Revision

$$x^{(n)} + P_{n-1} x^{(n-1)} + P_{n-2} x^{(n-2)} + \dots + P_0 x = 0$$

if x_1 is a soln $\Rightarrow x_1^{(n)} + P_{n-1} x_1^{(n-1)} + \dots + P_0 x = 0$

if $x_1, x_2 \dots$ are soln $\Rightarrow$

$$\varphi = \sum_{i=1} c_i x_i \text{ is also a soln}$$

$$= c_1 x_1 + c_2 x_2 + c_3 x_3 + \dots$$

if n soln.

$$W(x_1, x_2, \dots, x_n) = \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ \vdots & & & \\ x_1^{(n-1)} & x_2^{(n-1)} & \dots & x_n^{(n-1)} \end{bmatrix}$$

if $|W| \neq 0 \rightarrow$
n soln. are L.I.

Any other soln $\varphi(t) = \sum_{i=1}^{n} x_i c_i$

$x^{(n)} + p_{n-1} x^{(n-1)} + \dots + p_0 x = 0$

Assume $x = e^{rt}$

C.E. $r^n + p_{n-1} \cdot r^{n-1} + \dots + p_0 = 0$

$\downarrow$ roots

$\mathbb{R} \to$ real
$\mathbb{C} \to$ complex

- $r_1 \neq r_2 \neq r_3 \dots \in \mathbb{R}$ or $\mathbb{C}$

$r_1 \to e^{r_1 t} \to x_1$, $r_2 \to e^{r_2 t} \to x_2$

$$x = \sum_{i=1} c_i x_i \quad , x \in \mathbb{R}$$

- $r_k = r_{k+1} = \dots$

$$x_k = e^{r_k t}$$
$$x_{k+1} = e^{r_k t} \cdot t$$
$$x_{k+2} = e^{r_k \cdot t} \cdot t^2$$

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n$$
$$\vdots$$
$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n$$

$$\dot{x} = A\cdot x, \quad A \in \mathbb{R}^{n\times n}$$
$$x \in \mathbb{R}^n$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

A sol^n $x^{(k)}$

$$\dot{x}^{(k)} = A\cdot x^{(k)}$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad x^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t}$$

$$\dot{x}^{(1)} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t}$$

$$A\cdot x^{(1)} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{-t}$$

if $X^{(1)}, X^{(2)} \dots X^{(k)} \in \mathbb{R}^{n\times 1}$ are sol$_n$. (15)

$$\varphi(t) = \sum_{i=1}^{k} c_i X^{(i)}$$

if $X^{(1)}, X^{(2)}, \dots, X^{(n)}$

$$W = [X^{(1)}, X^{(2)} \dots X^{(n)}]$$

$|W| \neq 0 \Rightarrow$ my sol$_n$s are L.I.

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{cases} \to X^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} \\ \to X^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} \end{cases}$$

$$W(X^{(1)}, X^{(2)}) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

$$|W| = 0 \Rightarrow e^{-t} \cdot e^{-2t} \neq 0$$

How to find $X^{(1)}, X^{(2)} \dots$

$X = e \cdot e^{\lambda t}$, $e \in \mathbb{R}^{n\times 1}$, $\lambda \in \mathbb{R}$ or $\mathbb{C}$

$(A - \lambda I) \cdot e = 0$, $|A - \lambda I| = 0$

- $\lambda_1 \neq \lambda_2 \neq \lambda_3 \dots$

$\lambda_1 \to e^{(1)} \to X^{(1)} = e^{(1)} e^{\lambda_1 t}$

$\lambda_2 \to e^{(2)} \dots \to X^{(2)} = e^{(2)} e^{\lambda_2 t}$

L.C.

