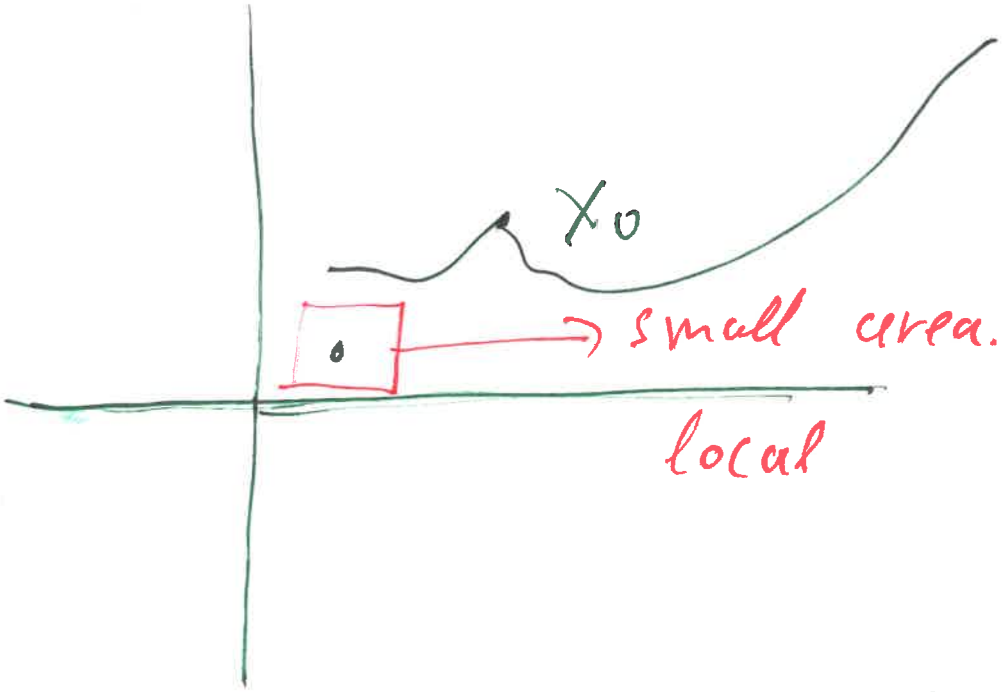
Revision

vector field

$\dot{X} = f(X)$, $X, f \in \mathbb{R}^n$

Study its state space.



- $X_{EP} = ?$ $\quad \dot{x} = f(x) = 0 \Rightarrow \dots \quad X_{EP} = \dots$
- Approximate f at X_{EP}

$$\Delta \dot{x} = A \cdot \Delta X$$

Jacobian of f

$$A = \left.\frac{\partial f}{\partial x}\right|_{X_{EP}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \vdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

→ stability → eigenvalues

→ S.S. → eigenvectors

$$\left.\begin{aligned}\dot{x} &= x - y\\ \dot{y} &= x + y - 2\cdot x\cdot y\end{aligned}\right\} \Rightarrow \begin{aligned}(x,y) &= (0,0)\\ (x,y) &= (1,1)\end{aligned}$$

$$A = \begin{bmatrix} 1 & -1 \\ 1-2y & 1-2x \end{bmatrix} \nearrow A(0,0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$\lambda = 1 \pm 2i$

