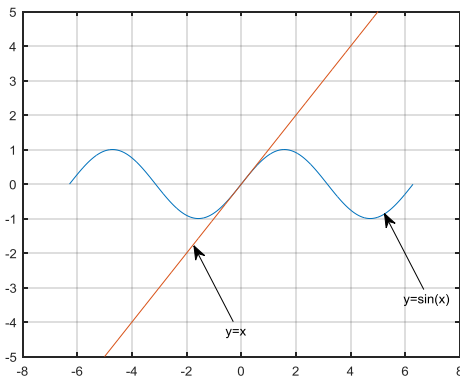
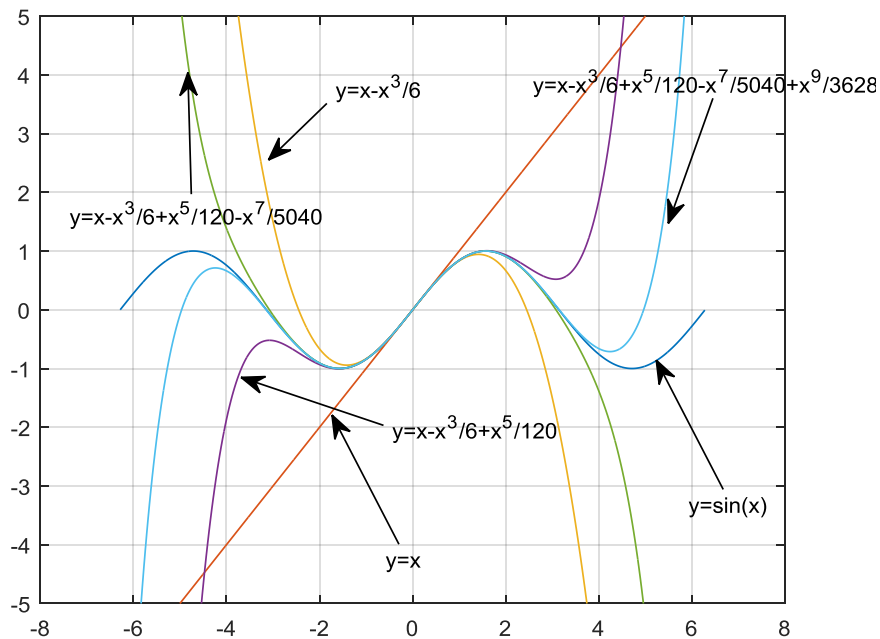---

[2] Notice how the command solve returns the results.

Example 1.3: Approximate the function $y=\sin(x)$ with respect to x around $x=0$, using several expansion orders.
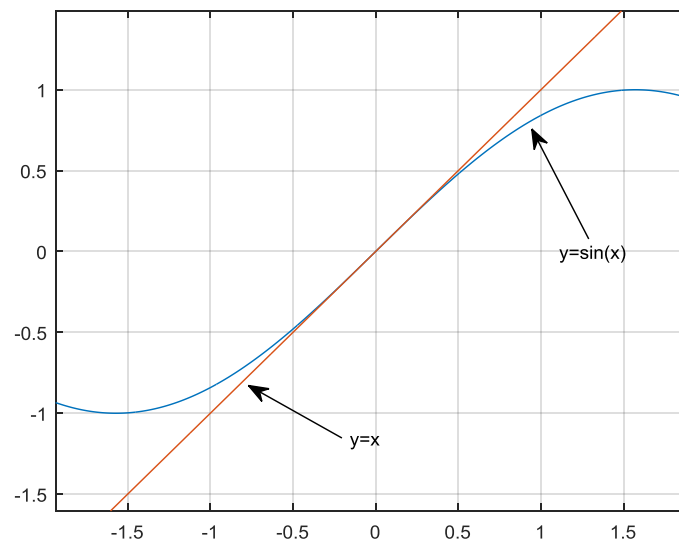The TS of $\sin(x)$ around $x=0$ is:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + O\left(x^{11}\right)$$

Now let's try to approximate $\sin(x)$:

So by increasing the order of the Taylor Series we get a better approximation. But let's get back to the first term $y=x$, obviously it is not a good approximation but if we focus close to the origin then:

