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Section 2 Transformations

2.1 Forward transformation

The final step is now to combine all these rotations and translations. To find the effect of translations and/or rotations simply multiply the matrices that correspond to them. All the transformations are with respect to the original frame. For example if the frame was rotated along the z-axis and then translated along the vector $r(r_x, r_y, r_z)$ the final position is:

$$\text{Trans}(r_x, r_y, r_z)\text{Rot}(z, \phi) = \begin{bmatrix} 1 & 0 & 0 & r_x \\ 0 & 1 & 0 & r_y \\ 0 & 0 & 1 & r_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 & 0 \\ \sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 & r_x \\ \sin(\phi) & \cos(\phi) & 0 & r_y \\ 0 & 0 & 1 & r_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now this matrix fully defines the new reference frame with respect to the original. **Note that the last factor of the above product is the operation that took place first.** Finally this transformation from one reference frame to another is usually denoted as: ${}^R\mathbf{T}_N = \text{Trans}(r_x, r_y, r_z)\text{Rot}(z, \phi)$, and it means the transformation from the original reference frame to the new frame, always with respect to the original one.

2.1.1 Example 1

A resultant transformation is given by the next equation. Find the resultant transformation matrix and draw the new frame:

$${}^R\mathbf{T}_N = \text{Rot}(y, 90^\circ)\text{Trans}(y, \alpha)$$

2.1.2 Solution: Example 1

Initially the new or moving frame is attached on the original. Hence the two origins coincide. The original reference frame and the first operation i.e. the translation along the y-axis for “a” units is:

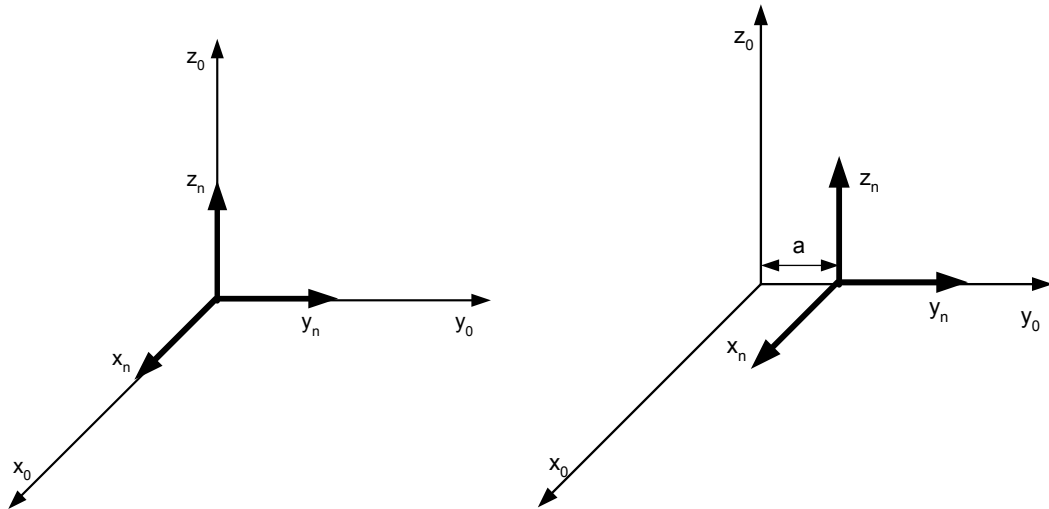


Figure 2.1 Original reference frame and y-axis translation

The translation vector is $\mathbf{r} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}$.

Hence, the transformation matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Thus, if there was a vector with coordinates (0,0,10) in the original frame, and if this vector is translated along the y-axis “a” units with the new frame then the new coordinates of this vector would be (0,a,10) according to the old reference frame:

$$\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} = \begin{bmatrix} x_o \\ y_o + a \\ z_o \end{bmatrix} \Rightarrow \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ 10 \end{bmatrix}$$

To distinguish between the two reference frames; the original reference frame is denoted as R(x,y,z) and the new as N(x,y,z). Therefore the original point was R(0,0,10). Then the origin was moved with this point and hence the new point is now R(0,a,10). Be careful: the original point (0,0,10) remains unchanged according to the new reference frame, i.e. N(0,0,10) before and after the translation. To better visualise this see Fig. 1.40:

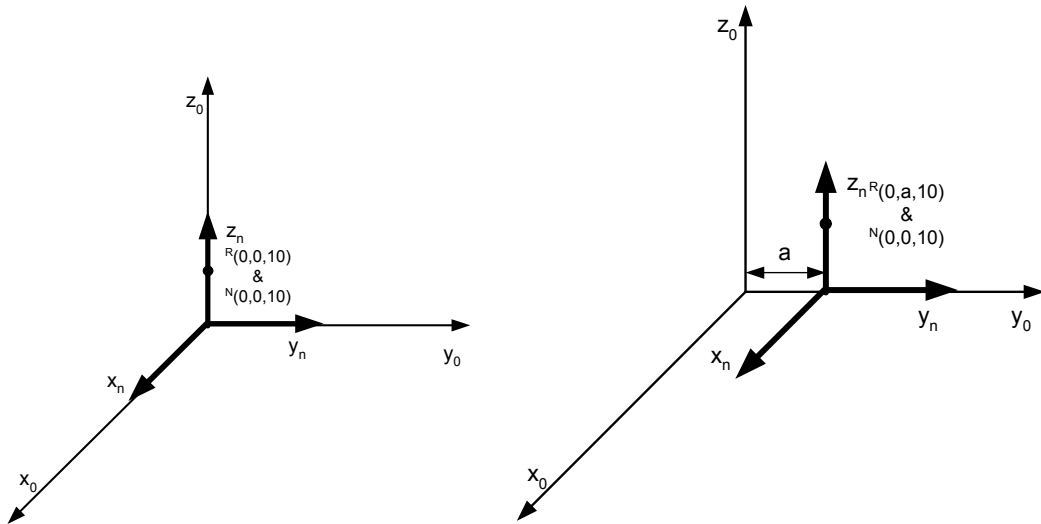


Figure 2.2 Point coordinates according to new and original reference frame

The second step is to rotate the y-axis for 90° :