$\bullet\ \lambda_1 = \lambda_2 = \lambda$ $\quad X^{(1)} = e^{(1)} \cdot e^{\lambda t}$ (16)

$\downarrow$ $\quad X^{(2)} = t \cdot e^{(1)} \cdot e^{\lambda t}$

$e^{(1)}$ $\quad \dot{x} = Ax$

$$\dot{X}^{(2)} = \underbrace{e^{(1)} \cdot \lambda \cdot e^{\lambda t} \cdot t}_{} + e^{(1)} \cdot e^{\lambda t} = \underbrace{A \cdot t e^{(1)} \cdot e^{\lambda t}}_{}$$

$+ A \cdot C$

$$a \cdot t + b = c \cdot t + d.$$

$$a = c, \quad b = d.$$

$$e^{(1)} \lambda \cancel{e^{\lambda t}} = A \cdot e^{(1)} \cancel{e^{\lambda t}} \Rightarrow (A - \lambda I) \cdot e^{(1)} = 0$$

$$e^{(1)} \cdot \cancel{e^{\lambda t}} = 0 \Rightarrow e^{(1)} = 0$$

$$X^{(2)} = t e^{\lambda t} e^{(1)} + e^{\lambda t} \cdot e^{(2)}$$

$$\dot{X}^{(2)} = \cancel{t}\lambda e^{\lambda t} \cdot e^{(1)} + e^{\lambda t} \cdot e^{(1)} + \lambda e^{\lambda t} \cdot e^{(2)}$$

$$A \cdot X^{(2)} = A \cdot t e^{\lambda t} \cdot e^{(1)} + A \cdot e^{\lambda t} e^{(2)}$$

$$\lambda \cdot \cancel{e^{\lambda t}} e^{(1)} = A \cdot \cancel{e^{\lambda t}} e^{(1)} \Rightarrow (A - \lambda I) e^{(1)} = 0$$

$$\cancel{e^{\lambda t}} \cdot e^{(1)} + \lambda \cancel{e^{\lambda t}} \cdot e^{(2)} = A \cancel{e^{\lambda t}} e^{(2)}$$

$$e^{(1)} = (A - \lambda I) \cdot e^{(2)}$$

I know $\quad (A - \lambda I) e^{(1)} = 0 \Rightarrow$

$$(A - \lambda I)^2 \cdot e^{(2)} = 0$$

If $\lambda_1 = \lambda_2 = \lambda_3 = \dots = \lambda_k$

$$(A - \lambda I)^k \cdot e^{(k)} = 0$$

$\dot{x} = A \cdot x \Rightarrow e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$ (17)

$P = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ $|P - \lambda I| = 0$

$a \neq b$

$\begin{vmatrix} a-\lambda & 0 \\ 0 & b-\lambda \end{vmatrix} = 0$

$(a-\lambda)(b-\lambda) = 0$

$\lambda_1 = a \quad , \lambda_2 = b$

• $\lambda = a$ $(P - \lambda I) \cdot e = 0$

$(P - aI) \cdot e = 0$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} a-a & 0 \\ 0 & b-a \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$0 \cdot v_1 + 0 v_2 = 0$

$0 \cdot v_1 + (b-a) \cdot v_2 = 0$

$v_2 = 0 \quad , v_1 = \text{any} = 1$

• $\lambda = b \Rightarrow e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2 L.I. eigenvectors

$$e^{Pt} = I + Pt + \frac{(Pt)^2}{2!} + \frac{(Pt)^3}{3!} + \dots$$

$$= I + Pt + \frac{P^2 t^2}{2!} + \frac{P^3 \cdot t^3}{3!}$$

$$P^2 = P \cdot P = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$$

$$P^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$$

$$e^{Pt} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} at & 0 \\ 0 & bt \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} a^2t^2 & 0 \\ 0 & b^2t^2 \end{bmatrix} + \frac{1}{3!} \begin{bmatrix} a^3t^3 & 0 \\ 0 & b^3 \cdot t^3 \end{bmatrix} + \dots$$

elements 2, 3: $= 0$

$$1: \; 1 + at + \frac{1}{2!}(at)^2 + \frac{1}{3!}(at)^3 + \dots$$

$$= e^{at}$$

$$4: \; = e^{bt}$$

$$e^{Pt} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}$$

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$$P = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & a_k \end{bmatrix} \to \begin{array}{l} \lambda_1 = a_1 \\ \lambda_2 = a_2 \\ \lambda_3 = a_3 \\ \vdots \\ \lambda_k = a_k \end{array}$$

