# Chapter #5

# EEE8086-EEE8115

# Robust and Adaptive Control Systems

# Robust Control

## 1. Ideal Systems

Assume that we have a 2*nd* order system:

$$\ddot{x} + A\dot{x} + Bx = u \quad (1)$$

and that we want to follow a specific signal $x_d$

then if we choose $u = \ddot{x}_d + A\dot{x}_d + Bx_d$ :

$$\ddot{x} + A\dot{x} + Bx = \ddot{x}_d + A\dot{x}_d + Bx_d \Leftrightarrow$$
$$(\ddot{x} - \ddot{x}_d) + A(\dot{x} - \dot{x}_d) + B(x - x_d) = 0$$

We can define now as a tracking error:

$$\tilde{x} = x - x_d \quad (2)$$

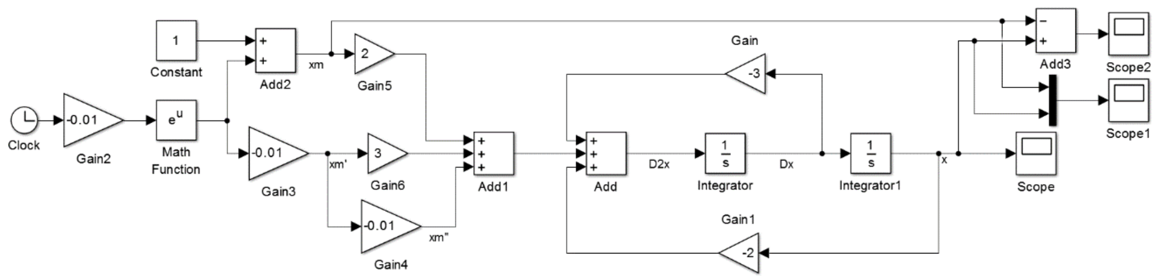
and hence the error dynamics are given by:

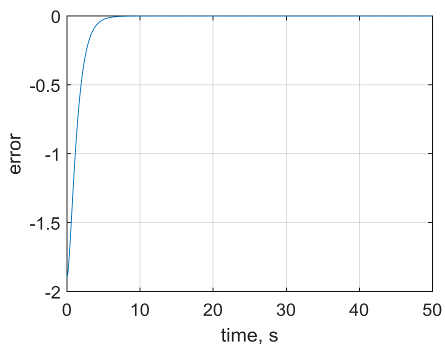
$$\ddot{\tilde{x}} + A\dot{\tilde{x}} + B\tilde{x} = 0 \quad (3)$$

So if the scalars $A$ and $B$ define a stable system the tracking error will converge to zero and hence the system will have the desired response.

**Example 1.1**: A system is given by the following 2nd order ODE: $\ddot{x} + 3\dot{x} + 2x = u$ and we want it to track the desired trajectory $x_d = 1 + e^{-0.01t}$

Then we can set the control signal $u = \ddot{x}_d + 3\dot{x}_d + 2x_d$ and the response is:




If on the other hand the matrices $A$ and $B$ are not stable (or fast enough) we can use:

$$u = A\dot{x} + Bx - C\dot{\tilde{x}} - D\tilde{x} + \ddot{x}_d \tag{4}$$

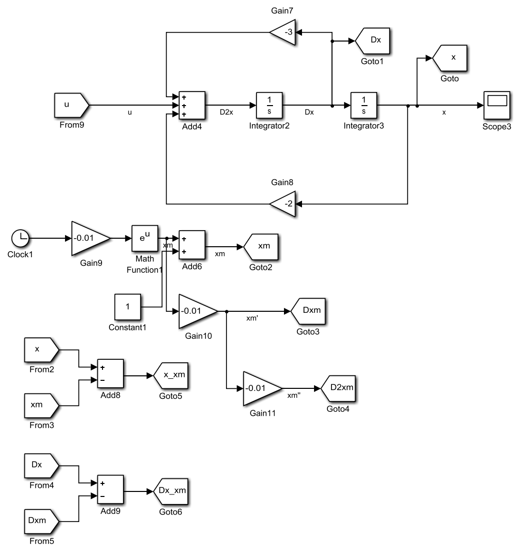
which will give me:

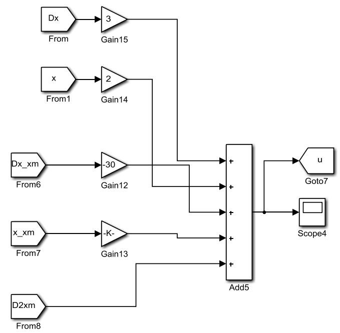
$$\begin{aligned}
&\ddot{x} + A\dot{x} + Bx = A\dot{x} + Bx - C\dot{\tilde{x}} - D\tilde{x} + \ddot{x}_d \Leftrightarrow \\
&\ddot{x} - \ddot{x}_d + C\dot{\tilde{x}} + D\tilde{x} = 0 \Leftrightarrow \\
&\ddot{\tilde{x}} + C\dot{\tilde{x}} + D\tilde{x} = 0
\end{aligned}$$

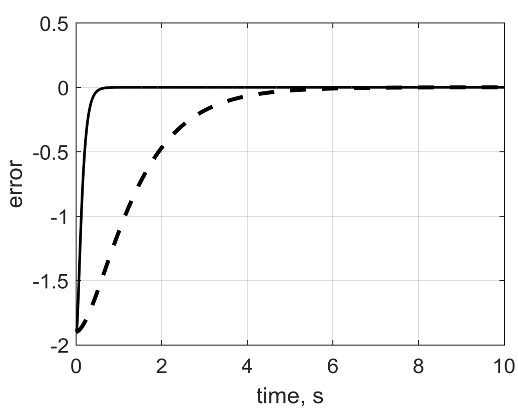
and hence if we properly choose $C$ and $D$ in order to have fast and stable dynamics for the error we can again ensure that the system will have the desired response.

