Revision

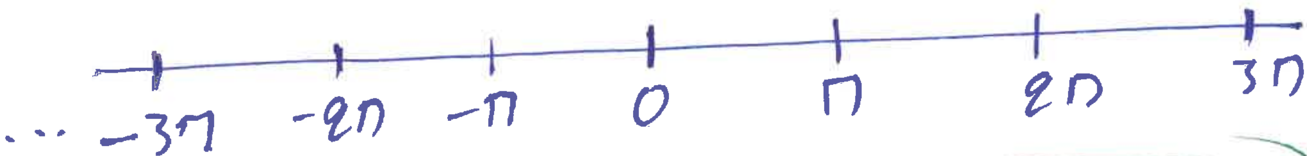
$$\dot{x} = f(x, u)$$

E.P. $\dot{x} = 0 \Rightarrow f(x_{EP}) = 0 \Rightarrow \dots x_{EP}$

e.g. $\dot{x} = \sin(x)$

$\dot{x} = 0 \Rightarrow \sin(x) = 0 \Rightarrow x_{EP} = \pi \pm k\pi$

$k = 0, 1, 2, \dots$

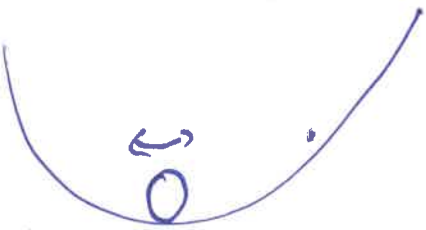


Linear

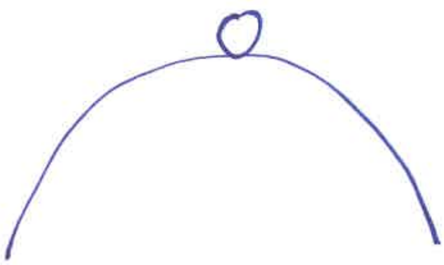
$\dot{x} = Ax + Bu$

$x_{EP} = -A^{-1} \cdot B \cdot u$

E.P. are invariant



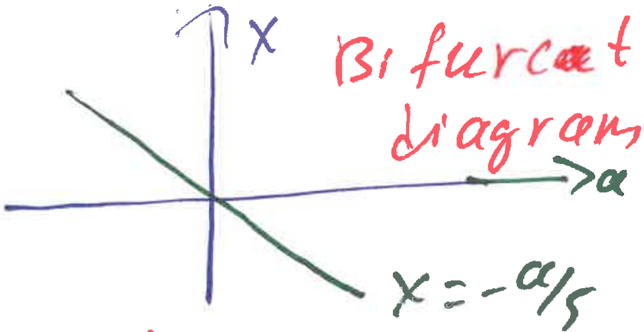
STABLE



UNSTABLE

$\dot{x} = 5x + a$

$\dot{x} = 0 \Rightarrow x = -a/5$

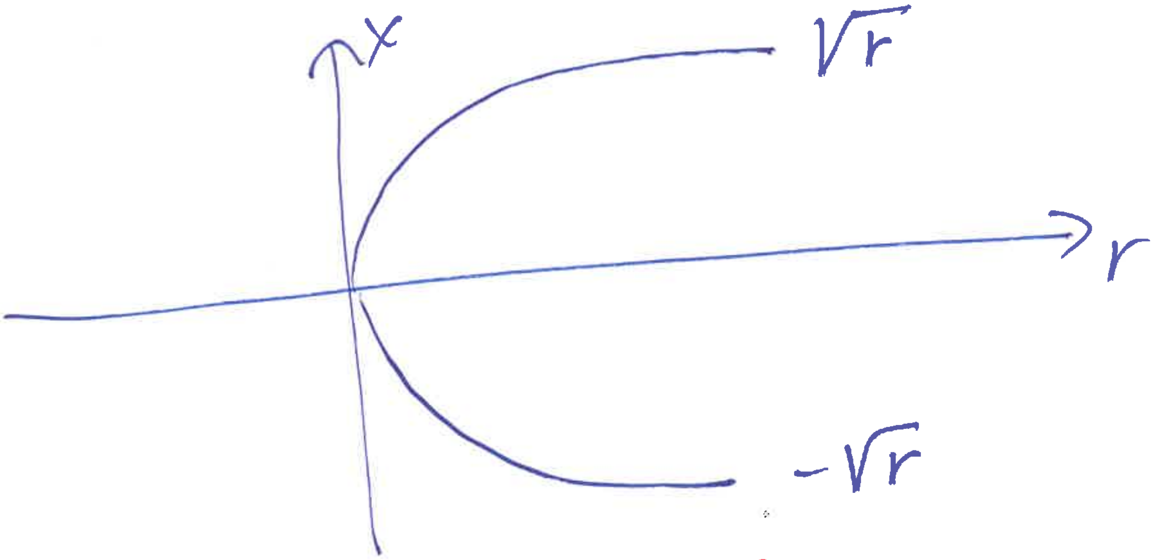


Change Location of F.P

$\dot{x} = -r + x^2 \Rightarrow x_{FP} = \pm\sqrt{r}\ ,\ r>0$

No F.P. if $r<0$

$x_{FP} = 0$, 1 F.P. if $r=0$



Change the number of F.P.s

---

start / finish on F.P.