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[3] x=-2*pi:0.01:2*pi; plot(x,sin(x)), hold on, plot(x,x), plot(x,x-x.^3/6), plot(x,x-x.^3/6+x.^5/120), plot(x,x-x.^3/6+x.^5/120-x.^7/5040), plot(x,x-x.^3/6+x.^5/120-x.^7/5040+x.^9/362880)

So we see that close to the origin effectively the two traces coincide, i.e. our approximation is good "close" to the point of expansion. This is exactly how we will study a nonlinear system and the stability of their EPs. By approximating them locally using a Taylor series expansion and by keeping only the linear term:

Assume that we have a nonlinear system $\dot{x}(t) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ \vdots \end{bmatrix} = f(x)$ and a

EP at $x = x_{EP}$, then the TS expansion is:

$$\dot{x}(t) = f(x_{EP}) + \frac{\partial f(x(t))}{\partial x(t)}\bigg|_{x=x_{EP}} (x(t) - x_{EP}) + HOT$$

By keeping only the linear term we have:

$$\dot{x}(t) \approx f(x_{EP}) + \frac{\partial f(x(t))}{\partial x(t)}\bigg|_{x=x_{EP}} (x(t) - x_{EP}) \tag{4}$$

As $f(x_{EP}) = 0$:

$$\dot{x}(t) = \frac{\partial f(x(t))}{\partial x(t)}\bigg|_{x=x_{EP}} (x(t) - x_{EP})$$

Obviously we also have that: $\dfrac{dx_{EP}}{dt} = 0$ and hence:

$$\frac{d\left(x(t)-x_{EP}\right)}{dt} = \left.\frac{\partial f\left(x(t)\right)}{\partial x(t)}\right|_{x=x_{EP}} \left(x(t)-x_{EP}\right)$$

We can define now a new variable: $\overline{x}(t) = x(t) - x_{EP}$ and we have that:

$$\frac{d\overline{x}}{dt} = \left.\frac{\partial f\left(x(t)\right)}{\partial x(t)}\right|_{x=x_{EP}} \overline{x}$$

Also, $\left.\dfrac{\partial f\left(x(t)\right)}{\partial x(t)}\right|_{x=x_{EP}} = \begin{bmatrix} \dfrac{\partial f_1\left(x(t)\right)}{\partial x_1(t)} & \dfrac{\partial f_1\left(x(t)\right)}{\partial x_2(t)} & \cdots & \dfrac{\partial f_1\left(x(t)\right)}{\partial x_n(t)} \\ \dfrac{\partial f_2\left(x(t)\right)}{\partial x_1(t)} & \cdots & & \\ \vdots & & & \\ \dfrac{\partial f_n\left(x(t)\right)}{\partial x_1(t)} & & \cdots & \dfrac{\partial f_n\left(x(t)\right)}{\partial x_n(t)} \end{bmatrix}_{x=x_{EP}} = A$ (this

matrix is called the Jacobian of f) and hence:

$$\frac{d\overline{x}}{dt} = A\overline{x} \tag{5}$$

Which is a linear equation that we know how to solve and how to determine its stability.

Similarly if $\dot{x}(t) = f(x,u)$ then:

$$\frac{d\overline{x}}{dt} = \left.\frac{\partial f\left(x(t),u(t)\right)}{\partial x(t)}\right|_{x=x_{EP},u=u_{EP}} \overline{x} + \left.\frac{\partial f\left(x(t),u(t)\right)}{\partial u(t)}\right|_{x=x_{EP},u=u_{EP}} \overline{u} \tag{6}$$

with $\overline{u} = u - u_{EP}$

which will give us: $\dfrac{d\bar{x}}{dt} = A\bar{x} + B\bar{u}$

which is the form for a linear state space model.

Example 1.4: Find the EPs and determine the stability of its EPs $\dot{x} = -x^2 + 1$

For the EPs we set $\dot{x} = 0 \Rightarrow -x^2 + 1 = 0 \Rightarrow x_{EP} = \pm 1$.