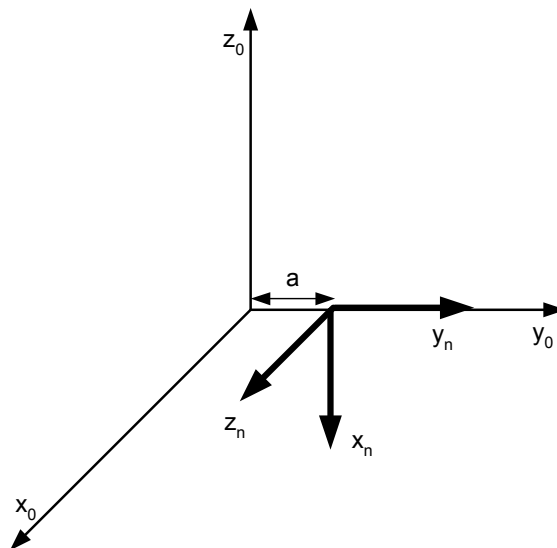


Figure 2.3 y-axis rotation of 90°

The transformation matrix is:

$$\begin{aligned}
 {}^R\mathbf{T}_N &= \begin{bmatrix} \cos(90^\circ) & 0 & \sin(90^\circ) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(90^\circ) & 0 & \cos(90^\circ) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(90^\circ) & 0 & \sin(90^\circ) & 0 \\ 0 & 1 & 0 & a \\ -\sin(90^\circ) & 0 & \cos(90^\circ) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & a \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

Hence if there was a vector with coordinates R(1,2,3) with respect to the original frame and if this vector is transformed as ${}^R\mathbf{T}_N = \text{Rot}(y, 90^\circ)\text{Trans}(y, \alpha)$ then the new coordinates of this vector would be R(3,2+a,-1), again the point remains unchanged with respect to the new frame N(1,2,3):

$$\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & a \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} = \begin{bmatrix} z_o \\ y_o + a \\ -x_o \end{bmatrix} \Rightarrow \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} = \begin{bmatrix} 3 \\ 2+a \\ -1 \end{bmatrix}$$

2.1.3 Example 1

A resultant transformation is given by the next equation. Find and plot the position and orientation of the new frame:

$${}^R\mathbf{T}_N = \text{Rot}(y, 90^\circ)\text{Trans}(z, \alpha)\text{Rot}(x, 90^\circ)\text{Trans}(y, \alpha)$$

2.1.4 Solution: Example 2

The first step is translation along the y-axis:

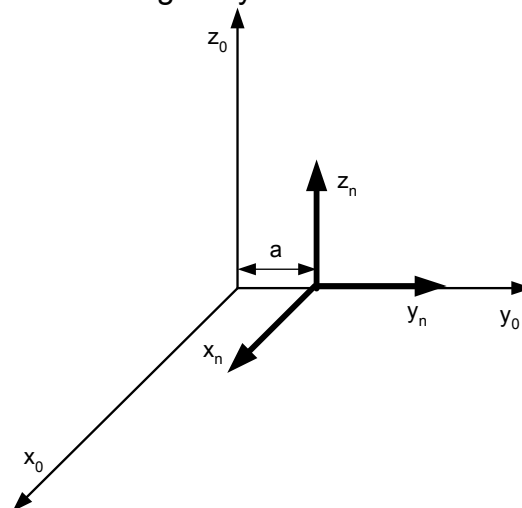


Figure 2.4 y-axis translation

The transformation matrix is ${}^R\mathbf{T}_N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ since the translation vector

is $\mathbf{r} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}$.

The second step is to rotate the x-axis for 90° :

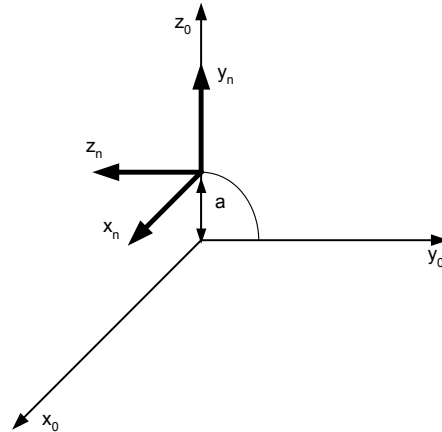


Figure 2.5 x-axis 90° rotation

$${}^R\mathbf{T}_N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) & 0 \\ 0 & \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^R\mathbf{T}_N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The third step is translation along the z-axis:

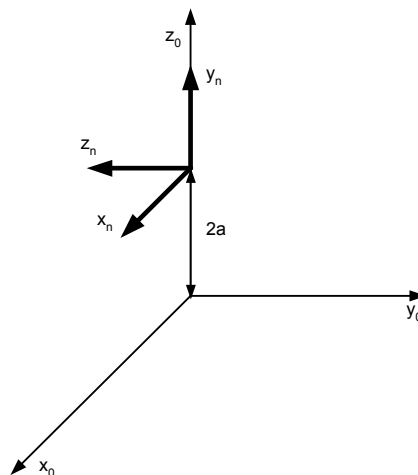


Figure 2.6 z-axis translation

The translation vector is $\mathbf{r} = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}$, hence

$${}^R\mathbf{T}_N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) & 0 \\ 0 & \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The last step is rotation around the y-axis:

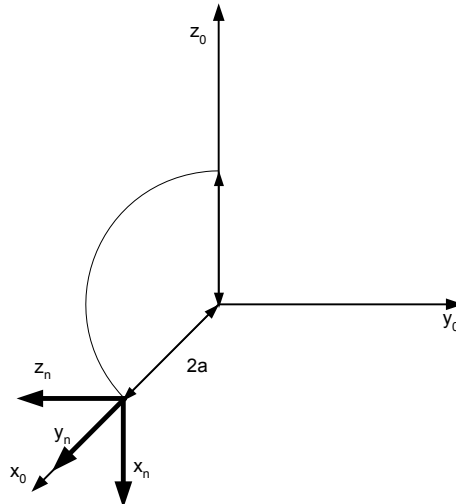


Figure 2.7 y-axis rotation

$${}^R\mathbf{T}_N = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) & 0 \\ 0 & \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^R\mathbf{T}_N = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^R\mathbf{T}_N = \begin{bmatrix} 0 & 1 & 0 & 2a \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The last matrix is very important since

- The first column describes the direction of the new x-axis, in this case is the opposite of the original z.
- The second column describes the direction of the y-axis, in this case the same as the original x.