Attracted → stable

repelled → unstable

$$\begin{rcases} \dot{x}_1 = x_1 - x_2 \\ \dot{x}_2 = x_1^2 + x_2^2 - 2 \end{rcases} \Rightarrow \begin{rcases} x_1 - x_2 = 0 \\ x_1^2 + x_2^2 - 2 = 0 \end{rcases} \Rightarrow$$

2 unknowns

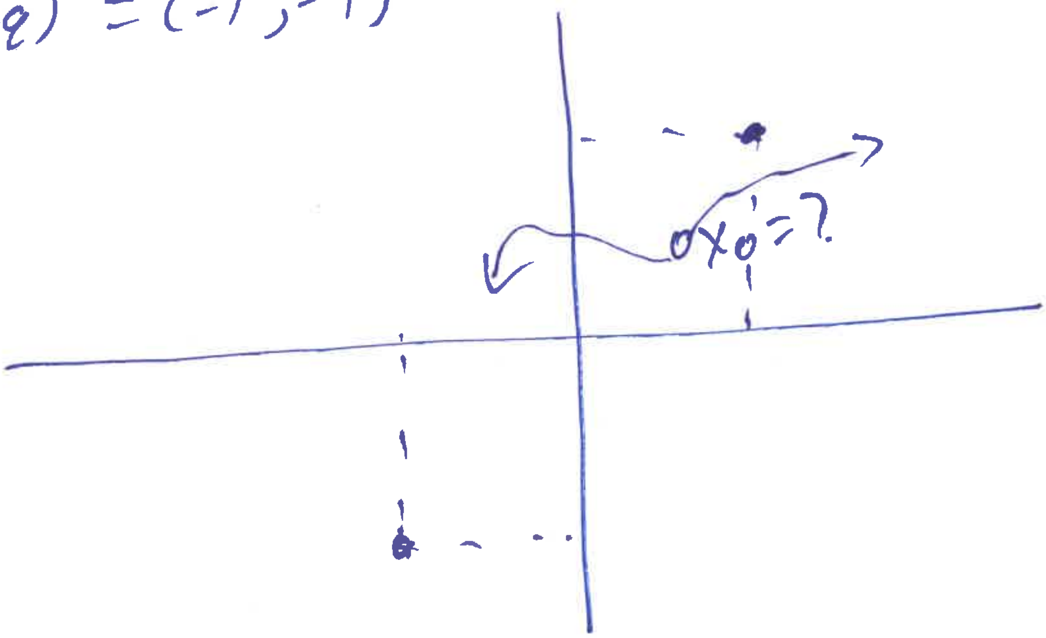
2 eqns.

