$\dot{x} = f(x, t) \rightarrow$ 1st order ODE ①

$\ddot{x} = f(\dot{x}, x, t) \rightarrow$ 2nd -"- -"-

ODE + I.C = I.V.P.

e.g. $\dot{x} = x^2$ $\quad x(t) = \dfrac{1}{1/x_0 - t}$

$$\dot{x} = \frac{-1}{(1/x_0 - t)^2} \cdot (1/x_0 - t)'$$

$$= \frac{\cancel{-}1}{(\frac{1}{x_0} - t)^2} \cdot (\cancel{-}1) = \frac{1}{(\frac{1}{x_0} - t)^2} = x^2$$

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$\dot{x} = -3x \qquad x_1 = e^{-3t} \qquad x_2 = 10\, e^{-3t}$

$(e^{-3t})' = -3 \cdot (e^{-3t})$

$-3 e^{-3t} = -3 e^{-3t}$

$(10 e^{-3t})' = -3 (\cdot 10 \cdot e^{-3t})$

$-30 e^{-3t} = -30 e^{-3t}$

$+ x_0 = 10$

$x_1(0) = e^{-0} = 1 \neq x_0$

$x_2(0) = 10 \cdot e^0 = 10 = x_0$

$$\ddot{x} + 3\cdot\dot{x} + 2x = 0$$

$$x = e^{-t}$$

$$(e^{-t})'' + 3\cdot(e^{-t})' + 2e^{-t} = 0$$

$$e^{-t} - 3e^{-t} + 2e^{-t} = 0$$

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1st Linear ODE

$$\dot{x} + kx = u \Rightarrow x = e^{-kt}\cdot x_0 + e^{-kt}\int_0^t e^{k\tau} u \, d\tau.$$

$u = \text{const.}$

$$= e^{-kt}\cdot x_0 + \frac{u}{k}(1 - e^{-kt})$$

- $k > 0$ $t \to \infty$ $x_{ss} = 0\cdot x_0 + \frac{u}{k}(1-0)$

  stable $= u/k$

- $k < 0$ $t \to \infty$ $x_{ss} \to \pm\infty$

  unstable

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$u = 0$ $x(t) = e^{-kt}\cdot x_0$

$\ddot{x} + A\dot{x} + Bx = u.$

$\Big\downarrow u=0$

$\ddot{x} + A\dot{x} + Bx = 0$

Assume $x = e^{rt}$

$r^2 \cancel{e^{rt}} + Ar\cancel{e^{rt}} + B\cancel{e^{rt}} = 0$

$r^2 + Ar + B = 0 \longrightarrow$ C.E.

$$r_{1,2} = \frac{-A \pm \sqrt{\Delta}}{2} \qquad \Delta = A^2 - 4B$$

- $\Delta > 0$, $r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$

  $x_1 = e^{r_1 t}$, $x_2 = e^{r_2 t}$

- $\Delta = 0$ $r_1 = r_2 = r \in \mathbb{R}$

  $x_1 = e^{rt}$ $x_2 = te^{rt}$

- $\Delta < 0$ $r = a \pm bi$, $a, b$ $\mathbb{R}$.