- The third describes the direction of the z-axis, in this case is the opposite to the original y.
- Finally, the last column has the displacement vector that connects the original with the new origin. Or simply, ${}^R\mathbf{T}_N = \left[\begin{array}{c|c} {}^A R_B & Trans(x, y, z) \\ \hline \mathbf{0}_{3 \times 3} & 1 \end{array} \right]$. This matrix is called homogeneous transform.

The question that was asked in the beginning of section 1.4 can be answered now by using the transformation matrix ${}^R\mathbf{T}_N$. For example assume that a point $R(0,0,b)$ (q) was on the original set of axis and then it was transformed through ${}^R\mathbf{T}_N$ to p . To find the new location simply multiply the vector that is described by $(0,0,b)$ with the transformation matrix:

$${}^R\mathbf{p} = {}^R\mathbf{T}_N \begin{matrix} R \\ \mathbf{q} \end{matrix} = {}^R\mathbf{T}_N \begin{matrix} N \\ \mathbf{p} \end{matrix} = \begin{bmatrix} 0 & 1 & 0 & 2a \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ b \\ 1 \end{bmatrix} = \begin{bmatrix} 2a \\ -b \\ 0 \\ 1 \end{bmatrix}$$

Therefore the point $R(0,0,b)$ was transformed to $R(2a,-b,0)$, but according to the new frame is still $N(0,0,b)$.

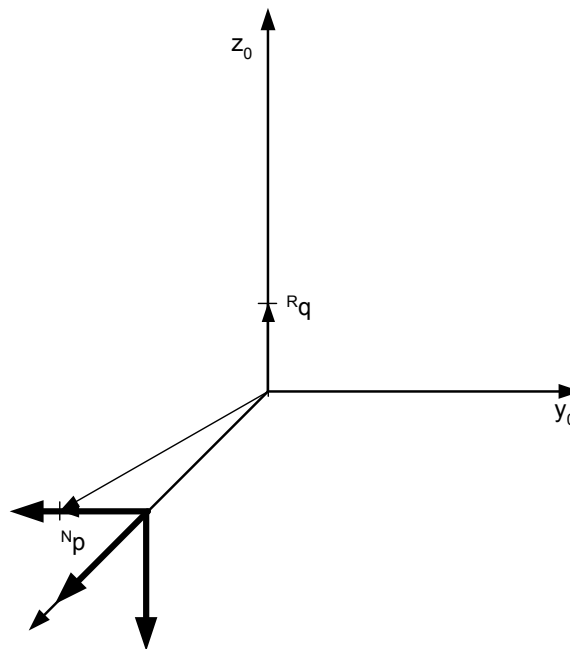


Figure 2.8 Point translation

The next examples will demonstrate the usefulness of the frame transformations and will show that the transformation matrix is not just another mathematical tool with no practical significance.

2.1.5 Example 1

Assume that a cubic object is located in a Cartesian space:

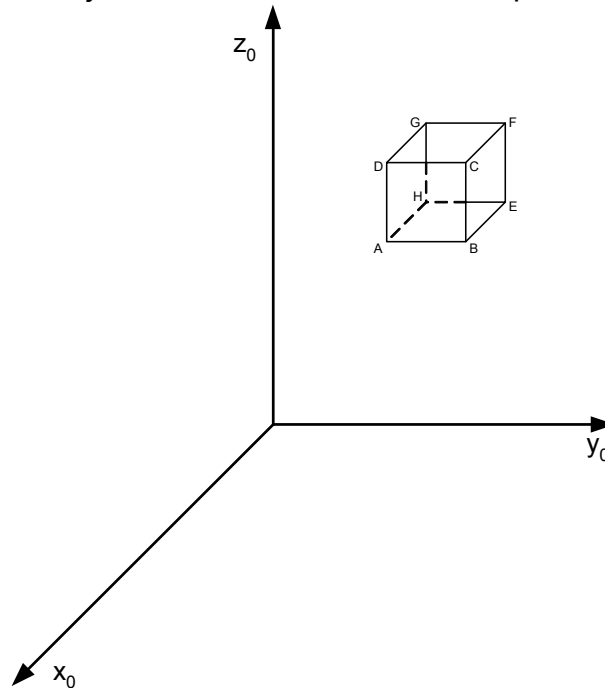


Figure 2.9 A cubic object in a Cartesian space

To describe this object 8 vectors will be needed (a, b, c, d, e, f, g, h). If this object is translated along the Y-axis for “k” units then another 8 vectors will be needed for its description. These will be found by using the transformation matrix to (a, b, c, d, e, f, g, h) and the new vectors will be created (a', b', c', d', e', f', g', h'). Hence to describe this motion 16 vectors and 8 multiplications are needed.

The other way is to use a new frame that is attached on the object (for example point A) and then eight vectors (including the zero vector) will describe the object with respect to the new frame. Another vector will describe the new frame with respect to the original. Hence there are 9 vectors: (Ra, Na, Nb, Nc, Nd, Ne, Nf, Ng, Nh). The transformation now will only be applied once on the vector Ra. This will result to a new vector Ra' and the others will remain the same: (Ra', Na, Nb, Nc, Nd, Ne, Nf, Ng, Nh). With this way only 10 vectors are needed one multiplication and 8 additions. As it can be seen it is considerably simpler to describe the object motion via frames that simply to use vectors.