unstable focus

$$\downarrow$$

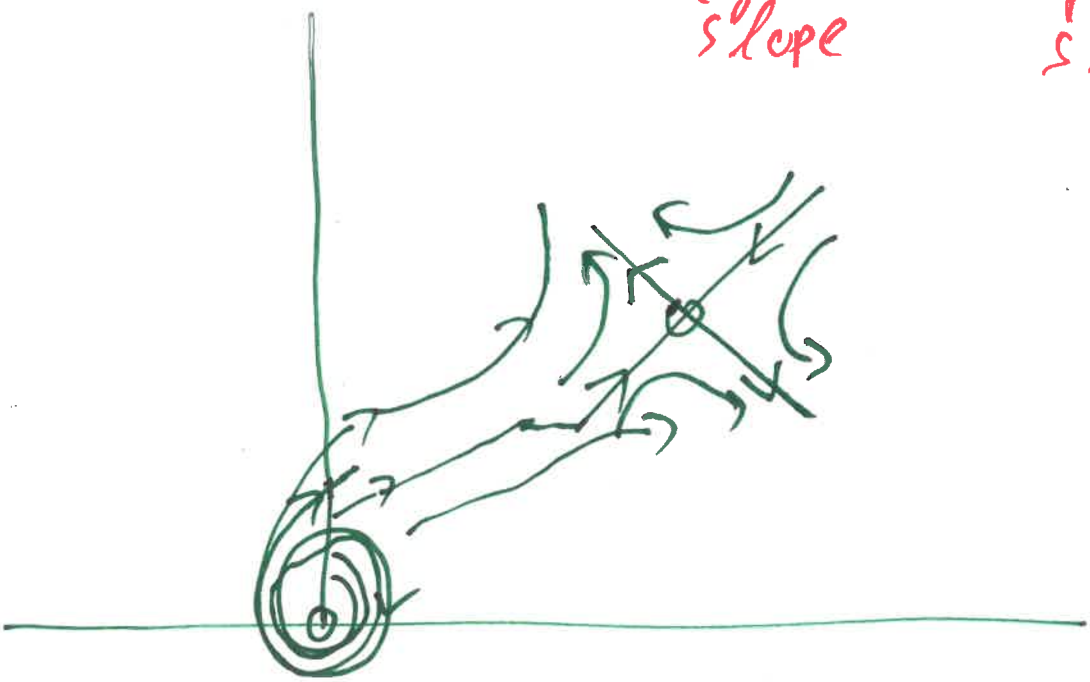
$$A(1,1) = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

$\lambda_1 = 1.41$, $\lambda_2 = -1.41$

saddle

$e_1 = \begin{bmatrix} -2.41 \\ 1 \end{bmatrix}$, $e_2 = \begin{bmatrix} -1 \\ -2.41 \end{bmatrix}$

negative slope   positive slope



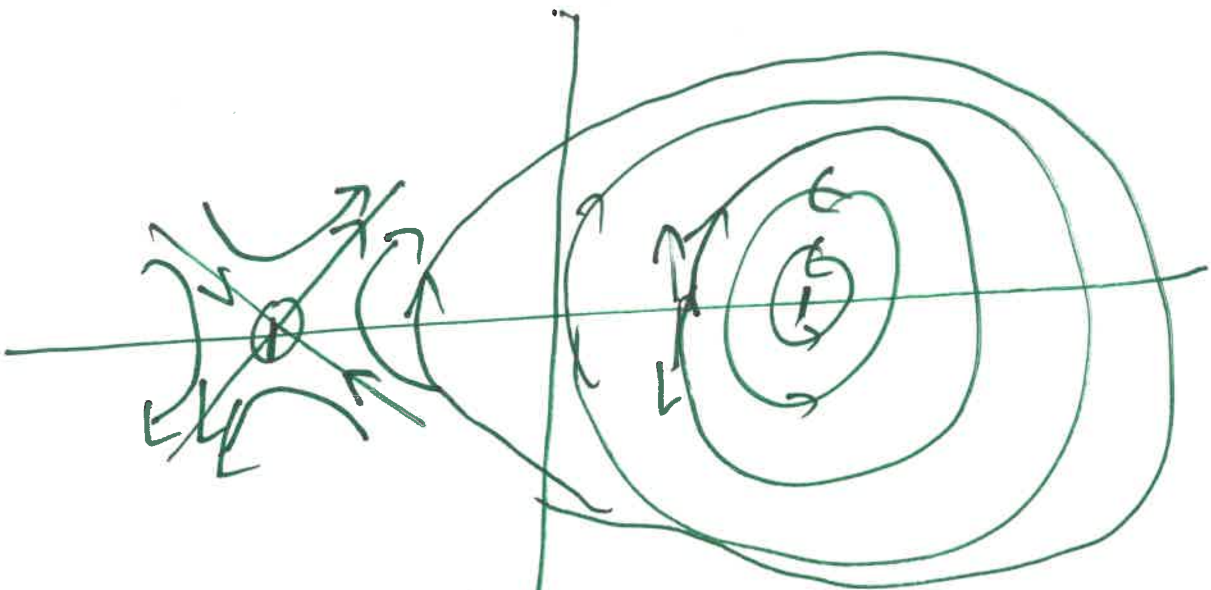
$\dot{x} = y \cdot e^{y} = f_1(x,y)$

$\dot{y} = 1 - x^2 = f_2(x,y)$

$f_1 = 0 \Rightarrow y \cdot e^{y} = 0 \Rightarrow y = 0$

$f_2 = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$

$(x, y) = (1, 0)$, $(x, y) = (-1, 0)$



$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & e^{y} \cdot 1 + y e^{y} \\ -2x & 0 \end{bmatrix}$$