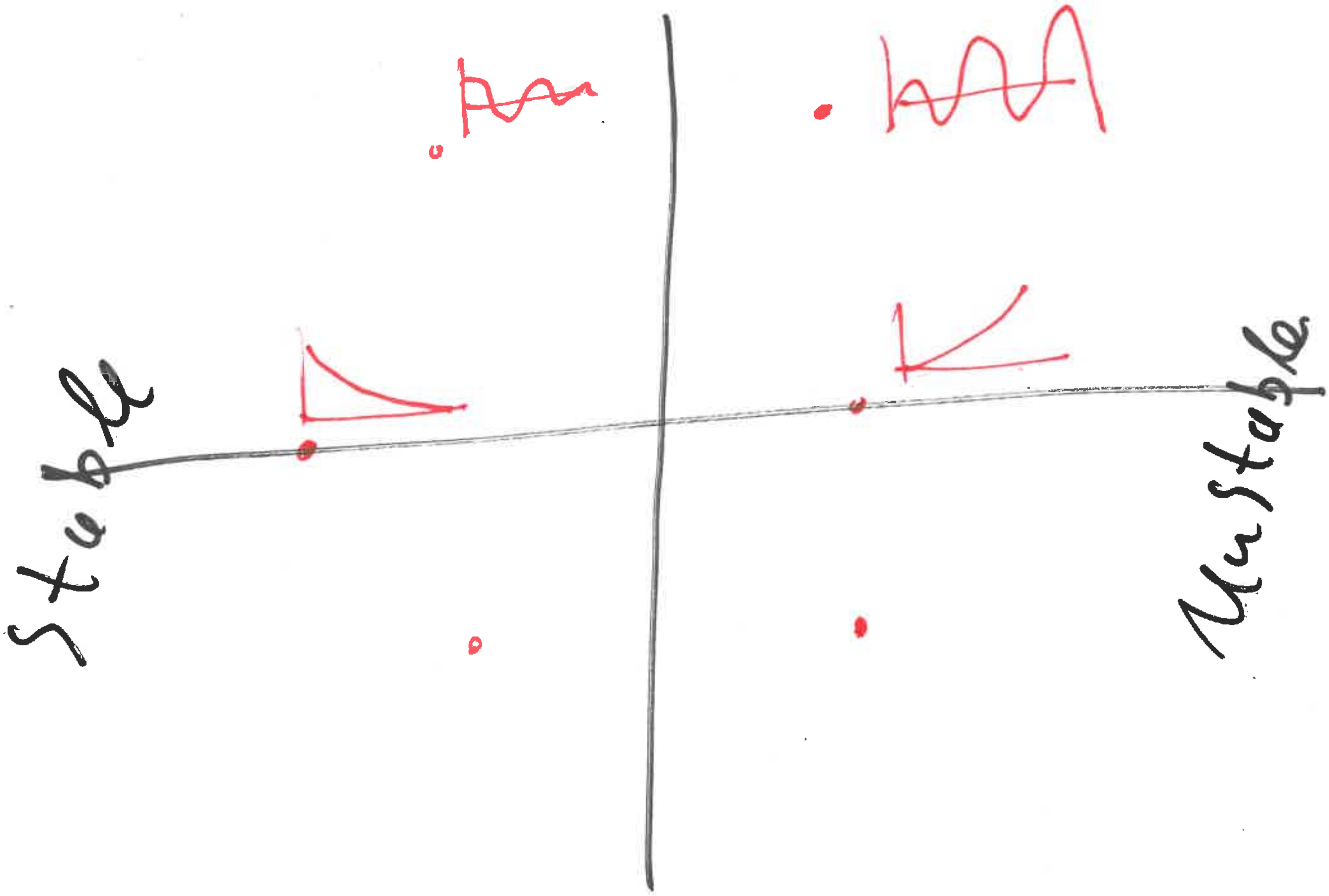
  $x_1 = e^{rt}$, $x_2 = e^{\bar{r}t}$

$X = C_1 x_1 + C_2 \cdot x_2$

$x_1 = e^{(a+bi)t} = e^{at} \cdot e^{bit}$

4

- $a < 0$  stable
- $a > 0$  unstable



2nd ODE

$x_1$ is a soln. $\rightarrow$ $k \cdot x_1$ is also a soln.

ad $x_2$ $\rightarrow$ $c_1 \cdot x_1 + c_2 \cdot x_2$ is .....

If $x_1, x_2$ are L.I.

All other solns $x = c_1 \cdot x_1 + c_2 \cdot x_2$

$\nexists \; k : \; x_1 = k \cdot x_2$

$$W(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{bmatrix}$$

$|W| \neq 0 \Longrightarrow x_1, x_2$ are L.I.

⑥

$$\frac{d^{(n)}x(t)}{dt} = f(x^{(n-1)}, x^{(n-2)}, \dots, x, t)$$

$\downarrow$ Linear.

$$x^{(n)} + p_{n-1}x^{(n-1)} + p_{n-2}x^{(n-2)} + \dots + p_0 x = 0$$

if $n=2$

$$x'' + p_1 x' + p_0 x = 0$$

if $n=3$

$$x''' + p_2 x'' + p_1 x' + p_0 x = 0$$

$x_1, x_2, \dots \dots x_k$, $x = \sum_{i=1}^{k} c_i \cdot x_i$

also a soln.