2.1.6 Example 2

Assume that there is an industrial process like the one of Figure 2.10:

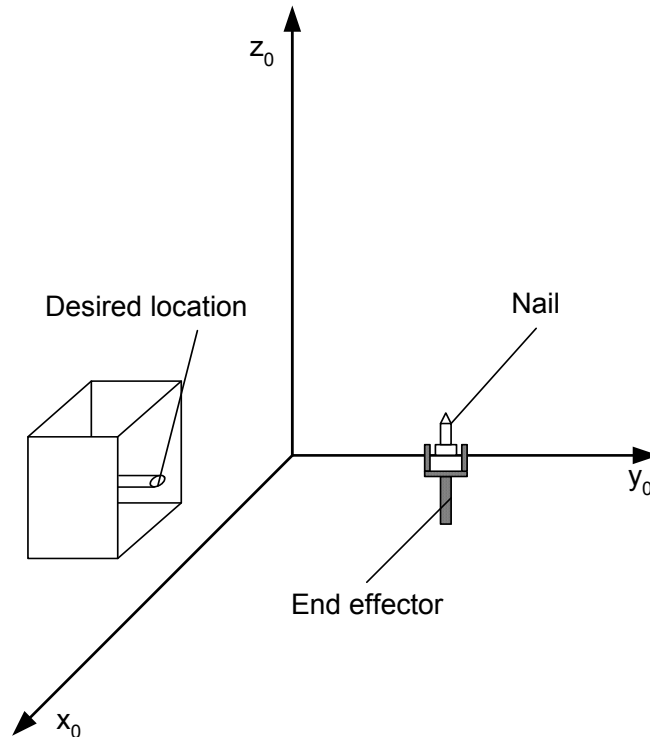


Figure 2.10 Industrial process

The goal here is to take the nail with the end effector and place it in the hole. The coordinates are: The centre of the nail head is at $(0, a, 0)$ i.e. is on the y -axis and the desired location is $(b, -c, 0)$ i.e. is on the x - y plane:

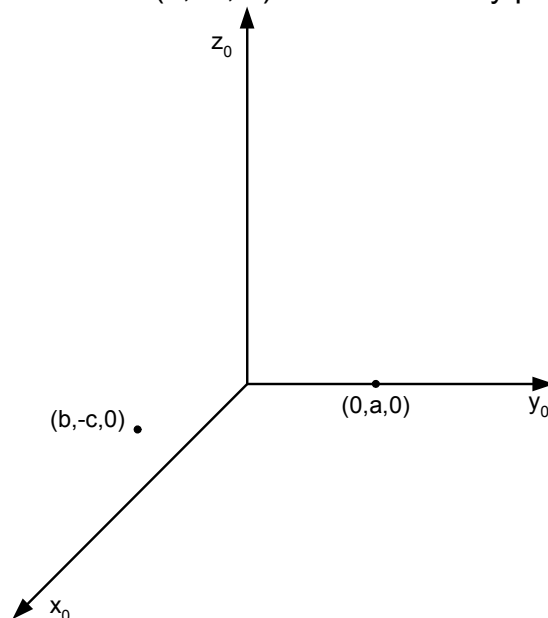


Figure 2.11 Coordinates of initial and desired points

2.1.7 Solution: Example 2

The first operation is to locate a new frame on the nail head:

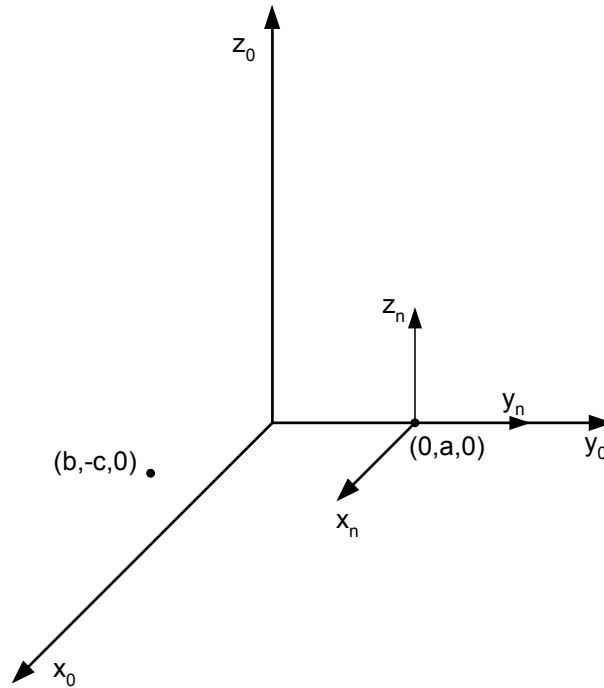


Figure 2.12 New coordinate frame on the nail head

The sequence now is:

1. $\text{Rot}(x, 90^\circ)$:

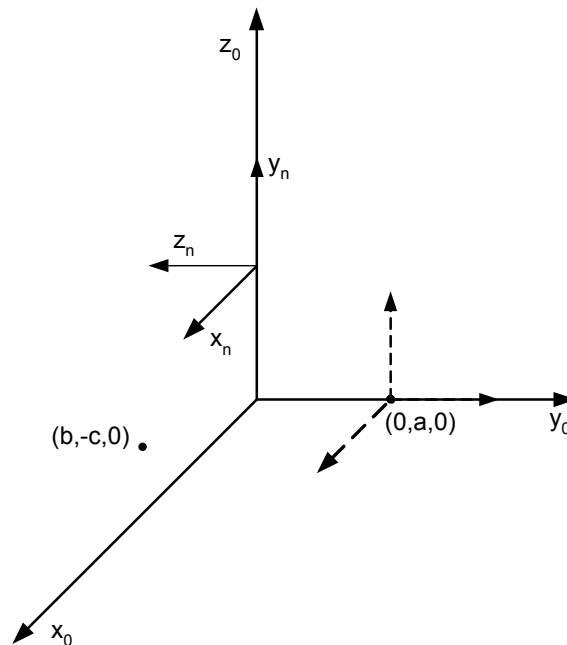


Figure 2.13 Rotation around x-axis

2. $\text{Trans}(x, b)$

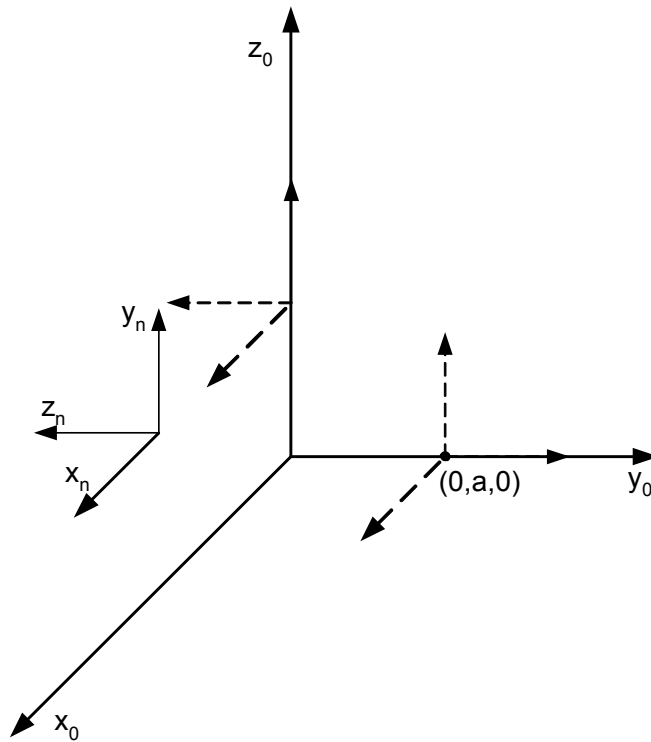


Figure 2.14 x-axis translation

3. $\text{Trans}(z, -a)$

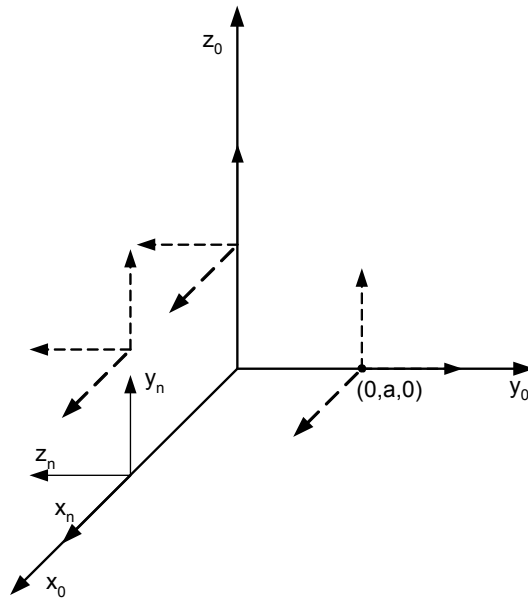


Figure 2.15 z-axis translation

4. Trans(y,-c)

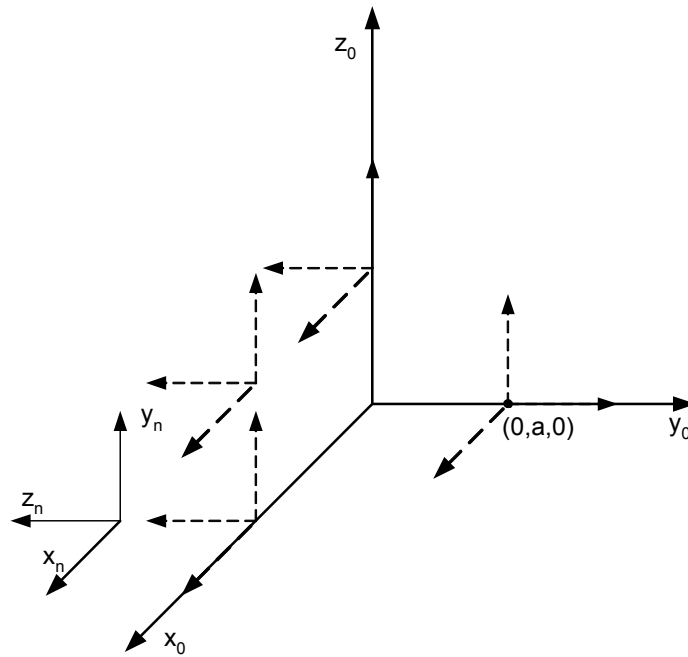


Figure 2.16 y-axis translation

The transformation matrix is:
$$\begin{bmatrix} 1 & 0 & 0 & b \\ 0 & 0 & -1 & -c \\ 0 & 1 & 0 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
, the original point was (0, a, 0)

and hence the new point is:
$$\begin{bmatrix} x_n \\ y_n \\ z_n \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & b \\ 0 & 0 & -1 & -c \\ 0 & 1 & 0 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ -c \\ 0 \\ 1 \end{bmatrix}$$

2.2 Relative transformation

Up until now it has been said that the correct sequence of the matrix multiplication, to calculate the ${}^R\mathbf{T}_N$, is to put the last action as the first factor of the matrix product.

For example: ${}^R\mathbf{T}_N = \text{Rot}(x,90^\circ) \text{Trans}(a,b,c)$

This transformation implies that the **first operation is the translation according to (a,b,c)** and then rotation around the original x-axis.

It must be mentioned again that this method assumes that all translations and rotations are given according to the original reference frame. As it can be seen in this method the matrices are pre-multiplied. This method is called absolute or forward. **If the sequence is reversed then the results will be different, unless the second transformation is according to the new frame that was created with the first operation.**

For example: ${}^R\mathbf{T}_{N_{\text{Relative}}} = \text{Rot}(x,90^\circ) \text{Trans}(a,b,c)$

This transformation means that **the rotation is the first operation** and then the translation of (a,b,c) is according to the frame that was created by the first rotation.

In this method the matrices are post-multiplied and is called **relative transformation**.

The relative transformation is a very important concept that will be used extensively in the forward kinematics in the following chapter.