**Example 1.2**: Assume that we want to make the speed of the error 10 times faster, i.e. to place the poles of the error dynamics at -10 and -20:

$$u = 3\dot{x} + 2x - 30\dot{\hat{x}} - 200\hat{x} + \ddot{x}_d$$







This method can be applied now to a nonlinear $n^{\text{th}}$ order system as if

$$x^{(n)} = f\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right) + g\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right)u$$

We can always choose:

$$u = \frac{-h\left(\tilde{x},\dot{\tilde{x}},\ddot{\tilde{x}},\cdots,\tilde{x}^{(n-1)},t\right) + x_d{}^{(n)} - f\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right)}{g\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right)}$$

$$\begin{aligned}
&x^{(n)} = f\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right) + \\
&g\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right)\left(\frac{-h\left(\tilde{x},\dot{\tilde{x}},\ddot{\tilde{x}},\cdots,\tilde{x}^{(n-1)},t\right) + x_d{}^{(n)} - f\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right)}{g\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right)}\right) \\
&x^{(n)} = -h\left(\tilde{x},\dot{\tilde{x}},\ddot{\tilde{x}},\cdots,\tilde{x}^{(n-1)},t\right) + x_d{}^{(n)} \Leftrightarrow \\
&x^{(n)} - x_d{}^{(n)} = -h\left(\tilde{x},\dot{\tilde{x}},\ddot{\tilde{x}},\cdots,\tilde{x}^{(n-1)},t\right) \Leftrightarrow \\
&\tilde{x}^{(n)} + h\left(\tilde{x},\dot{\tilde{x}},\ddot{\tilde{x}},\cdots,\tilde{x}^{(n-1)},t\right) = 0
\end{aligned}$$

Hence if we properly choose the function *h* we can make sure that regardless of the desired signal the system will behave satisfactory.

## 2. Sliding mode control

### 2.1 Ideal systems

Assume that we have a system:

$$x^{(n)} = f\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right)+g\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right)u \tag{5}$$

with a desired tracking trajectory $x_d(t)$, the error between the desired and real trajectory is defined as:

$$\tilde{x}(t)=x(t)-x_d(t) \tag{6}$$

We know that studying an $n^{th}$ order nonlinear system is a cumbersome task, while linear systems are much easier to handle. So the first question that we have here is, can a linear system represent our system given in (5)? Let's denote the variable "*s*"[1] whose ODE describes our system. We impose 2 properties on *s* here:

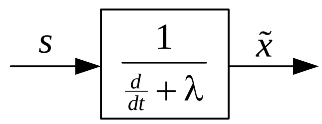
1. We need to differentiate *s* only once in order to have an expression of the control signal *u*.
2. When $s\rightarrow 0\Rightarrow \tilde{x}(t)\rightarrow 0$

If we have a 2[nd] order system: $\ddot{x}=f\left(x,\dot{x},t\right)+g\left(x,\dot{x},t\right)u$ then the conditions are verified if we choose:

[1] Do not be confused, this *s* has nothing to do with the Laplace variable.

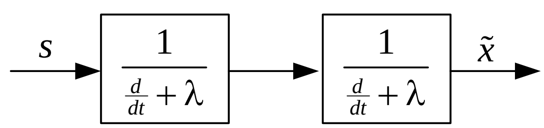
$$s = \dot{\tilde{x}} + \lambda\tilde{x} = \left(\frac{d}{dt} + \lambda\right)\tilde{x} \tag{7}$$

which can be seen as a stable linear filter:

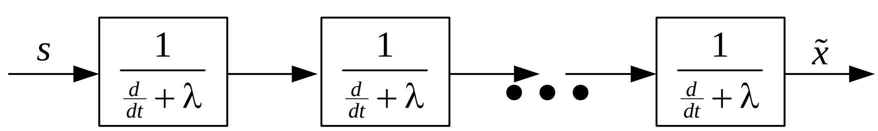



For 3rd order systems:

$$s = \left(\frac{d}{dt} + \lambda\right)^2 \tilde{x} = \left(\frac{d^2}{dt^2} + 2\lambda\frac{d}{dt} + \lambda^2\right)\tilde{x} = \frac{d^2\tilde{x}}{dt^2} + 2\lambda\frac{d\tilde{x}}{dt} + \lambda^2\tilde{x}$$




For the general case $s = \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x}$ (8)

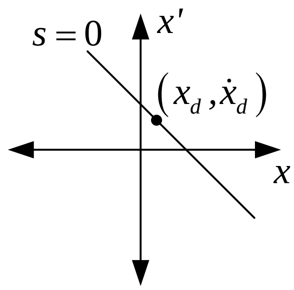



So by studying the error dynamics (given by (8)) we have replaced a nonlinear system with a linear one (and of a smaller order).