$x_1 = x_2$

$$x_1^2 + x_1^2 - 2 = 0 \Rightarrow x_1 = \pm 1$$

$(x_1, x_2) = (1, 1)$

$(x_1, x_2) = (-1, -1)$



$$\dot{x}_1 = x_1^2 x_2 + 3 \cdot x_1 x_2 - 10 x_2$$

$$\dot{x}_2 = x_1^2 x_2 - 4 x_1$$

$$\Rightarrow x_2 \left(x_1^2 + 3x_1 - 10 \right) = 0$$

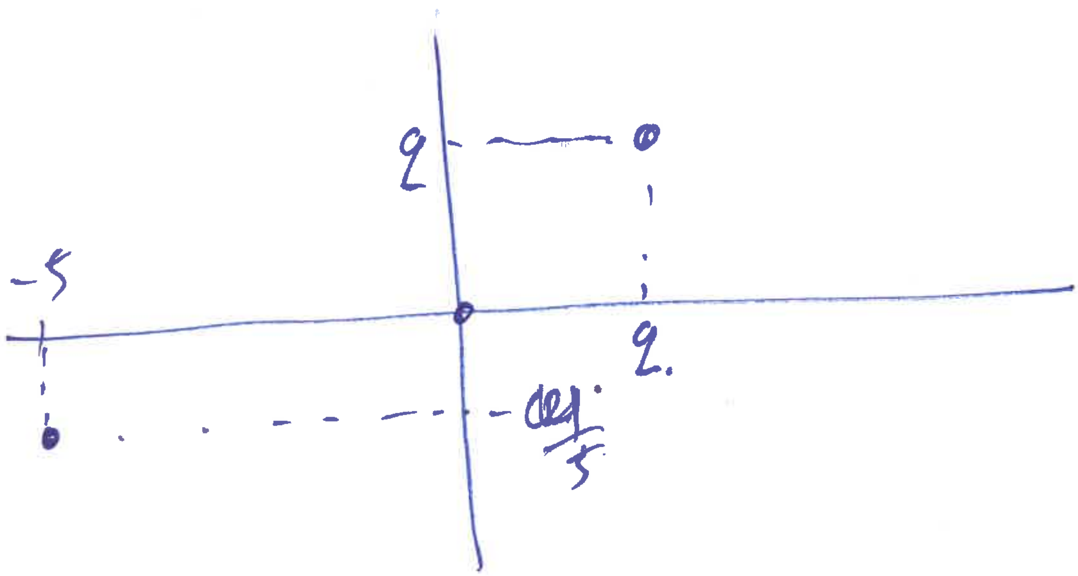
OR $x_2 = 0$

OR $x_1^2 + 3x_1 - 10 = 0 \rightarrow x_1 = 2$, $x_2 = -5$

- $x_2 = 0 \Rightarrow x_1^2 \cdot 0 - 4 \cdot x_1 = 0 \Rightarrow (x_1, x_2) = (0, 0)$
- $x_1 = 2 \Rightarrow 4 \cdot x_2 - 8 = 0 \Rightarrow x_2 = 2$

  $(x_1, x_2) = (2, 2)$
- $x_1 = -5 \Rightarrow \ldots\ldots \; x_2 = -4/5$

  $(x_1, x_2) = (-5, -4/5)$



Taylor Series

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin(x) \simeq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}, \quad x \in [-2, 2]$$

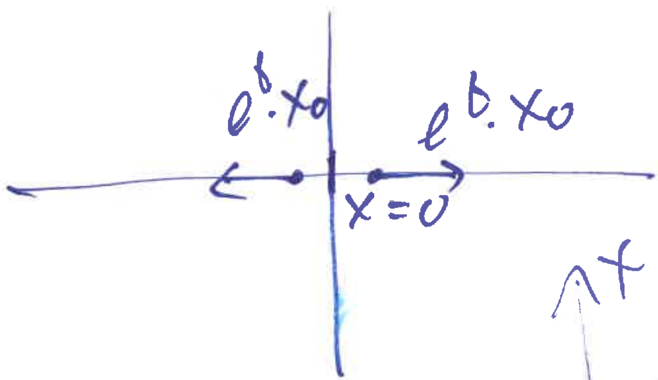
$$\sin(x) \simeq x - \frac{x^3}{3!} + \frac{x^5}{5!}, \quad x \in [-1.8, 1.8]$$

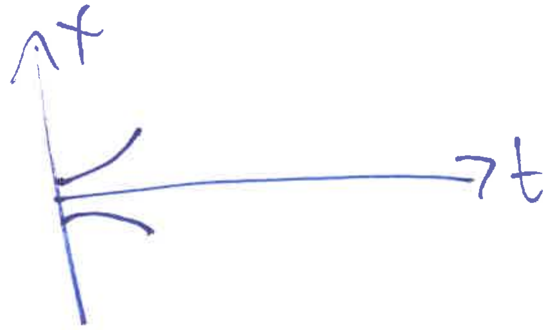
$$\sin(x) \simeq x - \frac{x^3}{3!}, \quad x \in [-1, 1]$$

$$\sin(x) \simeq x, \quad x \in [-0.5, 0.5]$$

$\sin(x) \simeq x$, Locally at zero

$$\dot{x} = \sin(x) \xrightarrow{\text{Locally}} \dot{x} = x \Rightarrow x = e^t . x_0$$





$\dot{x} = f(x)$, ~~x_0~~ $x \in \mathbb{R}^{n\times 1}$

$f \in \mathbb{R}^{n\times 1}$

Taylor Series $f(x)$ at around x_0

$$f(x) = f(x_0) + \frac{\partial f}{\partial x}\Big|_{x=x_0}(x-x_0) + \cancel{\frac{\partial^2 f}{\partial x^2}\Big|_{x=x_0}\cdot\frac{(x-x_0)^2}{2!}} + \dots$$

$$f(x) \simeq \underbrace{f(x_0)}_{\text{const.}} + \underbrace{\frac{\partial f}{\partial x}\Big|_{x=x_0}}_{\text{fun of } x_0 \text{ const}} \underbrace{(x-x_0)}_{\text{fun with respect } x}$$

e.g. $f(x) \simeq 5 + 3\cdot(x-2)$

$\dot{x} = f(x)$

$\dot{x} = 5+3(x-2)$ Locally

For any x_0

$$\dot{x} = f(x) \simeq f(x_0) + \frac{\partial f}{\partial x}\bigg|_{x=x_0} (x - x_0)$$