2.2.1 Example

Find the location of a vector $\mathbf{a}=[0,0,1,1]^T$ after the following transformations:

a. ${}^R\mathbf{T}_N = \text{Rot}(x,90^\circ) \text{Trans}(y,10)$

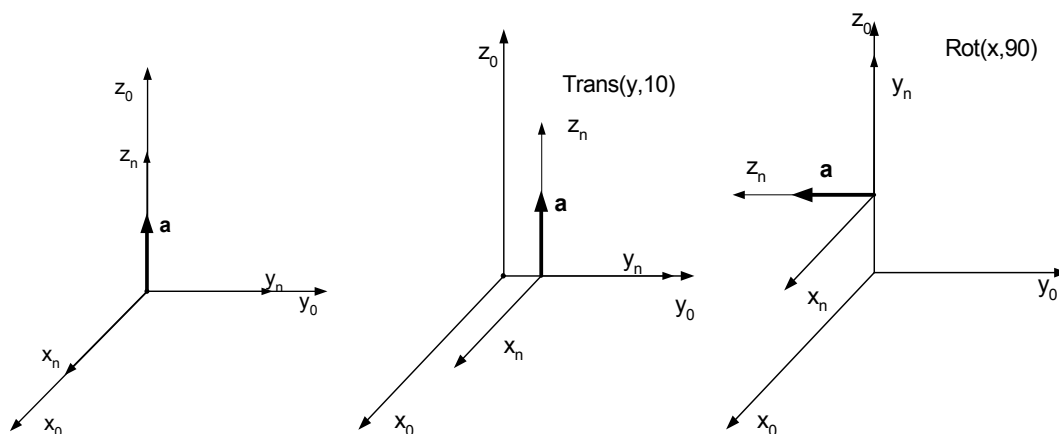


Figure 2.17 Forward Transformation

b. ${}^R\mathbf{T}_{N_{Relative}} = \text{Rot}(x,90^\circ)\text{Trans}(y,10)$

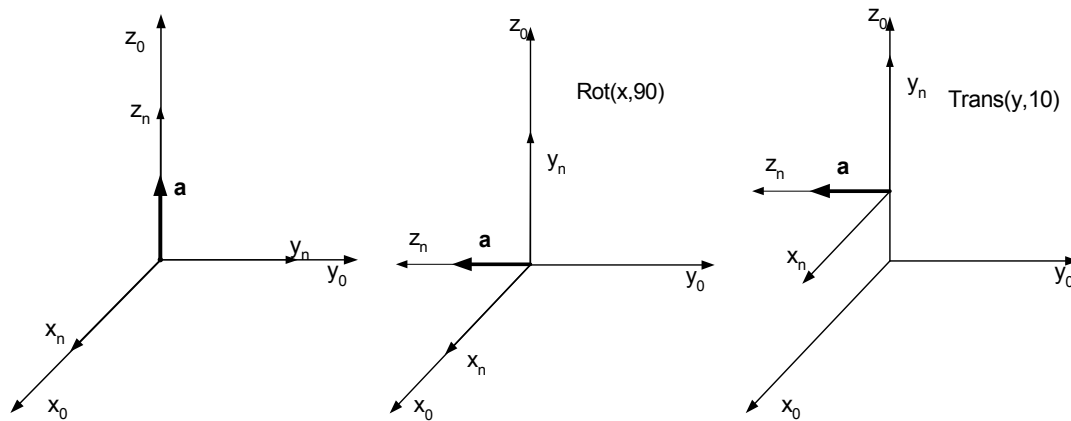


Figure 2.18 Relative Transformation

2.3 Orientation transformations

It has been shown that the transformation matrix is: ${}^R\mathbf{T}_N = \begin{bmatrix} \text{Rot} & \text{Trans} \\ 3 \times 3 & 3 \times 1 \\ \mathbf{0} & 1 \end{bmatrix}$

which can be considered to be: $\begin{bmatrix} x_x & y_x & z_x & p_x \\ x_y & y_y & z_y & p_y \\ x_z & y_z & z_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$, the first vector is going

to give the direction of the new x-axis with respect to the old, the second the y and the third the z-axis, the last column will give the translation. For example the new x-axis according to the original frame is (x_x, x_y, x_z) . Let's now see how the rotation of the new frame can be found by ${}^R\mathbf{T}_N$.

Assume that ${}^R\mathbf{T}_N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, hence the new x axis will be $[1,0,0]^T$, the

new y $[0,0,1]^T$ and the new z $[0 -1 0]^T$. Therefore the new axis will coincide with the old x axis, the new y will coincide with the old z and the new z will be the opposite of the old y, see Figure 2.19. Effectively the transformation was simply an x rotation for 90 degrees.

Assume now that ${}^R\mathbf{T}_N = \text{Rot}\left(x, \frac{\pi}{2}\right)\text{Rot}\left(y, \frac{\pi}{4}\right) = \begin{bmatrix} 0.7071 & 0 & 0.7071 & 0 \\ 0.7071 & 0 & -0.7071 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then

the new y axis will coincide with the old z, but the new x axis will belong to the plane $\{x_0, y_0\}$ at an angle of 45° ($\cos(45^\circ)=0.7071$). The new z axis will belong to the plane $\{x_0, -y_0\}$ at an angle of 45° , Figure 2.20. Now it can be

seen that drawing the new frame is very tricky even for this very simply transformation matrix.

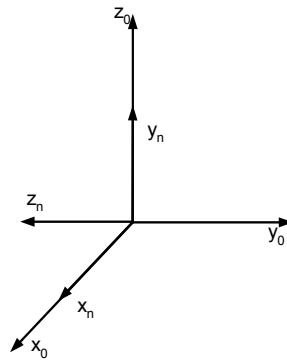


Figure 2.19 Old axis

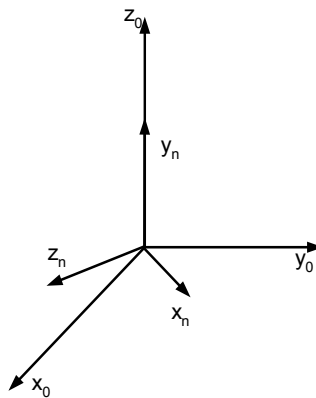


Figure 2.20 New axis

Therefore it is wise not to use a vector analysis but to again use matrices. The transformation matrix can be considered to be:

$${}^R\mathbf{T}_N = \begin{bmatrix} \text{Rot} & \text{Trans} \\ 3 \times 3 & 3 \times 3 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \text{Translation} \\ \text{Transformation} \end{bmatrix} \begin{bmatrix} \text{Rotation} \\ \text{Transformation} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_z \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \text{Rotation} \\ \text{Transformation} \end{bmatrix}$$

The rotation transform can be found with many ways that may suite a specific robot manipulator. Here three methods will be briefly described.