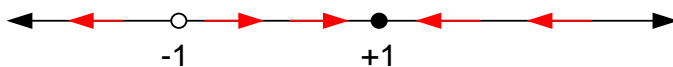Now the Jacobian is: $A = \dfrac{\partial f(x)}{\partial x} = \dfrac{\partial\left(-x^2 + 1\right)}{\partial x} = -2x$

And when this is evaluated at the 2 EPs: $A = \begin{cases} -2 \\ 2 \end{cases}$
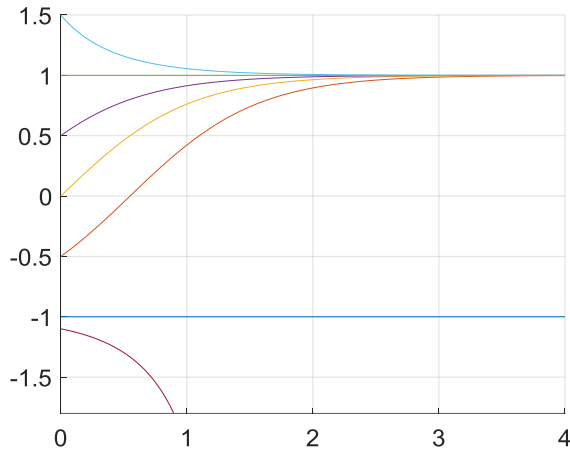
Hence, the 2 state space models are:

$\dot{\bar{x}} = -2\bar{x}$ for $x_{EP} = 1$

$\dot{\bar{x}} = +2\bar{x}$ for $x_{EP} = -1$

This implies that the eigenvalue of the 1st EP is -2 and hence this EP is stable, while the eigenvalue of the 2nd EP is +2 and hence that EP is unstable. So we expect the following 1D state space:

It is interesting to note here, that if we start at the left of the EP -1, I will diverge to -∞. While otherwise I will converge to +1:

■4

Example 1.5: Find the EPs and determine their stability for $\begin{aligned} x_1{}' &= x_1 - x_2 \\ x_2{}' &= x_1^2 + x_2^2 - 2 \end{aligned}$

We have seen that the EPs are: $\begin{cases} (x_1, x_2) = (1,1) \\ (x_1, x_2) = (-1,-1) \end{cases}$

So we have:

$$\left.\begin{aligned} x_1{}' &= x_1 - x_2 \\ x_2{}' &= x_1^2 + x_2^2 - 2 \end{aligned}\right\} \Rightarrow A = \begin{bmatrix} \dfrac{\partial(x_1 - x_2)}{\partial x_1} & \dfrac{\partial(x_1 - x_2)}{\partial x_2} \\ \dfrac{\partial(x_1^2 + x_2^2 - 2)}{\partial x_1} & \dfrac{\partial(x_1^2 + x_2^2 - 2)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2x_1 & 2x_2 \end{bmatrix}$$

Now the Jacobian evaluated at the EPs:

[4] hold on, for x0=-1:0.5:1.5, sim('non_1D.slx'), plot(x(:,1),x(:,2)), end

$$A_1 = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \Rightarrow eigs = \frac{3 \pm \sqrt{17}i}{2}$$ which implies that (1,1) is an unstable focus.

$$A_2 = \begin{bmatrix} 1 & -1 \\ -2 & -2 \end{bmatrix} \Rightarrow eigs = \frac{-1 \pm \sqrt{17}}{2}$$ which implies that (1,1) is a saddle.

Now in order to understand what happens we have to draw the eigenvectors of the saddle (the eigenvectors of the focus do not give us anything as we have seen). The saddle has 2 eigenvalues at 1.56 and -2.56 with the corresponding eigenvectors being:

$$e_1 = \begin{bmatrix} 1 \\ -0.56 \end{bmatrix} \text{ and } e_2 = \begin{bmatrix} 0.28 \\ 1 \end{bmatrix}$$