**Our task now is to find the control law that will make the ODE given by (8) a stable one, i.e. that the error tends to zero (in finite time).**

From this point we will focus on 2nd order systems, but the same analysis can be carried out in a general *n*th order system.

Geometrically the condition $s=\dot{\tilde{x}}+\lambda\tilde{x}$ with $s$ is zero we have $\dot{\tilde{x}}+\lambda\tilde{x}=0$ or $(\dot{x}-\dot{x}_d)+\lambda(x-x_d)=0$. Which is the equation of a straight line (or a surface in $n^{th}$ dimensional systems) in the state space:




Now let's try to solve the ODE of the error dynamics (by assuming that $s$=0): $\dot{\tilde{x}}+\lambda\tilde{x}=0\Rightarrow\tilde{x}(t)=\tilde{x}(t_0)e^{-\lambda t}$ which implies that $\dot{\tilde{x}}=0,\tilde{x}=0$. Hence, if the trajectory at some point hits the surface defined by $s$=0 at $t$=$t_0$ we have that $\tilde{x}(t)=0,\forall t>t_0$. Hence the surface defined by $s$ is invariant and this implies that we will have the desired response $\forall t>t_0$.

Note: At this point we have **NOT** solved our control problem. We have just changed a nonlinear problem with a linear one and we have seen its geometric interpretation in the state space. The task now is to find the control law $u$ that will make $s$=0 in finite time.

Now assume that you have a $2^{nd}$ order system:

$$\ddot{x}=f(x,\dot{x},t)+g(x,\dot{x},t)u$$

We want to converge to $s=0$ by the appropriate choice of $u$. In order to guarantee that the above equation is stable we can look for a Lyapunov function like: $V(s)=\frac{1}{2}s^2$ with $V(0)=0$ and $V(s)>0, s\neq 0$. So now we have to find the appropriate $u$ such as $\frac{dV(s)}{dt}=s\dot{s}<0$ and hence according to Lyapunov to have a stable system. One obvious way to make sure that is to set $\dot{s}=-s$:

$$s=\dot{\tilde{x}}+\lambda\tilde{x}\Leftrightarrow\dot{s}=\ddot{\tilde{x}}+\lambda\dot{\tilde{x}}=\ddot{x}-\ddot{x}_d+\lambda\dot{\tilde{x}}=f(x,\dot{x},t)+g(x,\dot{x},t)u-\ddot{x}_d+\lambda\dot{\tilde{x}}$$

Hence: $f(x,\dot{x},t)+g(x,\dot{x},t)u-\ddot{x}_d+\lambda\dot{\tilde{x}}=-\dot{\tilde{x}}-\lambda\tilde{x}\Leftrightarrow$

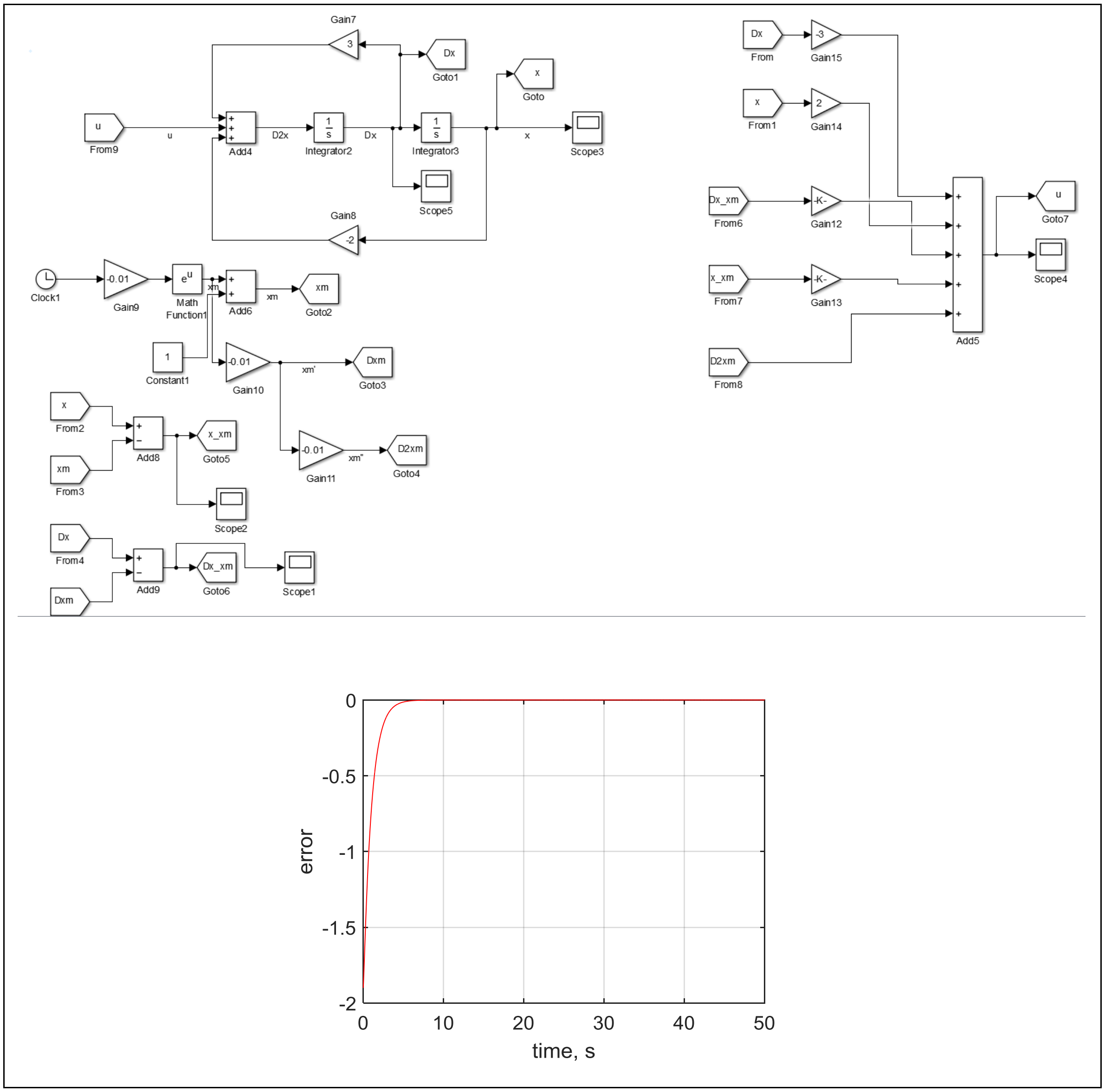
$$u=\frac{1}{g(x,\dot{x},t)}\left(-\dot{\tilde{x}}-\lambda\tilde{x}+\ddot{x}_d-\lambda\dot{\tilde{x}}-f(x,\dot{x},t)\right)$$