2.3.1 Roll-Pitch-Yaw (RPY) transformation

The first is the Roll-Pitch-Yaw transformation. This transform assumes firstly a rotation around the x-axis (Roll), then a rotation around the y-axis (Pitch) and finally a rotation around the z-axis (Yaw). All this rotations are according the original reference frame and hence the pre-multiplication method is used:

$$\text{RPY}(\phi, \theta, \psi) = \text{Rot}(z, \phi)\text{Rot}(y, \theta)\text{Rot}(x, \psi)$$

2.3.2 Euler transformation

The second way is the rotation around the new frame. Hence the first rotation is along the z-axis, the second along the new y-axis and the last along the new x-axis. Since every transformation with respect to the previous reference frame this is a post-multiply method:

$$\text{Euler}(\phi, \theta, \psi) = \text{Rot}(z, \phi)\text{Rot}(y, \theta)\text{Rot}(x, \psi)$$

2.3.3 Transformation with respect to a general vector

Until now all rotations were defined with respect to a unit vector of a predefined frame ($\{O\}$ or $\{N\}$ in our case), the next logical step is to ask if it is possible to define a rotation with respect to a general vector when this is expressed at the frame $\{O\}$. Let's assume that we have two frames $\{O\}$ and $\{N\}$ and that on $\{O\}$ we define a vector ${}^O K = [k_x \ k_y \ k_z]$. On $\{O\}$ we attach another frame $\{N\}$ that initially coincides with $\{O\}$. Then we rotate $\{N\}$ with respect to $\{O\}$ and the axis of rotation (previously \hat{x} , \hat{y} & \hat{z}) is the vector ${}^O K$. The positive angle is defined exactly as before with the right hand rule where the thumb is pointing at the positive sense of ${}^O K$ and the fingers define the positive angle:

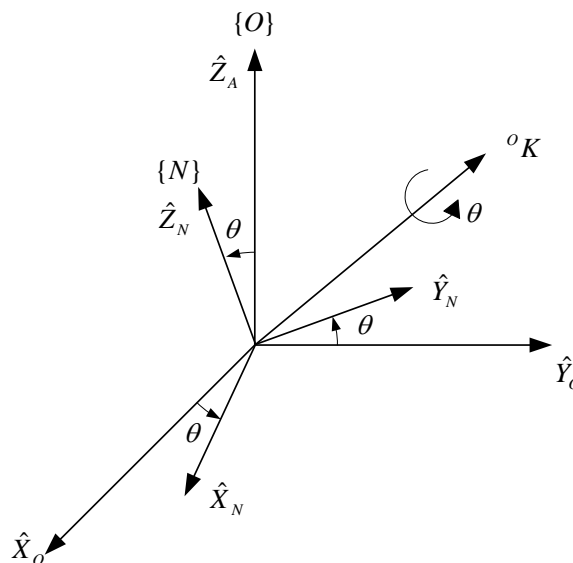


Figure 2.21 Orientation along an arbitrary vector.

It can be shown that the rotation matrix is:

$$R_K(\theta) = \begin{bmatrix} k_x k_x (1 - \cos(\theta)) + \cos(\theta) & k_x k_y (1 - \cos(\theta)) - k_z \sin(\theta) & k_x k_z (1 - \cos(\theta)) + k_y \sin(\theta) \\ k_x k_y (1 - \cos(\theta)) + k_z \sin(\theta) & k_y k_y (1 - \cos(\theta)) + \cos(\theta) & k_x k_z (1 - \cos(\theta)) - k_x \sin(\theta) \\ k_x k_z (1 - \cos(\theta)) - k_y \sin(\theta) & k_y k_z (1 - \cos(\theta)) + k_x \sin(\theta) & k_z k_z (1 - \cos(\theta)) + \cos(\theta) \end{bmatrix}$$

2.4 Inverse transformation

In the forward transformation a point is known with respect to the original frame and a series of translations and rotations occur along the axes of the original frame. The task is to find the new location of this point with respect to the initial original frame. For example assume the point P, described by the vector p . Then the reference frame that this vector is attached on is translated and rotated. What is going to be the location of the new vector with respect to the original frame?

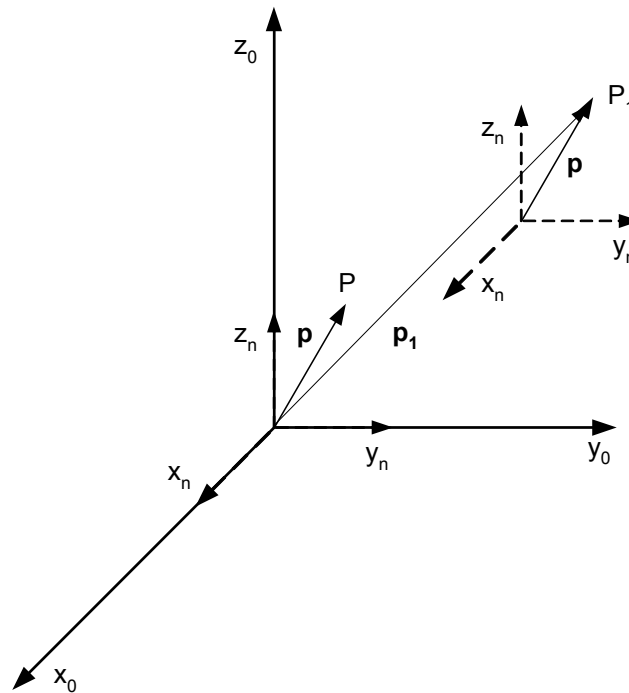


Figure 2.22 Forward transformation

Initially the point P according to the original frame is ${}^R P$ and according to the new is ${}^N P$. Before the translation ${}^R P$ is the same as ${}^N P$. After the translation the point ${}^R P$ has moved to ${}^R P_1$. To find the point ${}^R P_1$, the initial point ${}^R P$ must be multiplied with the transformation matrix ${}^R T_N$.

The inverse problem is to know the position of a point with respect to the original frame, to know the location and orientation of a new frame and to try to find the representation of the point to this new frame.