Hence, the state space is:
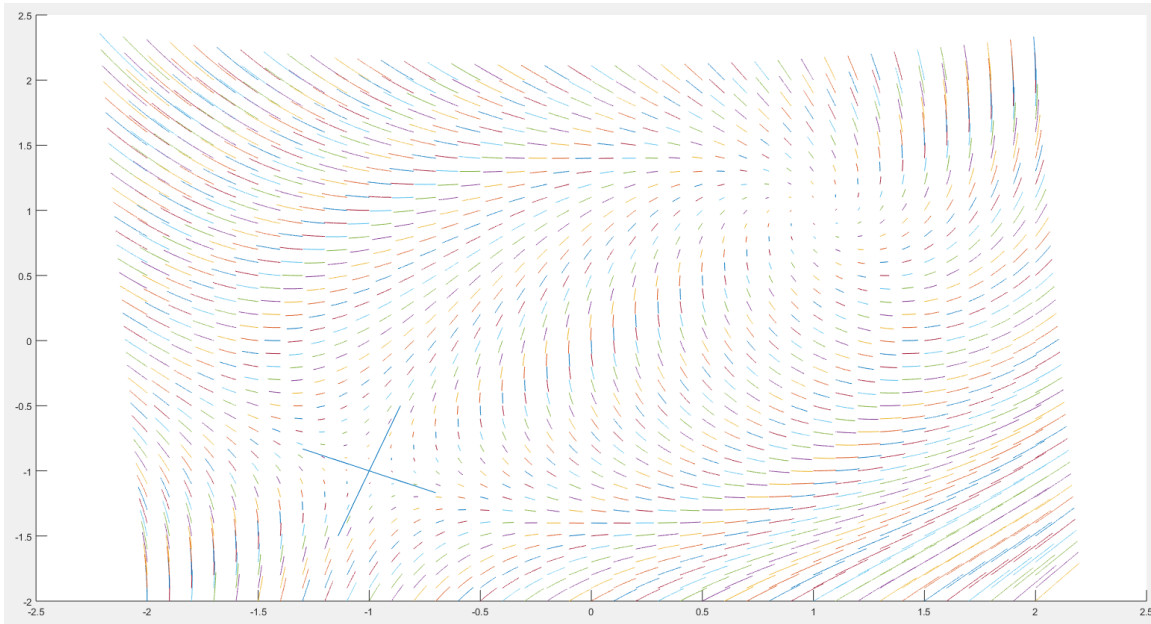
■[5]

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[5] syms x1 x2, f1=x1-x2, f2=x1^2+x2^2-2, [x1s x2s]=solve(f1,f2), J=jacobian([f1 f2], [x1 x2]);
A1=subs(J,{x1,x2},{x1s(1) x2s(1)}), A2=subs(J,{x1,x2},{x1s(2) x2s(2)}), eigsA1=eval(eig(A1))
eigsA2=eval(eig(A2))

4. Limit cycles

Apart from the previous difference of having multiple equilibria in nonlinear systems, another feature of these systems is that it is possible to have a persistence periodic motion, a limit cycle. Assume the Van der Pol system $\ddot{x} + \left(x^2 - 1 \right) \dot{x} + x = 0$. The time and the state space responses are:
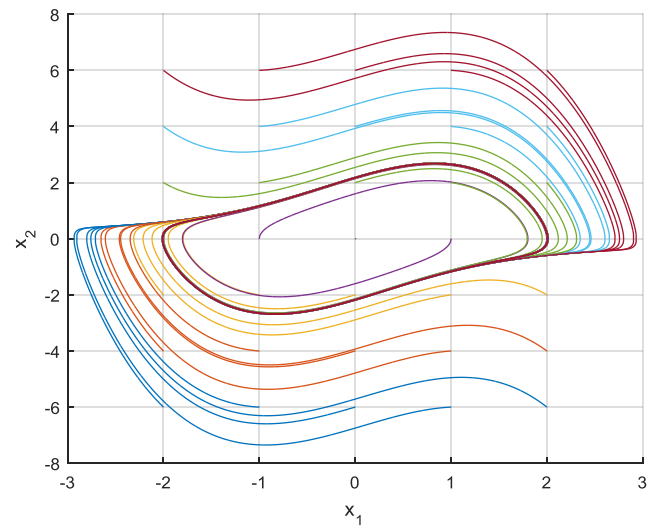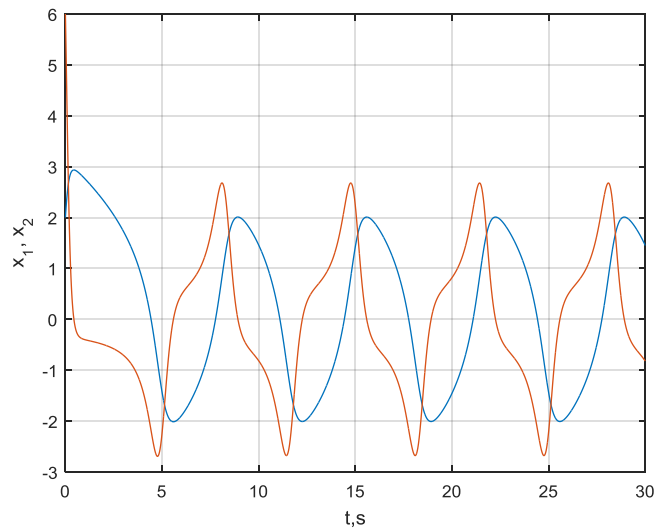
5. Bifurcations