$A(1,0) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \rightarrow \lambda = \pm 1.41 \cdot i$ centre

$A(-1,0) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ $\rightarrow \lambda_1 = -1.4,\ e_1 = [-0.5\ \ 0.8]^T$

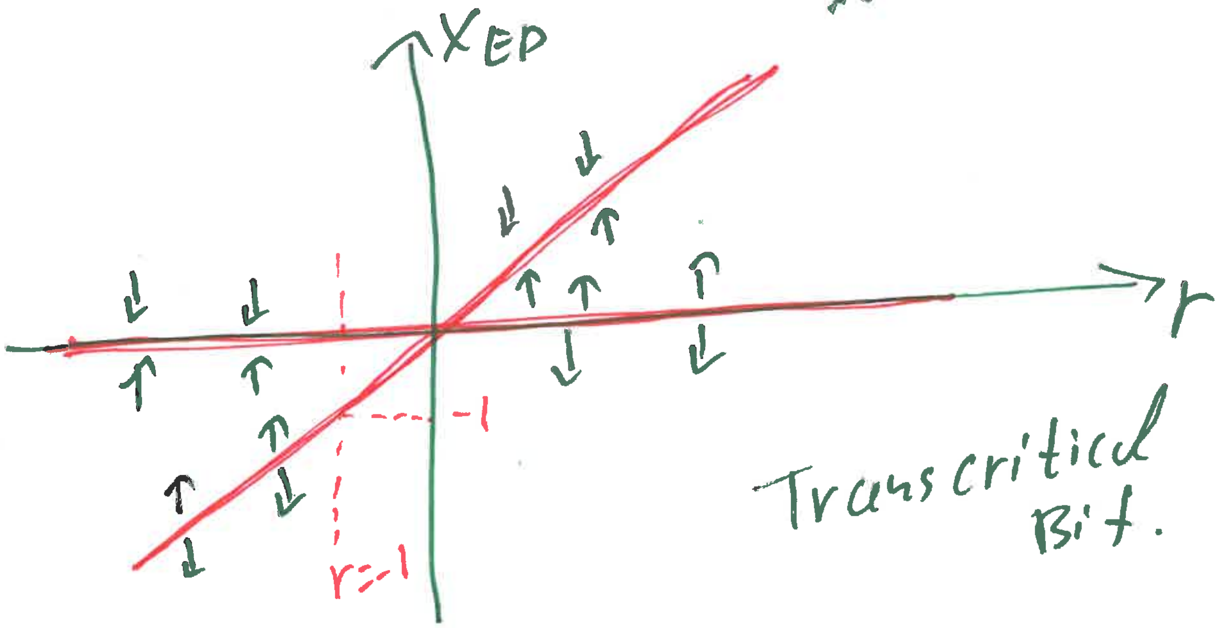
$\rightarrow \lambda_2 = 1.4,\ e_2 = [0.5\ \ 0.8]^T$

saddle

Bifurcations

$\dot{x} = r\cdot x - x^2$

$\dot{x} = 0 \Rightarrow r\cdot x - x^2 = 0 \Rightarrow \begin{cases} x = r \\ x = 0 \end{cases}$



Transcritical Bif.

