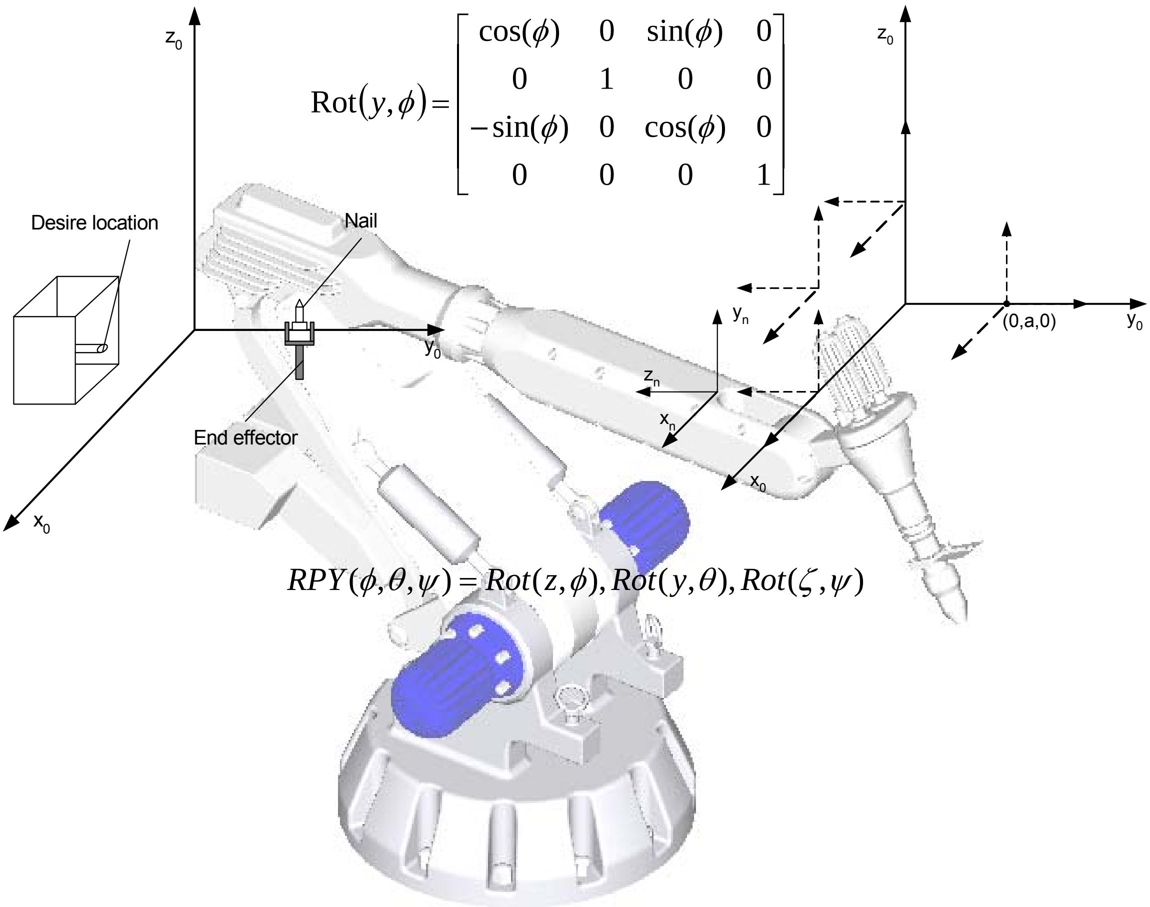
School of Electrical, Electronic and Computer Engineering

EEE 8005 – Student Directed Learning (SDL)

Industrial Automation– Fuzzy Logic



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8005– Industrial Automation

Self Directed Learning (SDL) – Artificial Intelligence

1. Introduction

Conventional control schemes require accurate models in order to create the appropriate compensators to control a process. These models, depending on the control method, may be fully deterministic (the model is assumed to be perfect) or stochastic (when some uncertainty is allowed and is represented by random signals). But the model is only an approximation of the real system; it is a mathematical expression, which is based on a number of assumptions. Therefore while on a simulation of the system the controller may give a satisfactory performance, the realisation of the compensating scheme on the actual system usually gives poorer results. On the other hand experienced practical engineers may well be able to control a process without having the slightest knowledge of its mathematical model. They can control the system by knowing only its physical characteristics and using their experience. This is exactly what the Artificial Intelligence (**AI**) methods try to mimic. They try to understand human reasoning and to create controllers that will not need a model but still be able to perform satisfactorily. The main methods that constitute AI are Fuzzy Logic (**FL**), Artificial Neural Networks (**ANN**) and Genetic Algorithms (**GAs**). The SDL part of EEE807 Industrial Automation will study FL control schemes and how they perform compared to normal control schemes. There will be a number of tutorials and then each student will have to design a number of FL Controllers (**FLC**s) in Matlab and show understanding of all the parameters of the controller.