In this case we have:

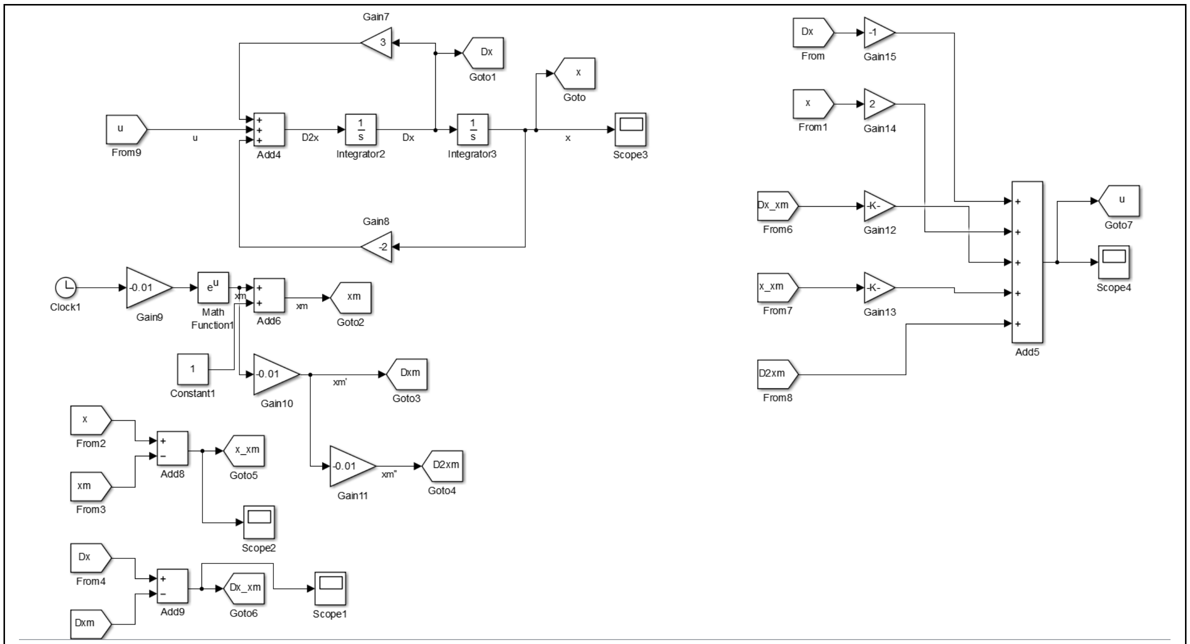
$$\begin{aligned}\ddot{x}&=f(x,\dot{x},t)+g(x,\dot{x},t)u\\&=f(x,\dot{x},t)+g(x,\dot{x},t)\frac{1}{g(x,\dot{x},t)}\left(-\dot{\tilde{x}}-\lambda\tilde{x}+\ddot{x}_d-\lambda\dot{\tilde{x}}-f(x,\dot{x},t)\right)\\&=-\dot{\tilde{x}}-\lambda\tilde{x}+\ddot{x}_d-\lambda\dot{\tilde{x}}\end{aligned}$$

Hence, $\ddot{x}+\dot{\tilde{x}}+\lambda\tilde{x}-\ddot{x}_d+\lambda\dot{\tilde{x}}=0\Leftrightarrow\ddot{\tilde{x}}+\dot{\tilde{x}}(1+\lambda)+\lambda\tilde{x}=0$