$A = \dfrac{\partial f}{\partial x}\Big|_{x = x_{EP}}$, $f = rx - x^2$

$f(x,r) = rx - x^2$

$A = r - 2x \begin{cases} \xrightarrow{x=r} A = -r \\ \xrightarrow{x=0} A = r \end{cases}$

- $x_{EP} = r$, $A = -r$, $r < 0$, unstable
  $A = -r$, $r > 0$, stable
- $x_{EP} = 0$ $A = r$, $r < 0$ stable
  $A = r$, $r > 0$ unstable

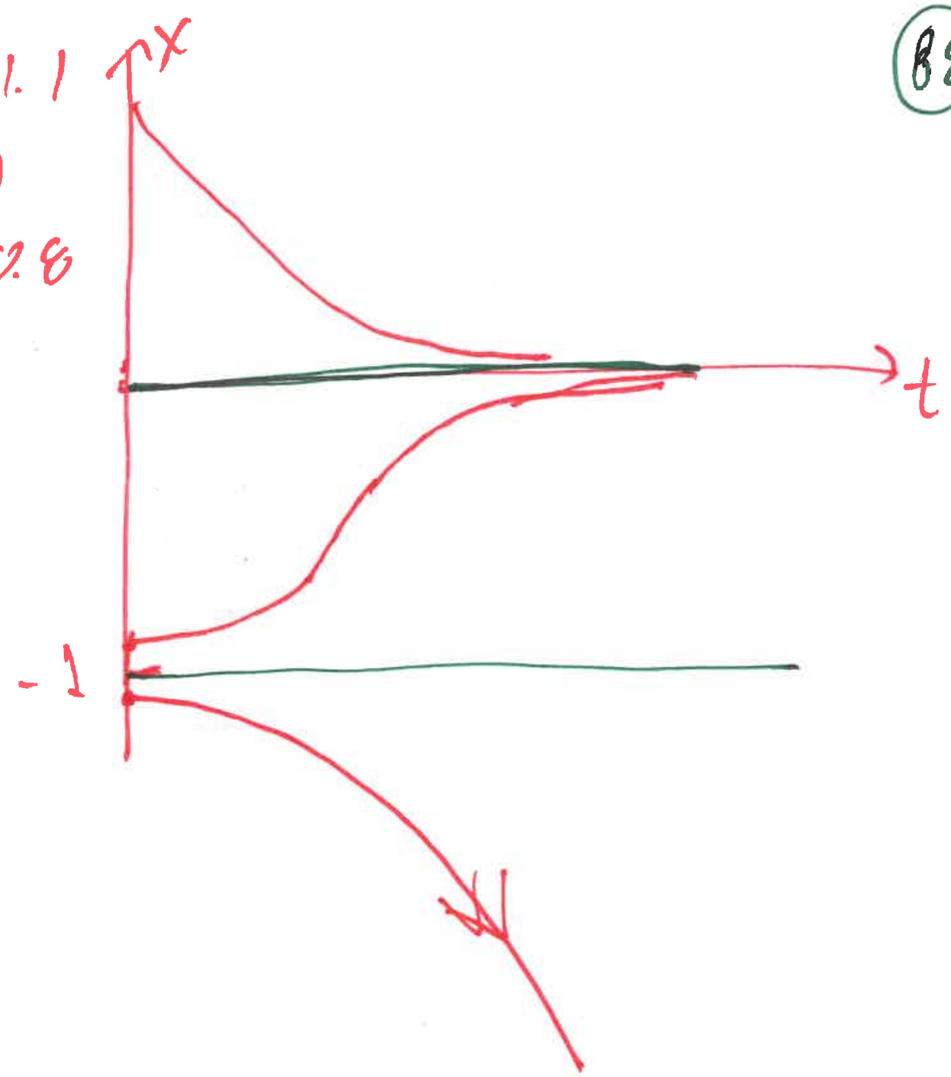
• $r = -1$ $\quad x_0 = -1.1$

$x_0 = -1$

$x_0 = -0.8$

$x_0 = 0$

$x_0 = 1$



$\dot{x} = r + x^2$ Saddle Node Bif.

Pitchfork Bif. → supercritical
→ subcritical

$\dot{x} = rx - x^2$

$\dot{y} = -y.$

H.W.

Lyapunov !!!