$$e_1 = [1 \ 0 \ 0 \ \cdots \ 0]^T$$

$$e_2 = [0 \ 1 \ 0 \ \cdots \ 0]^T$$

$$\vdots$$

$$e_n = [0 \ \cdots \cdots \ 1]^T$$

$$P = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \qquad \lambda_1 = \lambda_2 = a$$

$$(P - aI)e = 0 \iff$$

$$\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} - \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \right) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$0 \cdot v_1 + 0 \cdot v_2 = 0$$

$$v_1 = \text{any} \qquad v_2 = \text{any}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2 L.I. eigenvectors

$$e_3 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$$

$$|P - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} a-\lambda & 1 \\ 0 & a-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$(a-\lambda)^2 + 0 = 0 \Rightarrow$$

$$\lambda_1 = \lambda_2 = a.$$

$$(P - aI) \cdot e = 0$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

1 L.I. eign.

$$0 \cdot v_1 + v_2 = 0$$

$$v_2 = 0, \quad v_1 = \text{any} = 1. \qquad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boxed{\text{if } v_1 = 5 : \quad 5 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}}$$

$$e = (P - aI) \cdot b$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Rightarrow \begin{array}{l} 1 = 0 \cdot b_1 + b_2 \\ 0 = 0 \\ b_1 = \text{any} = 1 \\ b_2 = 1 \end{array}$$

$$b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$\Downarrow$

$\lambda_1 = a$

$\lambda_2 = b$

2 L.I.

$e_1 = [1 \quad 0]^T$

$e_2 = [0 \quad 1]^T$

$$e^{Pt} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}$$

$$P = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

$\Downarrow$

$\lambda_1 = \lambda_2 = a$

2 L.I.

$e_1 = [1 \quad 0]^T$

$e_2 = [0 \quad 1]^T$

$$e^{Pt} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{at} \end{bmatrix}$$

$$P = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$$

$\lambda_1 = \lambda_2 = a$

$\boxed{1 \quad \text{L.I.}}$

$e = [1 \quad 0]^T$

$e^{Pt} = ?$

e.g.

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

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$$e^{Pt} = I + Pt + \frac{1}{2}(Pt)^2 + \frac{1}{3!}(Pt)^3$$

$$P^2 = P \cdot P = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \cdot \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2a \\ 0 & a^2 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} a^2 & 2a \\ 0 & a^2 \end{bmatrix} \cdot \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2 \\ 0 & a^3 \end{bmatrix}$$

$$P^k = \begin{bmatrix} a^k & k a^{k-1} \\ 0 & a^k \end{bmatrix}$$

$$e^{Pt} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} at & t \\ 0 & at \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} a^2t^2 & 2at^2 \\ 0 & a^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} a^3t^3 & 3a^2t^3 \\ 0 & a^3t^3 \end{bmatrix}$$

ele. 1, 4 $= 1 + at + \frac{1}{2!}(at)^2 + \frac{1}{3!}(at)^3 = e^{at}$

ele: 2: $0 + t + \frac{1}{2!}2at^2 + \frac{1}{3!}3a^2t^3 + \dots$

$$= t\left(1 + \frac{1}{\cancel{2}}\cancel{2}\,at + \frac{1}{1\cdot 2\cdot \cancel{3}}\cancel{3}\,a^2t^2\right)\dots$$

$$t\left(1 + at + \frac{1}{2!}a^2t^2 + \frac{1}{3!}a^3t^3 + \dots\right)$$ 22

$$= t\,e^{at}$$

$$e^{Pt} = \begin{bmatrix} e^{at} & te^{at} \\ 0 & e^{at} \end{bmatrix}$$

$$P = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$|P - \lambda I| = 0 \Rightarrow$$

$$\begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)^2 + b^2 = 0$$

$$\lambda^2 - 2a\lambda + (a^2 + b^2) = 0$$

$$\dots \quad \lambda_{1,2} = a \pm bi$$

$$e_1 = [1 \quad i]^T$$ 2 L.I. eig.

$$e_2 = [1 \quad -i]^T$$

$$e^{Pt} = e^{at} \cdot \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}$$