$x_0 = x_{EP} \rightarrow$ T.S approximation of f
around a F.P

*

$$\dot{x} = f(x) \simeq \cancelto{0}{f(x_{EP})} + \frac{\partial f}{\partial x}\bigg|_{x=x_{EP}} (x - x_{EP})$$

$$\dot{x} = \frac{\partial f}{\partial x}\bigg|_{x=x_{EP}} (x - x_{EP})$$

$$\dot{x} = \sin(x) = \cancelto{0}{\sin(0)} + \frac{\partial(\sin x)}{\partial x}\bigg|_{x=0} (x - 0)$$

$x_0 = 0$

$$= 0 + 1 \cdot x \Rightarrow x = e^{t} \cdot x_0$$

$$\dot{x} = \sin x = \cancelto{0}{\sin(\pi)} + \cos(\pi) \cdot (x - \pi)$$

$x_0 = \pi$

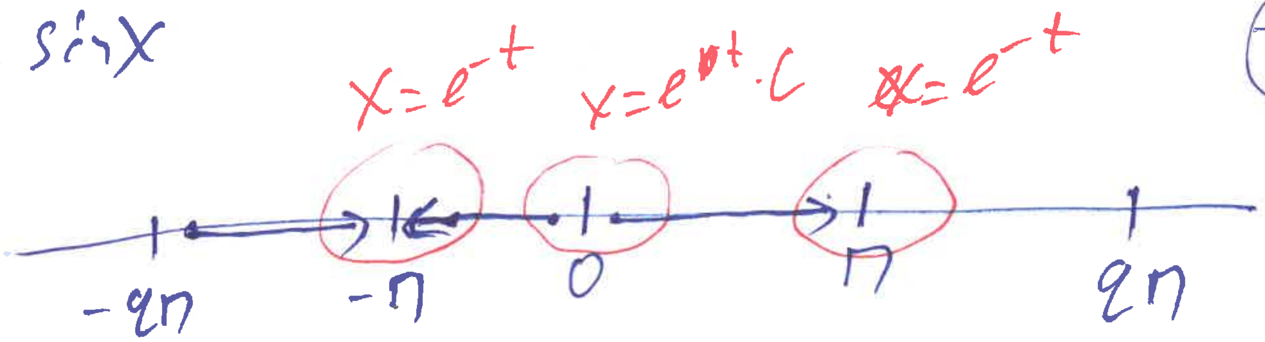
$$\dot{x} = -(x - \pi) \Rightarrow x = e^{-t} \cdot x_0 + \square$$