Example 1.16

$$x_1, \ldots\ x_n$$

if $\underbrace{\phantom{x_1 \ldots x_n}}_{\text{L.I}}$ $\Leftarrow$ $x = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

$$W(x_1, x_2, \ldots x_n) = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \\ \dot{x}_1 & \dot{x}_2 & \ldots & \dot{x}_n \\ \vdots & & & \\ x_1^{(n-1)} & x_2^{(n-1)} & \ldots & x_n^{(n-1)} \end{bmatrix}$$

$|W| \neq 0 \rightarrow$ Are L.I.

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$$x^{(n)} + p_{n-1} \cdot x^{(n-1)} + p_{n-2}\, x^{(n-2)} + \ldots\ p_0 x = 0$$

$$x = e^{rt}$$

$$r^n \cancel{e^{rt}} + p_{n-1} \cdot r^{n-1} \cdot \cancel{e^{rt}} + \ldots + p_0 \cdot \cancel{e^{rt}} = 0$$

$$r^n + p_{n-1} \cdot r^{n-1} + \ldots + p_0 = 0$$

$\big\downarrow$ roots

roots([1 $p_{n-1}$ $p_{n-2}$ .... $p_0$]) ↵

$r^n + p_{n-1} r^{n-1} + \dots + p_0 = 0$

- $r_1, r_2 \dots \quad r_1 \neq r_2, \ r_1, r_2 \in \mathbb{R}$

$x_1 = e^{r_1 t} \quad x_2 = e^{r_2 t}, \ x_3 = \dots \qquad x_p = \dots$

- $r_A = a \pm bi$

$x_{p+1} = e^{r_A t}, \ x_{p+2} = e^{\bar{r}_A t}$

- $r_B = c \pm di$

$x_{p+3} = e^{r_B t}, \ x_{p+4} = e^{\bar{r}_B t}$

- $r_1 = r_2 = r_C$

$x_{p+4} = e^{r_C t} \qquad x_{p+5} = t\, e^{r_C t}$

- $r_1 = r_2 = r_3 = r_D$

$x_{p+6} = e^{r_D t}, \ x_{p+7} = e^{r_D t} \cdot t$

$x_{p+8} = e^{r_D \cdot t} \cdot t^2$

- $r_1 = r_2 = a \pm bi, \ \bar{r}_1 = \bar{r}_2$

$x_{p+8} = e^{r_1 t} \qquad x_{p+9} = e^{\bar{r}_1 t}$

$x_{p+10} = e^{r_1 t} \cdot t \qquad x_{p+11} = e^{\bar{r}_1 t} \cdot t$

4th order ODE

$x = e^{rt} \rightarrow$ 4th orde pol. exp.

roots $\rightarrow r_1 = -1$, $r_2 = -2$ $r_3 = -3$, $r_4 = -4$.

$$x(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-3t} + c_4 e^{-4t}$$

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$r_1 = r_2 = r_3 = -1 \rightarrow$ G.S = ?

$$x = C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t}$$

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$r_1 = r_2 = r_3 = r_4 = -1$.

$x_1 = e^{-t}$ $\qquad x_3 = t^2 \cdot e^{-t}$

$x_2 = t e^{-t}$ $\qquad x_4 = t^3 \cdot e^{-t}$

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$r_1=-1$, $r_2=-2$ $r_3=-2$

$r_{4,5}=-3\pm i$, $r_{6,7}=-4\pm 2i$, $r_{8,9}=-4\pm 2i$

$r_{10,11}=-5\pm 3i$, $r_{12,13}=-5\pm 3i$

$r_{14,15}=-5\pm 3i$

$x_1=e^{-t}$, $x_2=e^{-2t}$ $x_3=e^{-2t}\,t$

$x_4=e^{(-3+i)t}$ $x_5=e^{(-3-i)\cdot t}$

$x_6=e^{(-4+2i)t}$ $x_7=e^{(-4-2i)t}$

$x_8=e^{(-4+2i)t}\cdot t$ $x_9=e^{(-4-2i)t}\cdot t$

$x_{10}=e^{(-5+3i)\cdot t}$ $x_{11}=e^{(-5-3i)t}$

$x_{12}=e^{(-5+3i)\cdot t}\cdot t$ $x_{13}=e^{(-5-3i)t}\cdot t$

$x_{14}=e^{(-5+3i)\cdot t}\,t^2$, $x_{15}=e^{(-5-3i)\cdot t}\cdot t^2$

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n$$

$$\Downarrow$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\dot{X} = A \cdot X, \quad X \in \mathbb{R}^{n\times 1}, \quad A \in \mathbb{R}^{n\times n}$$

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