Another difference between linear and nonlinear systems is the phenomenon of bifurcation, which (very briefly) is the creation (or destruction) of equilibria by changing some (one in our case) parameters of the system. This can be the inductance in an RL circuit, the mass of a mass-spring system, the force applied on a bridge, the temperature in a water tank… So now, we study systems of the form $\dot{x} = f(x, p)$, where p is a vector that contains the system's parameters that may change and effectively the algebraic equation $f(x, p) = 0$ has different number of solutions for different parameters in p.

Example 1.6: Assume[6] the system $\dot{x} = r + x^2$ for various values of r. The equilibria are: $\dot{x} = r + x^2 \Leftrightarrow x^2 = -r$
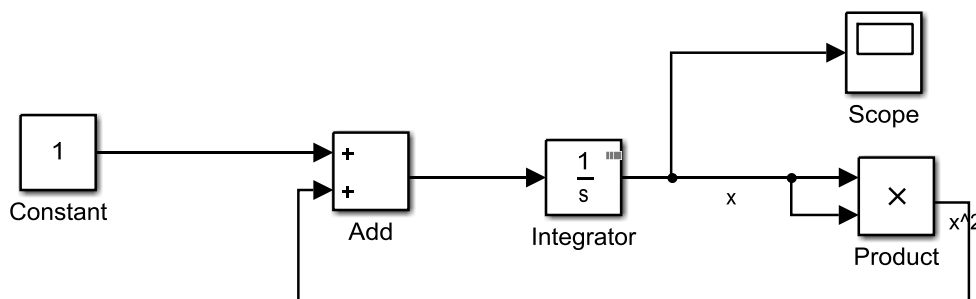
So if $r>0$ there are no equilibrium points

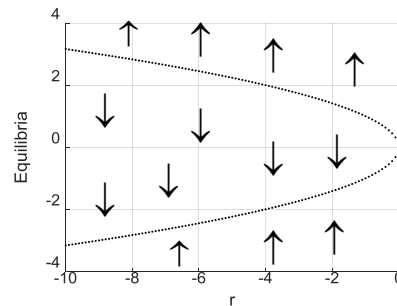[6] Example taken from https://en.wikipedia.org/wiki/Saddle-node_bifurcation
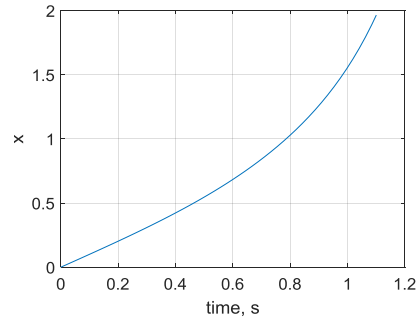
While if $r<0$:

$$\dot{x} = r + x^2 \Leftrightarrow x^2 = \pm\sqrt{-r}$$

i.e. 2 equilibria which for various values of r are:

```
hold on
for r=-10:0.1:0
    plot(r,sqrt(-r),'k.')
    plot(r,-sqrt(-r),'k.')
end
```