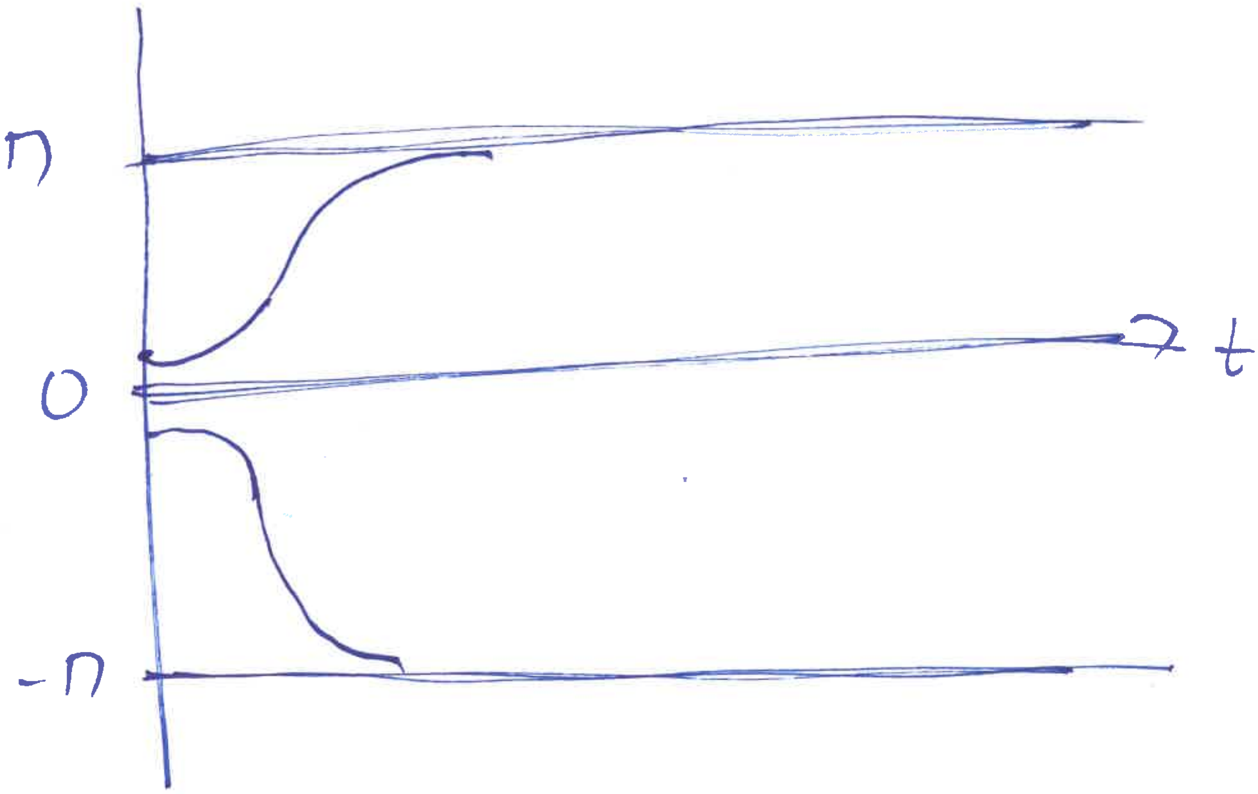
$$\dot{x} = \sin x = \cancelto{0}{\sin(-\pi)} + \cos(-\pi) \cdot (x + \pi)$$

$x_0 = -\pi$

$$\dot{x} = -(x + \pi) \Rightarrow x = e^{-t} \cdot x_0 + \square$$

$\dot{x} = \sin x$

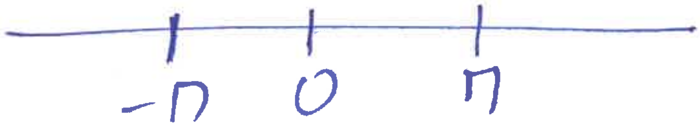




$\dot{x} = \sin x$ $\boxed{\dot{x} = Kx + u}$

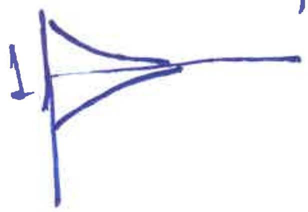
$x_{EP} = \pi \pm k\pi$

$k = 0, 1, 2, \dots$



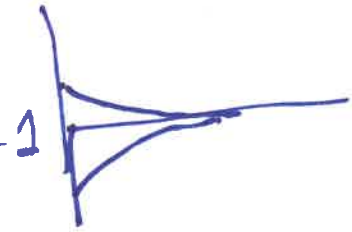
T.S. at $x = 0$ $\dot{x} = x \Rightarrow x = e^{t} \cdot x_0$