And hence we have a homogeneous ODE and with the appropriate choice of $\lambda$ we can make sure that $\tilde{x}\to 0$




But if there is an imperfection in the system then the convergence will not happen:


Gain7
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Integrator2
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Gain8
-2
Clock1
-0.01
Gain9
Math Function1
xm
Add6
xm
Goto2
1
Constant1
-0.01
Gain10
xm'
Dxm
Goto3
x
From2
Add8
x_xm
Goto5
xm
From3
Scope2
-0.01
Gain11
xm''
D2xm
Goto4
Dx
From4
Add9
Dx_xm
Goto6
Scope1
Dxm
Dx
From
-1
Gain15
x
From1
2
Gain14
Dx_xm
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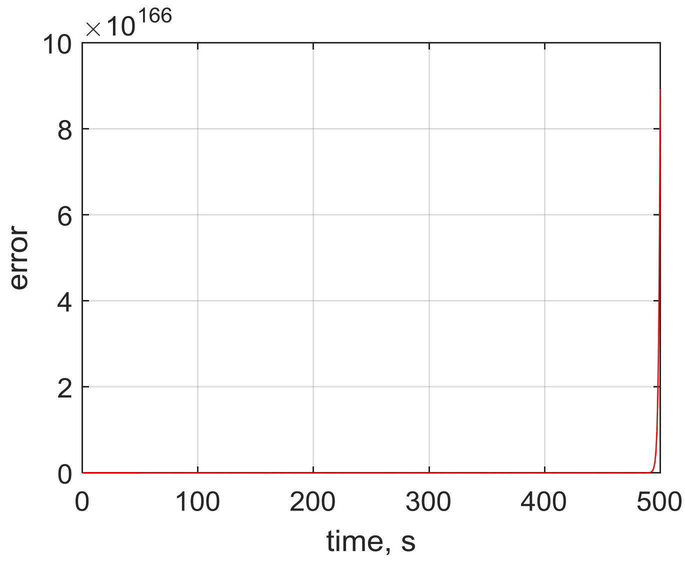
$\times 10^{166}$
10
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6
4
2
0
error
0
100
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400
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time, s

## 2.2 Nonsmooth control law

In order to make the controller more robust to parameter changes we impose a different condition on the Lypunov function:

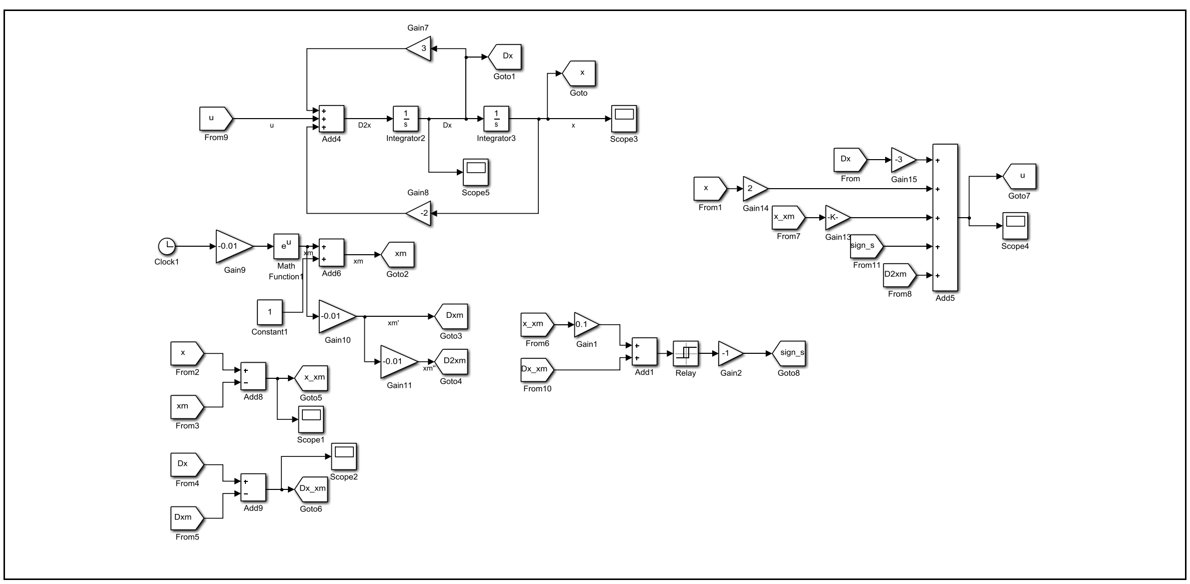
$$\dot{s} = -k \cdot \text{sign}(s)$$

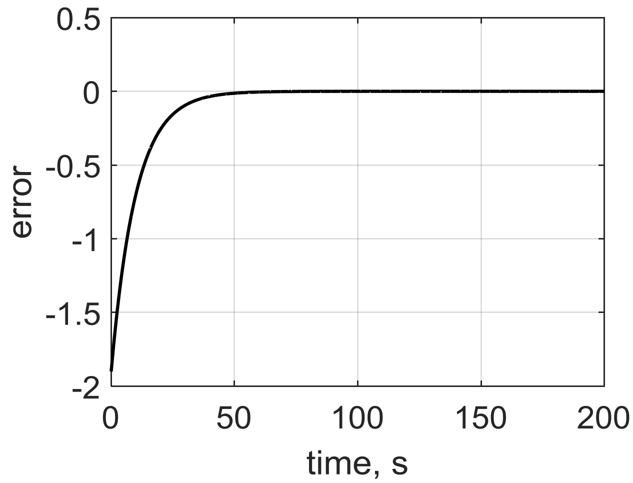
In this case $\dfrac{dV(s)}{dt} = s\dot{s} = -s \cdot k \cdot \text{sign}(s) = -k \cdot |s|$