The linearised system closed to these 2 points is:

$$\left.\begin{array}{l} \dot{\tilde{x}} = 2\sqrt{-r}\tilde{x} \\ \dot{\tilde{x}} = -2\sqrt{-r}\tilde{x} \end{array}\right\} \Rightarrow \left.\begin{array}{l} \tilde{x} = \tilde{x}(t_0)e^{2\sqrt{-r}} \\ \tilde{x} = \tilde{x}(t_0)e^{-2\sqrt{-r}} \end{array}\right\}$$

The response for $r=-1$:

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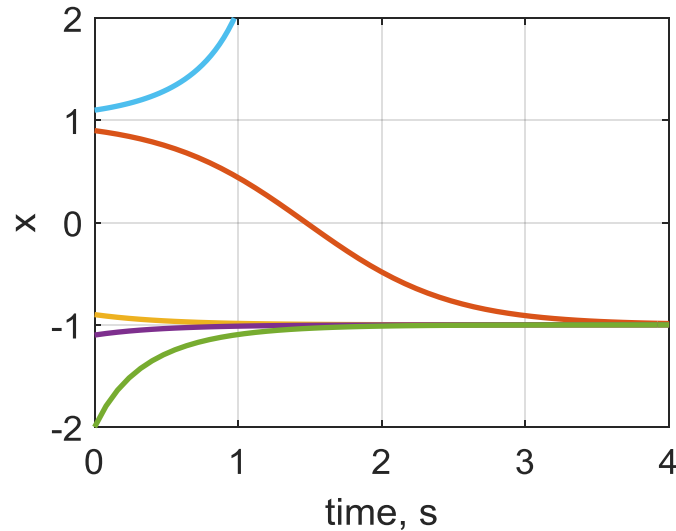
6. Stability

In linear systems we called a system stable if the orbit was not diverging to infinity (exponentially or with oscillations). In nonlinear systems, we have multiple equilibria and we must check the stability "locally" around each equilibrium point. But we need a more general definition that will allow us to focus on "global" stability. Assume that we have a general nonlinear ODE and a specific solution $x(t)$. Obviously as this will depend on the initial conditions we can denote it as $\phi\left(x(t_0),t\right)$. Now we want to test the stability of the general orbit φ (note that the EP can be one such orbit). To do that, we add a perturbation (say at $t=t_0$) and then we observe the perturbed orbit. If it stays "close" to the nominal then it is called stable. Effectively this means that for each required $\varDelta$ we can always find a suitable δ.
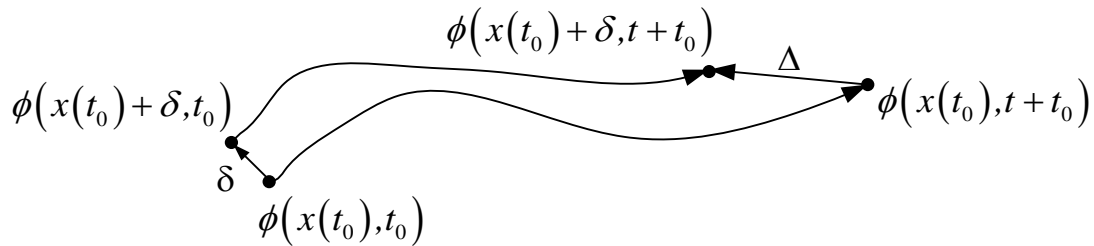
As we are focused on equilibria we have the following definition:

An equilibrium point x_E is stable if for each $R>0$ there exists an $r>0$ such that $\left\| x_E - x(t_0) \right\| < r$, then $\left\| x_E - x(t) \right\| < R$

One of the most important theorems that are used to study the stability of nonlinear systems is called: Lyapunov Stability Theorem

Let function V of the states x, such as:

- $V(x) > 0$

- $\dot{V}(x) < 0$

- $\lim\limits_{\|x\| \to \infty} V(x) = \infty$

then the equilibrium point is globally asymptotic stable.

The difficulty here is how to find the function V, so for the purposes of this module, this function V will be given.