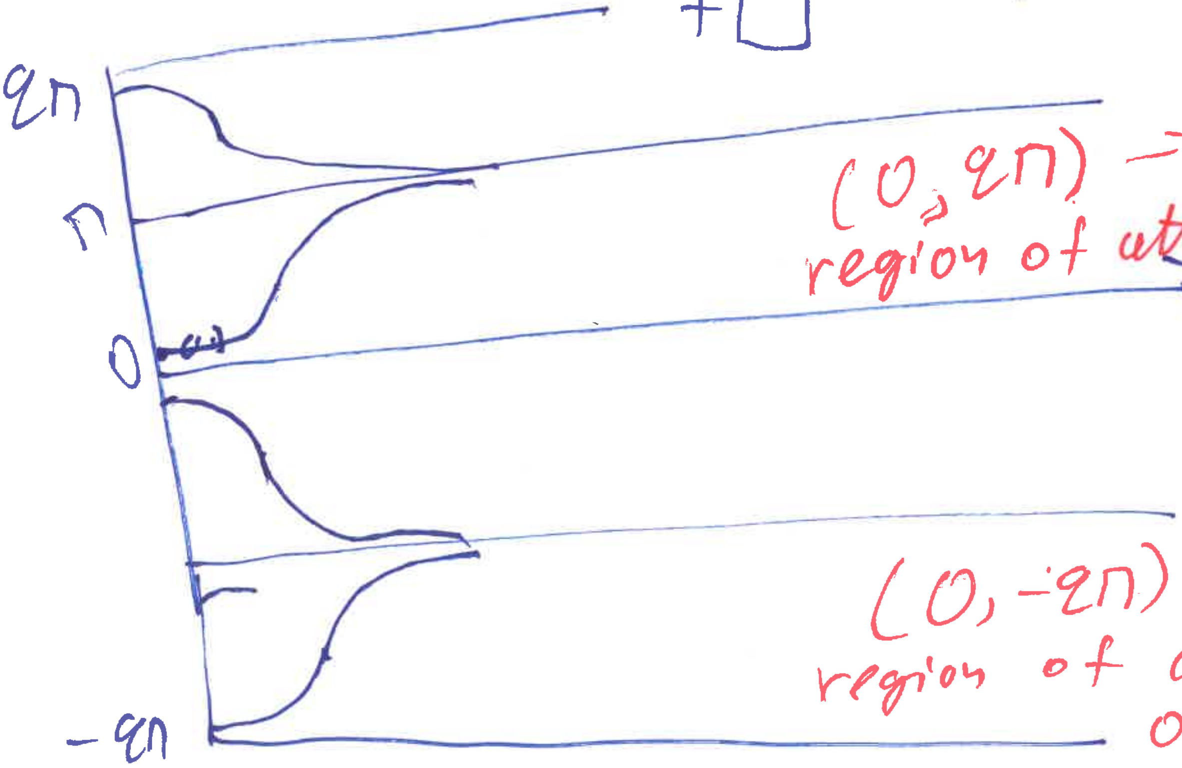
$x = \pi$ $\dot{x} = -x + \pi \to x = e^{-t} x_0 + \square$



$x = -\pi$ $\dot{x} = -x - \pi$

$\to x = e^{-t} \cdot x_0 + \square$





$(0, 2\pi) \to \pi$
region of attraction of π

$(0, -2\pi) \to \pi$
region of attraction of $-\pi$

79

$n=2$

$$x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad f=\begin{bmatrix} f_1(\,) \\ f_2(\,) \end{bmatrix}$$

$$\dot{x}=f(x)$$

$$\dot{x} \approx \left.\frac{\partial f}{\partial x}\right|_{x=x_{EP}} (x-x_{EP})$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=EP} = \left.\underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}}_{2\times 2}\right|_{x=x_{EP}}$$

$$\dot{x} = A \cdot (x-x_{EP})$$

$\to$ Jacobian of f at x_{EP}

$$\frac{d(x-x_{EP})}{dt} = A\,(x-x_{EP})$$

$$\Delta x = x - x_{EP}$$

$$\boxed{\Delta\dot{x} = A \cdot \Delta x}$$

ODE of perturbations around F.P

$\dot{x} = x - y$

$\dot{y} = x + y - 2 \cdot x \cdot y$

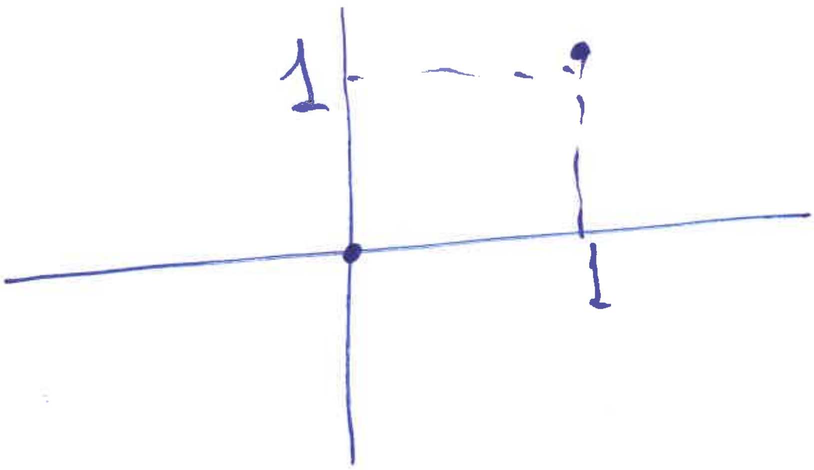
F.P.
Stability
S.S.