We know that:

$$\dot{s} = f(x,\dot{x},t) + g(x,\dot{x},t)u - \ddot{x}_d + \lambda\dot{\tilde{x}} = -k \cdot sign(s) \Leftrightarrow$$

$$u = \frac{1}{g(x,\dot{x},t)}\left(-k \cdot sign(s) - f(x,\dot{x},t) + \ddot{x}_d - \lambda\dot{\tilde{x}}\right)$$






## 2.3 Uncertainty

Assume that we have a system:

$$\ddot{x} = f\left(x,\dot{x},t\right) + g\left(x,\dot{x},t\right)u \tag{9}$$

with *f* and *g* being two **NOT "COMPLETELY"** known functions.

Our model based on an estimation of *f* and *g* is written as:

$$\ddot{x} = \hat{f}\left(x,\dot{x},t\right) + \hat{g}\left(x,\dot{x},t\right)u \tag{10}$$

And we only know that the difference between the real and the estimated functions is bounded:

$$\left|\hat{f} - f\right| \le F, \left|\hat{g} - g\right| \le G \tag{11}$$

To simplify the analysis we assume that $g\left(x,\dot{x},t\right) = \hat{g}\left(x,\dot{x},t\right) = 1$

As before we can find when $\dot{s} = 0$ but now

$$u = -\hat{f}(x,\dot{x},t) + \ddot{x}_d - \lambda\dot{\tilde{x}} - k \cdot sign(s) \quad (12)$$

Hence,

$$\dot{s} = f(x,\dot{x},t) + u - \ddot{x}_d + \lambda\dot{\tilde{x}} = f(x,\dot{x},t) + \left(-\hat{f}(x,\dot{x},t) + \ddot{x}_d - \lambda\dot{\tilde{x}} - k \cdot sign(s)\right) - \ddot{x}_d + \lambda\dot{\tilde{x}} =$$
$$= f(x,\dot{x},t) - \hat{f}(x,\dot{x},t) - k \cdot sign(s)$$

Or $\dfrac{dV(s)}{dt} = s\left(f(x,\dot{x},t) - \hat{f}(x,\dot{x},t) - k \cdot sign(s)\right)$

So if we choose $k > F$ we can be sure that $\dfrac{dV(s)}{dt} \le 0$

**2.4 Finite Convergence**

We have seen that $\dfrac{dV(s)}{dt} = s\dot{s} \le 0$. We can impose a stricter condition to ensure that $s$ becomes zero in finite time: $\dfrac{dV(s)}{dt} = s\dot{s} = -\eta|s| < 0, \eta > 0$

This is happening as if at some instant $t=t_0$ we have $s(t_0)>0$:

$$s\dot{s} = -\eta|s| \Leftrightarrow s\dot{s} = -\eta s \Leftrightarrow s = -\eta t + s(t_0)$$

which means that $s$ will decrease until it becomes zero at $\eta t = s(t_0) \Leftrightarrow t = \dfrac{s(t_0)}{\eta}$

Similarly if $s(t_0)<0$:

$$s\dot{s} = -\eta|s| \Leftrightarrow s\dot{s} = \eta s \Leftrightarrow s = \eta t + s(t_0) \Leftrightarrow t = -\frac{s(t_0)}{\eta} > 0, s(t_0) < 0$$

Hence, we have

$$\frac{dV(s)}{dt} = s\left( f(x,\dot{x},t) - \hat{f}(x,\dot{x},t) - k \cdot sign(s) \right)$$

And

$$\frac{dV(s)}{dt} \leq -\eta|s| \leq 0$$

Which is true if $k = F + \eta$

# 3. Model Reference Adaptive Control

Assume the following system:

$$\ddot{x} = f\left(x,\dot{x},t\right) + u \tag{13}$$

with

$$f\left(x,\dot{x},t\right) = f_1\left(x,\dot{x},t\right)p_1 + f_2\left(x,\dot{x},t\right)p_2 + \cdots \tag{14}$$

for example:

- $f\left(x,\dot{x},t\right) = \dot{x}p_1 + xp_2$ (linear 2nd order ODE)
- $f\left(x,\dot{x},t\right) = \dot{x}^2 p_1 + x|x||\dot{x}|\, p_2 + cos\left(\sqrt{x}\right)$ (nonlinear but time invariant)
- $f\left(x,\dot{x},t\right) = cos(t)\,\dot{x}\,cos(x)\,p_1$ (nonlinear and time varying)

Now, we assume that $f_1\left(x,\dot{x},t\right), f_2\left(x,\dot{x},t\right)\cdots$ are known functions of the state vector, while the $p_1, p_2\cdots$ are unknown **constants**.