Example 1.7: Determine the stability of the origin of
$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x + y - xy^2 \\ -2x - y - x^2 y \end{bmatrix}, \text{ by using } V(x,y) = x^2 + y^2 \text{ as a candidate}$$
Lyapunov Function and by using local linearization:

$$V(x, y) = x^2 + y^2 > 0$$

$$\dot{V}(x, y) = 2x\dot{x} + 2y\dot{y} = 2x\left(-x + y - xy^2\right) + 2y\left(-2x - y - x^2 y\right) =$$

$$= -2x^2 + 2xy - 2x^2 y^2 - 4xy - 2y^2 - 2x^2 y^2 =$$

$$= -2x^2 - 2xy - 2y^2 - 4x^2 y^2 = -2(x + y)^2 - 4x^2 y^2 < 0$$

$$F = \begin{bmatrix} -x + y - xy^2 \\ -2x - y - x^2 y \end{bmatrix} \Rightarrow A = \begin{bmatrix} -1 - y^2 & 1 - 2xy \\ -2 - 2xy & -1 - x^2 \end{bmatrix} \Rightarrow$$

$$A = \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix} \Rightarrow \begin{vmatrix} -1 - r & 1 \\ -2 & -1 - r \end{vmatrix} = 0 \Rightarrow (-1 - r)^2 + 2 = 0 \Leftrightarrow$$

$$r^2 + 2r + 3 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{4 - 12}}{2} \rightarrow$$

complex with negative real part, so stable

■[7]

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[7] syms x1 x2, f1=-x1+x2-x1*x2^2; f2=-2*x1-x2-x1^2*x2; [x1s x2s]=eval(solve(f1,f2)); J=jacobian([f1 f2], [x1 x2]); A1=subs(J,{x1,x2},{0 0}); eigsA1=eval(eig(A1))

7. Exercises

1) For the system $\dfrac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u(v-1) \\ 4-u^2-v^2 \end{bmatrix}$,

      i. Find the fixed points and classify them according to their stability.

      ii. Make a draft sketch showing the fixed points and a few trajectories.

2) For the system $\dfrac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x+y-xy^2 \\ -2x-y-x^2 y \end{bmatrix}$, prove that the system has a stable equilibrium

point at the origin, using:

      i. $V(x,y) = x^2 + y^2$.

      ii. Local linearization.

3) For the Lorenz equations:

    $\dot{x} = a(y-x),\ \dot{y} = bx - y - xz,\ \dot{z} = xy - cz$ where $a,b,c > 0$ are constants:

      i. Find the equilibrium point(s).

      ii. Show that if $b \le 1$, then the origin is the only equilibrium point, and that there

         three equilibrium points if $b > 1$. Discuss the stability of the origin for $b<1$ and

         for $b>1$. It is given that for $b<1$ we have that $(1+a)^2 > (1-a)^2 + 4ab$ while for

         $b>1$ we have that $(1+a)^2 < (1-a)^2 + 4ab$.

4) Determine the stability of the equilibrium points of $\dfrac{dx}{dt} = rx - x^2$ for $r<0$ and $r>0$.

Sketch a draft bifurcation diagram.

5) Assume an input matrix $B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and four state matrices:

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix},\ A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},\ A_3 = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix},\ A_4 = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

    i.   Find the equilibrium points of each system.

ii.   Find the eigenvalues and hence describe the response of each system starting from

a) $x\begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and b) from the equilibrium point.

iii.   Create the Simulink model for the above systems and hence crosscheck your previous answers.

6) For a given function $f(x) = \cos(x)$

i.   Find the 1st, 2nd, 3rd and 4th order TS expansion around x=0.

ii.   Using Matlab crosscheck that the fourth order is indeed a better approximation that the 1st order one.

7) A nonlinear system is given by $\dot{x} = x - y - x^2 + xy,\ \dot{y} = -x^2 - y$

i.   Find the equilibrium points.

ii.   Determine the stability of each point.

iii. Create a draft sketch of the state space.

iv.   Create the Simulink model of system and hence crosscheck your answers.

v.   For each EP, simulate the linearized approximation of the nonlinear system and compare the responses.

8) For the system $\dot{x} = r + x^2$:

i.   Describe its response based on the values of r.

ii.   Create a Simulink model that will crosscheck your answer.

9) Repeat question 8 for $\dot{x} = a - x^2,\ \dot{y} = -y$

8. Matlab Based Exercises for EEE8086

Crosscheck your analysis in section 7 using Matlab. This must include mfiles and Simulink models.