$$\left.\begin{aligned} x - y &= 0 \\ x + y - 2 \cdot x \cdot y &= 0 \end{aligned}\right\} \Rightarrow x = y$$

$$x + x - 2 \cdot x \cdot x = 0 \Rightarrow 2x - 2x^2 = 0 \Rightarrow$$

$$x = 0 \quad \text{or} \quad 1$$

$(0,0)$, $(1,1)$



$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$$

$f_1(x,y) = x - y$

$f_2(x,y) = x + y - 2 \cdot x \cdot y$

$$A(FP) = \left.\frac{\partial f}{\partial x}\right|_{x = x_{EP}}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$f_1 = x - y \qquad f_2 = x + y - 2 \cdot x \cdot y$$

$$\frac{\partial f_1}{\partial x} = 1 - 0 \qquad \frac{\partial f_1}{\partial y} = 0 - 1$$

$$\frac{\partial f_2}{\partial x} = 1 + 0 - 2 \cdot y \cdot 1 = 1 - 2y$$

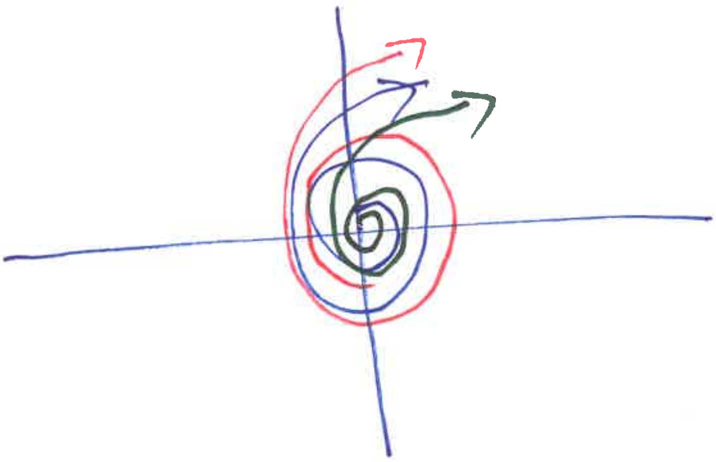
$$\frac{\partial f_2}{\partial y} = 0 + 1 - 2x = 1 - 2x$$

$$A = \begin{bmatrix} 1 & -1 \\ 1-2y & 1-2x \end{bmatrix}$$

$$A(0,0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\Delta\dot{x} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \Delta x$$ ODE of pert. around (0,0)

$\lambda = 1 \pm 2i$ unstable Focus



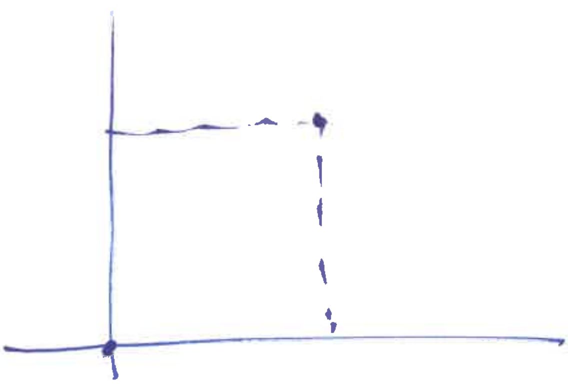
cross check

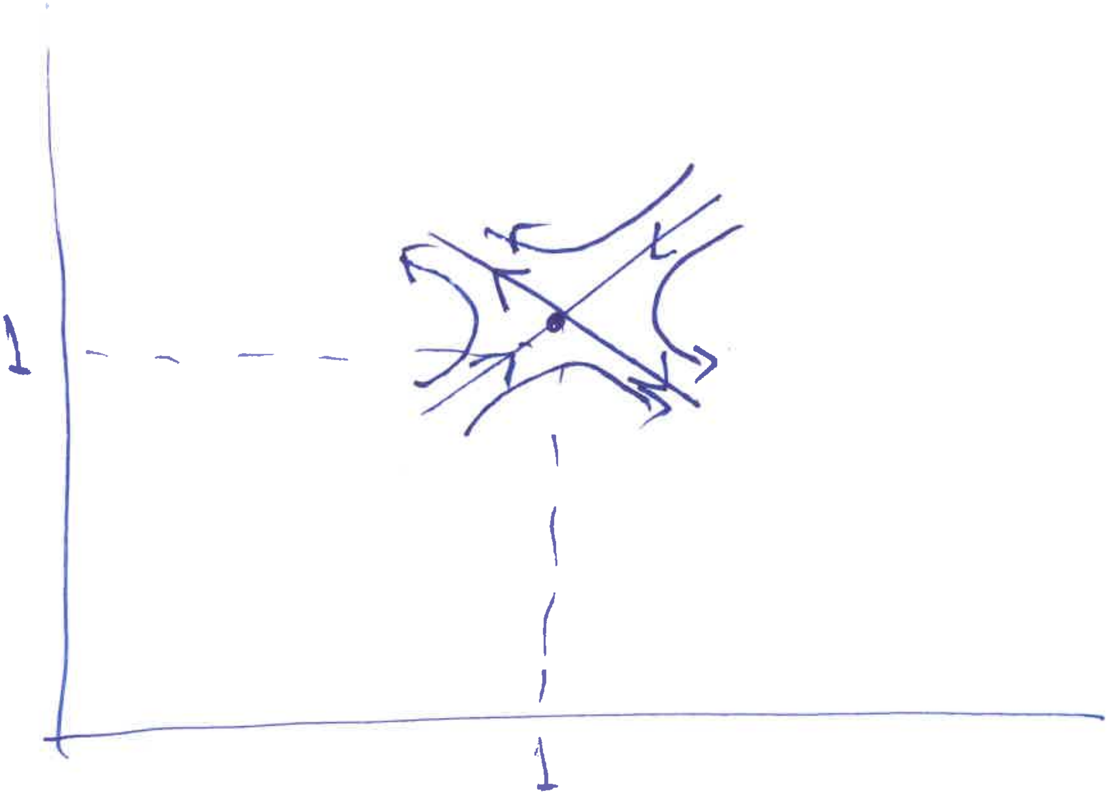
$$A(1,1) = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