Now as in the previous section let's choose a Lyapunov function such as:

$$V(s) = \frac{1}{2}s^2 \tag{15}$$

where $s = \dot{\tilde{x}} + \lambda\tilde{x}$

and hence: $\dot{s} = \ddot{\tilde{x}} + \lambda\dot{\tilde{x}} = \ddot{x} - \ddot{x}_d + \lambda\dot{\tilde{x}} = f\left(x,\dot{x},t\right) + u - \ddot{x}_d + \lambda\dot{\tilde{x}}$

from (14) we have that

$$\dot{s} = u + \left( f_1(x,\dot{x},t)\,p_1 + f_2(x,\dot{x},t)\,p_2 + \cdots \right) - \ddot{x}_d + \lambda\dot{\tilde{x}}$$

Now this can be written as:

$$\dot{s} = u + \left( f_1(x,\dot{x},t)\,p_1 + f_2(x,\dot{x},t)\,p_2 + \cdots - \ddot{x}_d + \lambda\dot{\tilde{x}} \right)$$
$$= u + F \cdot p$$

So: $\dot{V}(s) = s\dot{s} = s(u + F \cdot p)$

with $F = \left[ f_1(x,\dot{x},t) \quad f_2(x,\dot{x},t) \quad \cdots \quad \ddot{x}_d \quad \dot{\tilde{x}} \right]$ & $p = \left[ p_1 \quad p_2 \quad \cdots \quad -1 \quad \lambda \right]^T$

Hence, if we knew the vector $p$ we can choose: $u = -F \cdot p - ks, k > 0$

which would have given:

$$\dot{V}(s) = -ks^2 < 0$$

Unfortunately we do not the vector $p$ but we can have an estimate $\hat{p}(t)$ and hence we can define the parameter error $\tilde{p}(t) = p - \hat{p}(t)$

So actually our control signal is: $u = -F \cdot \hat{p} - ks$

So $\dot{V}(s) = s\dot{s} = s(u + F \cdot p) = s(-F \cdot \hat{p} - ks + F \cdot p) = -ks^2 + sF \cdot \tilde{p}$ (16)

Previously when we had only the tracking error we used as $V(s) = \frac{1}{2}s^2$, now we also have the parameter error and hence we can use:

$$V(s) = \frac{1}{2}s^2 + \frac{1}{2}\tilde{p}_1^2 + \frac{1}{2}\tilde{p}_2^2 + \cdots$$

As not all parameter errors are equally important:

$$V(s)=\frac{1}{2}s^2+h_1\frac{1}{2}\tilde{p}_1^2+h_2\frac{1}{2}\tilde{p}_2^2+\cdots$$

And in a matrix form[2]:

$$V(s)=\frac{1}{2}s^2+\frac{1}{2}\tilde{p}^T H\tilde{p}$$

Now, $\frac{d\left(\frac{1}{2}\tilde{p}^T H\tilde{p}\right)}{dt}=\frac{1}{2}\dot{\tilde{p}}^T H\tilde{p}+\frac{1}{2}\tilde{p}^T H\dot{\tilde{p}}=\dot{\tilde{p}}^T H\tilde{p}\ \left(\text{ or }\tilde{p}^T H\dot{\tilde{p}}\right)$

Also, $\tilde{p}(t)=p-\hat{p}(t)\Rightarrow\dot{\tilde{p}}(t)=\cancel{\dot{p}}-\dot{\hat{p}}(t)$

So $\frac{d\left(\frac{1}{2}\tilde{p}^T H\tilde{p}\right)}{dt}=-\dot{\hat{p}}^T H\tilde{p}$

Hence the time derivative of the chosen Lyapunov function is:

$$\dot{V}(s)=-ks^2+sF\cdot\tilde{p}-\dot{\hat{p}}^T H\tilde{p}$$

Hence, if we set $sF\cdot\tilde{p}=\dot{\hat{p}}^T H\tilde{p}\ \Leftrightarrow sF=\dot{\hat{p}}^T H\Leftrightarrow\dot{\hat{p}}^T=sFH^{-1}$

So the adaptation law is: $\dot{\hat{p}}^T=sFH^{-1}$ (17)

And the control law is: $u=-F\cdot\hat{p}-ks$ (18)

[2] In general the matrix H does not have to be diagonal, but only symmetric.

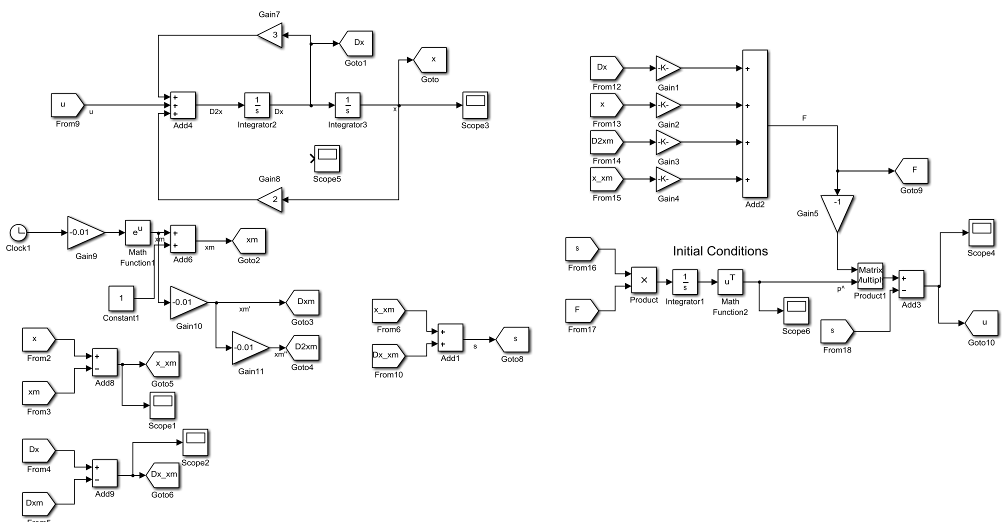
Another way to see this is to place (xx) into (xx):