Global stability

$\dot{x} = f(x)$, $x, f, \in \mathbb{R}^n$

if $V \ \exists : V(x) \to V(0) = 0$ ($V \in \mathbb{R}$)

$V(x) > 0$

$\dot{V}(x) < 0$

e.g. $\dot{x} = -x + y - x\cdot y^2$

$\dot{y} = -2x - y - x^2 y$

$V(x,y) = x^2 + y^2$

$V(0,0) = 0$

$V(x,y) = x^2 + y^2 > 0$

Chain rule

$$\dot{V} = \frac{\partial V}{\partial x} \cdot \dot{x} + \frac{\partial V}{\partial y} \cdot \dot{y}$$

$$= (2x + 0)\,(-x + y - x\cdot y^2) + 2y\cdot(-2x - y - x^2 y)$$

$$= \underset{\text{H.W.}}{\ldots\ldots} = -2(x+y)^2 - 4\cdot x^2 y^2 < 0$$

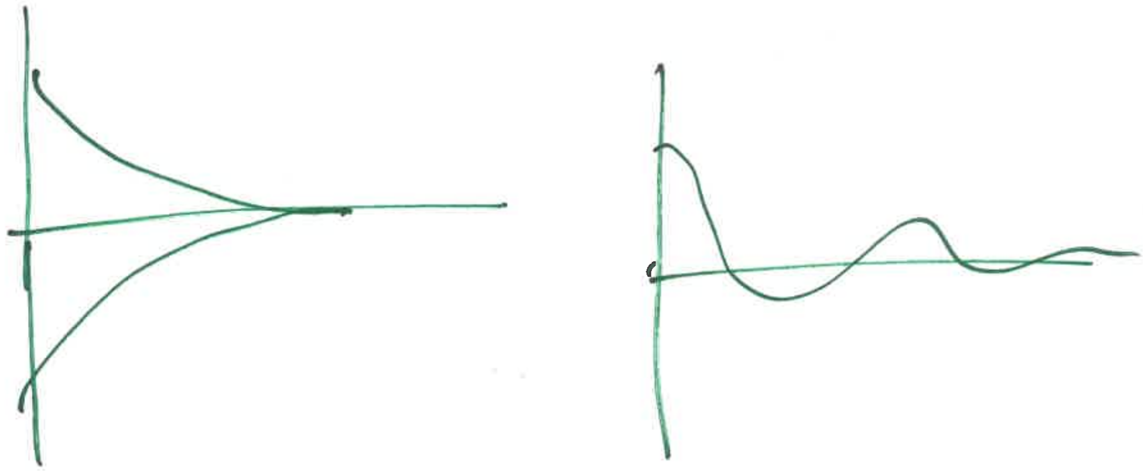
Ch. 5

sem. I, ch. 1

$\ddot{x} + A\dot{x} + Bx = u$

$\downarrow u = 0$

C.E. $r^2 + Ar + B = 0$ $\begin{cases} r_1 = \dots < 0 \\ r_2 = \dots < 0 \end{cases}$ stable



$u \neq 0$
const.

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sem I, ch 4.