For example the point ${}^R P_1$ is known, the location and orientation of the new frame is also known and is given by the matrix ${}^R T_N$, and the task is to find the point ${}^N P_1$. Or the vector p_1 is known and the task is to find the vector p .

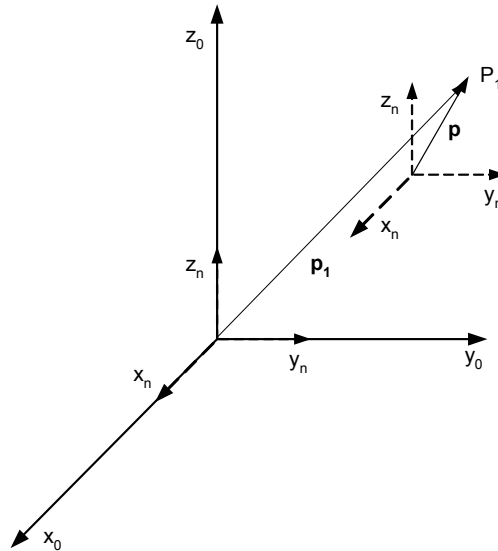


Figure 2.23 Inverse transformation, p_1 is known and the question is to find p .

Form the forward transformation: ${}^R P = {}^R \mathbf{T}_N {}^N P$, assuming that the transformation matrix is not singular: ${}^N P = ({}^R \mathbf{T}_N)^{-1} P$. Hence the problem of the inverse transformation is transformed in finding the matrix $({}^R \mathbf{T}_N)^{-1}$. It can be proved that this matrix is given by:

$$({}^R \mathbf{T}_N)^{-1} = \begin{bmatrix} x_X & y_X & z_X & p_X \\ x_Y & y_Y & z_Y & p_Y \\ x_Z & y_Z & z_Z & p_Z \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} x_X & x_Y & x_Z & -x_X p_X - x_Y p_Y - x_Z p_Z \\ y_X & y_Y & y_Z & -y_X p_X - y_Y p_Y - y_Z p_Z \\ z_X & z_Y & z_Z & -z_X p_X - z_Y p_Y - z_Z p_Z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It can be seen that the upper left part is simply the transpose of the upper left part of the forward matrix.

Student exercise: Prove that ${}^R \mathbf{T}_N ({}^R \mathbf{T}_N)^{-1} = ({}^R \mathbf{T}_N)^{-1} {}^R \mathbf{T}_N = \mathbf{I}_{4 \times 4}$

2.4.1 Example

A transformation is given by the following transformation matrix:

$${}^R \mathbf{T}_N = Rot(x, 90)$$

Calculate the transformation matrix and find the location of the vector ${}^R \mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$

with respect to the new frame that was created after the transformation ${}^R \mathbf{T}_N$.

Note the vector ${}^R \mathbf{a}$ did not transform.

The transformation matrix is ${}^R\mathbf{T}_N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, hence the vector \mathbf{a} with

respect to the new frame is: ${}^N\mathbf{a} = ({}^R\mathbf{T}_N)^{-1} {}^R\mathbf{a} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 0 \\ 1 \end{bmatrix}$

And this is exactly what was expected from the Cartesian space:

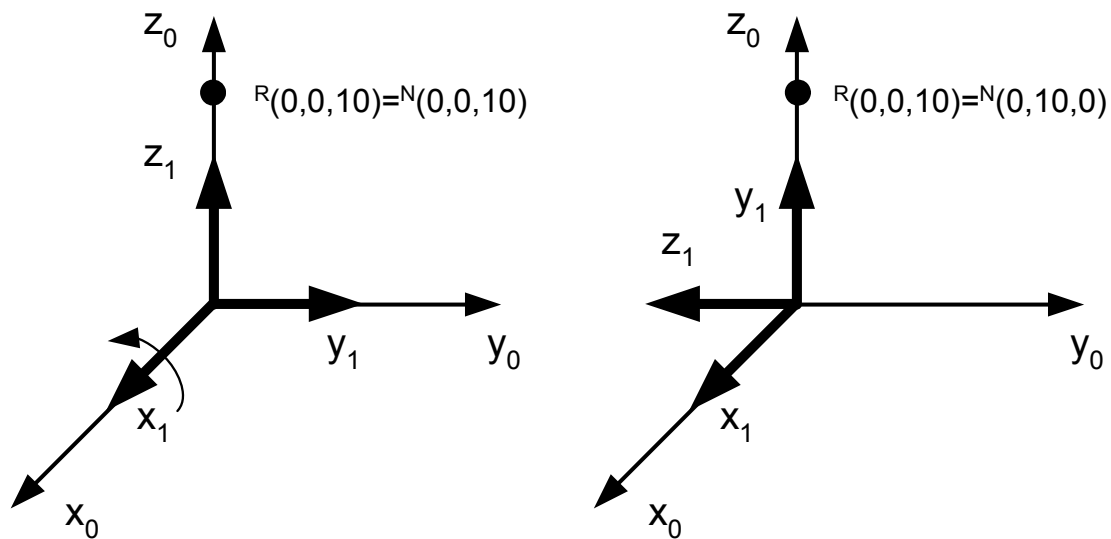


Figure 2.24 Inverse transformation

2.4.2 Example

A transformation is given by the following transformation matrix:

$${}^R\mathbf{T}_N = \text{Trans}(x, -5) \text{Rot}(x, -60) \text{Trans}(z, 6).$$

Calculate the transformation matrix and find the location of the vector ${}^R\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

with respect to the new frame that was created after the transformation ${}^R\mathbf{T}_N$.
Note the vector ${}^R\mathbf{a}$ did not transform.

Solution:

The transformation matrix is ${}^R\mathbf{T}_N = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 0.5 & 0.866 & 5.1962 \\ 0 & -0.866 & 0.5 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

Hence the vector \mathbf{a} with respect to the new frame is:

$${}^N\mathbf{a} = ({}^R\mathbf{T}_N)^{-1} \mathbf{a} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0.5 & -0.866 & 0 \\ 0 & 0.866 & 0.5 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -8.6603 \\ -1 \\ 1 \end{bmatrix}$$