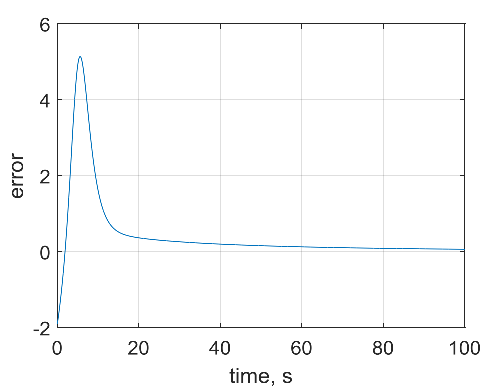
$$\hat{p} = \left(\int sFH^{-1}dt\right)^T \Rightarrow u = \underbrace{-F\cdot\left(\int sFH^{-1}dt\right)^T}_{Integral\ term} - \underset{PD\ term}{ks} \tag{19}$$

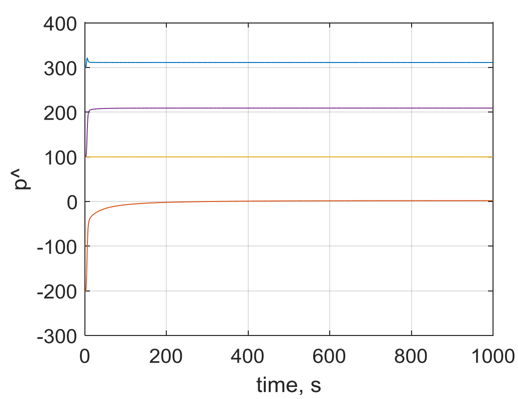
i.e. an adaptive PID controller.

Example 1:

$$\ddot{x} = 3\dot{x} + 2x + u$$

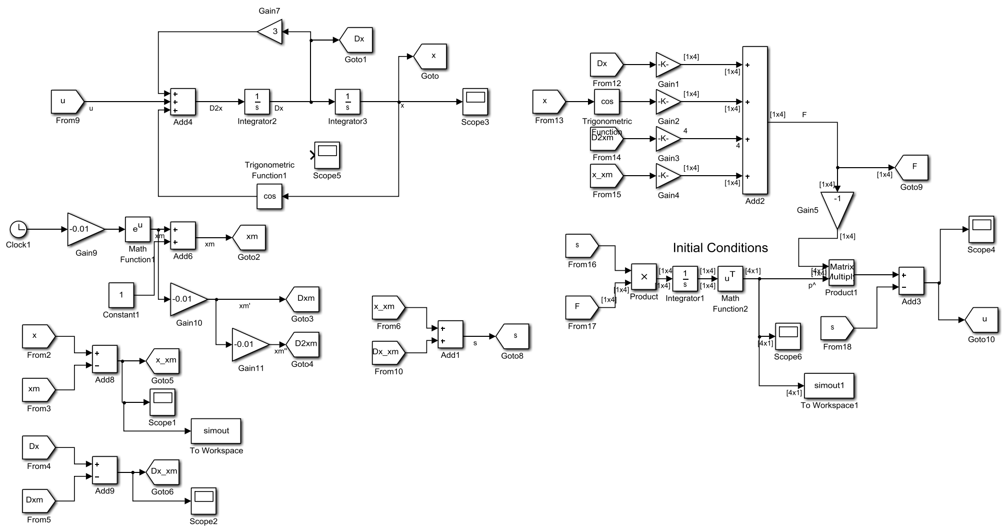


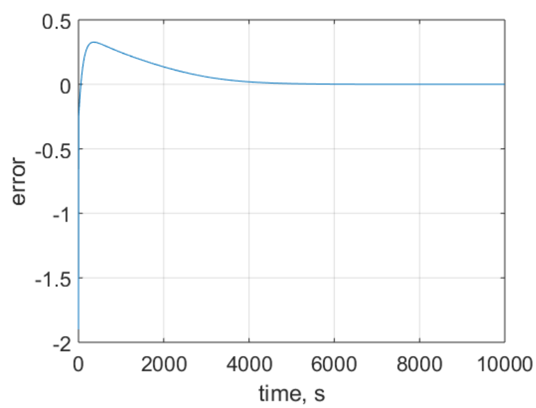



Example 2:

$$\ddot{x} = 3\dot{x} + \cos(x) + u$$



# Matlab Based Exercises for EEE8086

Reproduce all Simulink files of chapter 5.