$u = -Kx$ stabilise

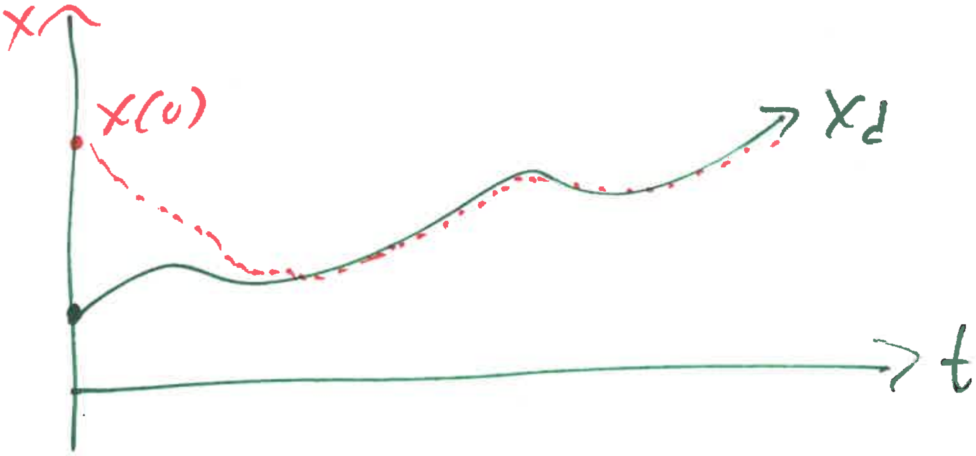
$u = K_1 R - Kx$, $R = \text{const}$

$x_d \to$ Desired trajectory

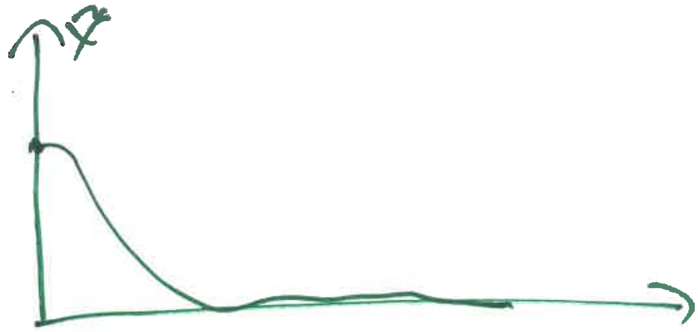
Anything, smooth, e.g. $x_d = \cos t$

$+ e^{-t}\sin t$

$\ddot{x} + A\dot{x} + Bx = u$, $u = ?$: $x \to x_d$.



$\tilde{x} = x - x_d$. $u = ?$: $\tilde{x} \to 0$



$\ddot{x} + A\dot{x} + Bx = u$, Assume A, B are known

$r_1 < 0$ $r_2 < 0$

$x \to x_d(t)$

If somehow $u = \ldots$

$\ddot{\tilde{x}} + A\dot{\tilde{x}} + B\tilde{x} = 0 \to$ ODE of error

As $r_1, r_2 < 0$, stable ODE (of the error)

$\tilde{x} = c_1 \cdot e^{r_1 t} + c_2 e^{r_2 t} \to 0$

$x \to x_d$

I have $\ddot{x} + A\dot{x} + Bx = u$ 86

$\tilde{x} = x - x_d$ $\downarrow u = ?$

I want $\ddot{\tilde{x}} + A\dot{\tilde{x}} + B\tilde{x} = 0$

$\dot{\tilde{x}} = \dot{x} - \dot{x}_d$

$\ddot{\tilde{x}} = \ddot{x} - \ddot{x}_d$

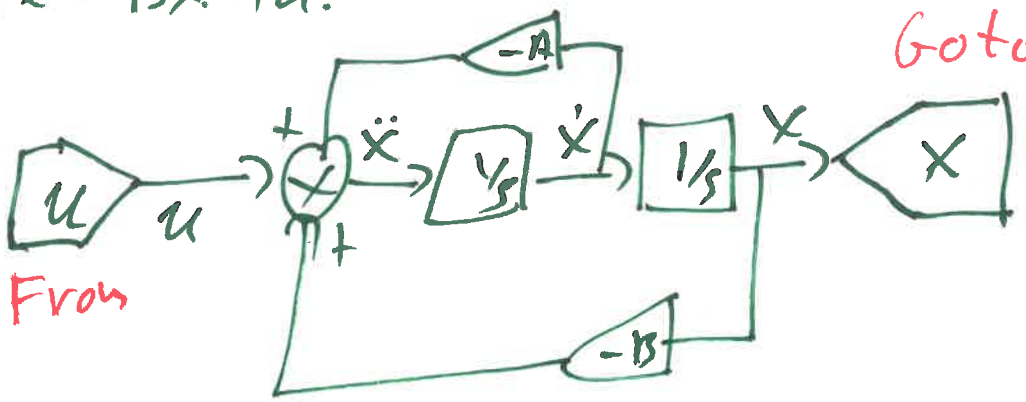
$\Rightarrow \ddot{x} - \ddot{x}_d + A\cdot(\dot{x} - \dot{x}_d) + B(x - x_d) = 0$

$$\ddot{x} + A\dot{x} + Bx = \underbrace{\ddot{x}_d + A\dot{x}_d + Bx_d}_{u}$$

This will work
if A, B are known
$r_1, r_2 < 0$

$\ddot{X} + A\dot{X} + BX = u$ $\qquad u = \ddot{X}_d + A\dot{X}_d + B X_d.$ $\quad$ 87

$\ddot{X} = -A\dot{X} - BX + u.$



$X_d = 1 + e^{-0.01t}$ $\quad$ assume.

$\dot{X}_d = -0.01 \cdot e^{-0.01t}$

$\ddot{X}_d = 10^{-4}\, e^{-0.01t}$