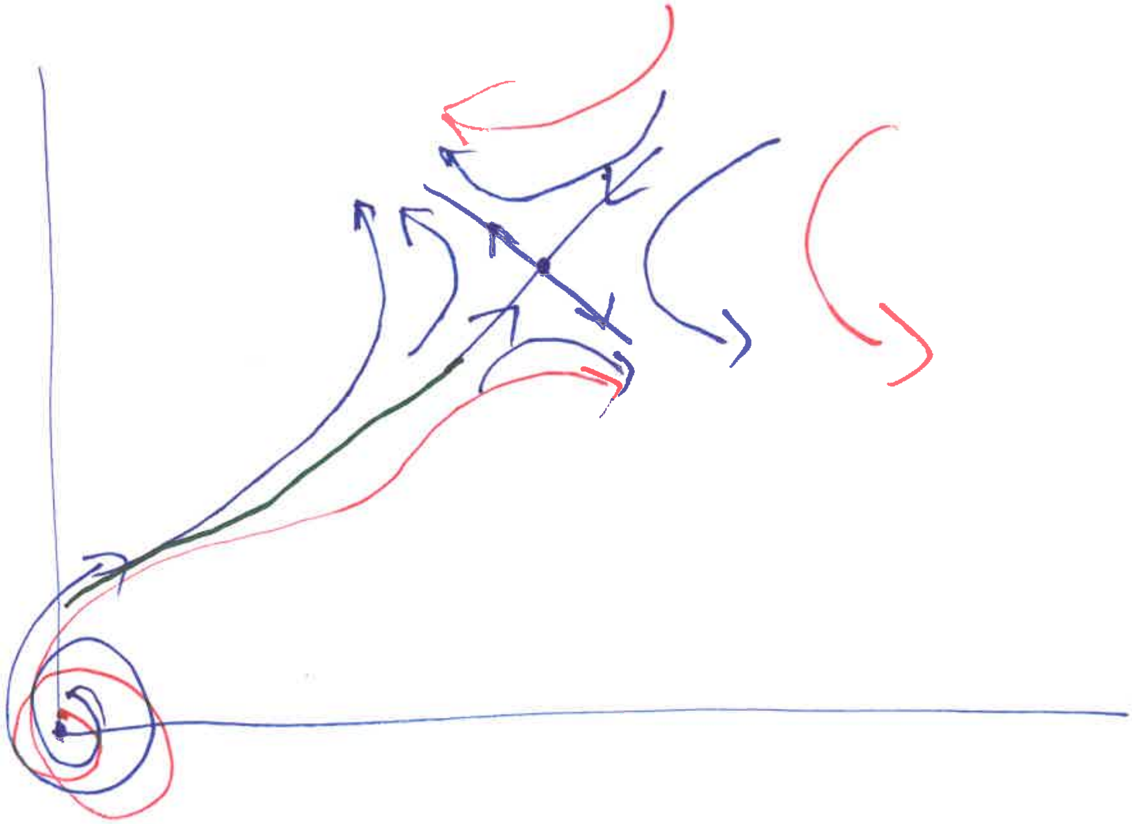
$\lambda_1 = 1.41,\ e_1 = \begin{bmatrix} -2.41 \\ 1 \end{bmatrix}$

$\lambda_2 = -1.41,\ e_2 = \begin{bmatrix} -1 \\ -2.41 \end{bmatrix}$

Saddle




1
1



$$\dot{x} = y \cdot e^{y}$$

homework

17

$$\dot{y} = 1 - x^{2}$$