2. The need for Fuzzy Logic

The idea of FL was born in July 1964 and was first published in 1965 by *Lofti A. Zadeh* (University of California, Berkeley). Initially this theory faced many criticisms and considerable scepticism from the scientific community. US government sponsorship of Zadeh's research led to reports to Congress that this was a waste of money. Later (from the 90s) and with the effort of many researchers around the world this theory has became an important part of control engineering.

The trigger for this new theory was the fact that the modern control systems can be extremely complicated which makes the relevant mathematics accessible only by experts. Also the mathematical model is rarely accurate and nonlinearities often influence the behaviour of the system. Furthermore it may not possible to model the disturbances that also influence the system. All these features make accurate control an extremely difficult and expensive task. Thus the need for a new theory, which does not depend on a model of the system, is strong.

In FL a model of the system is not necessary but the designer must have a very good idea about the system. The controller to be designed is going to “think” and “react” rather as a human whose experience on the specific problem is extensive. FLCs are used in many areas of control theory. One of their greatest advantages is that they are very robust and can control a nonlinear system. The irony is that FL can perform in areas that even the most complicated conventional scheme has failed. For example in space technology, FL contributed significantly and helped to overcome problems that originally seemed impossible. As shown later there is a significant drawback in the application of FL, since there is no clear method to design a controller.

3. Boolean and Fuzzy Logic

Assume that there are some objects of any shape, Fig. 1, and that the desired task is to distinguish between the shapes that have angles and shapes that only have curves:

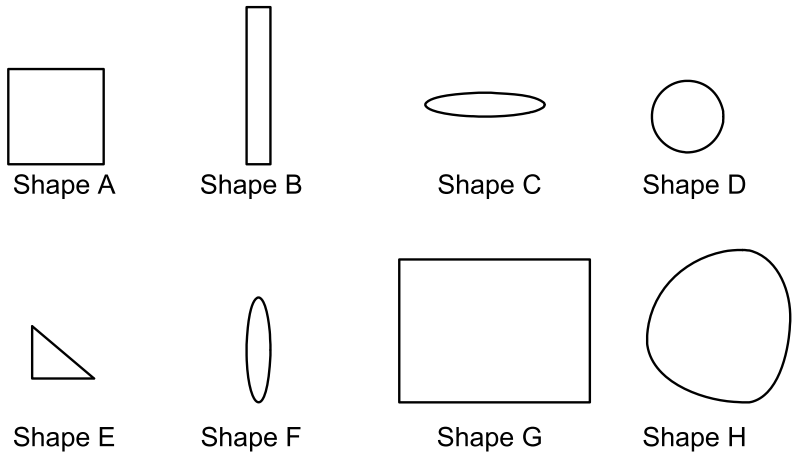

Fig. 1 *Objects with different shapes*

Clearly then the objects that have only curves are: Shape C, Shape D, Shape F and Shape H. Hence a set, A, can be created that has these objects.

Mathematically this is written as: $A = \{\text{Shape C}, \text{Shape D}, \text{Shape F}, \text{Shape H}\}$. The set that has all the other objects is: $B = \{\text{Shape A}, \text{Shape B}, \text{Shape E}, \text{Shape G}\}$.

The set that has all the objects is called universe of discourse and is:

$$C = \{\text{Shape A}, \text{Shape B}, \text{Shape C}, \text{Shape D}, \text{Shape E}, \text{Shape F}, \text{Shape G}, \text{Shape H}\}.$$

Subsets can also be created, for example a set that consists of all the objects with 4 angles is: $D = \{\text{Shape A}, \text{Shape B}, \text{Shape G}\}$ and clearly is a subset of *B*:

$$D \subset B.$$

Union is a set that has all the elements of two other sets:

$D \cup A = \{\text{Shape C, Shape D, Shape F, Shape H, Shape A, Shape B, Shape G}\}$

Obviously: $B \cup A = B \cup A = C$

Set intersection is another set that has the common elements in two sets:

$B \cap E = \{\text{Shape G}\}$, where $E = \{\text{Shape G, Shape H}\}$. Again $B \cap E = E \cap B$

The empty set is the set that has zero elements: $A \cap B = \varnothing$

Other properties are:

1 $A \cup A = A$
2 $A \cap A = A$
3 $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$
4 $A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$
5 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
7 $A \cup \varnothing = A$
8 $A \cap \varnothing = \varnothing$

In all the above an element was 100% a member of set or 0%. This logic is called Boolean or True and False logic.

Assume now that there is a set of temperatures. Further create a subset that will have all the temperatures that will correspond to a hot environment. Then by using the previous logic it might be said that hot temperatures are all the temperatures above 25 degrees: $Hot = \{temperature \mid temperature > 25\}$, this expression says that the set *Hot* has all the temperatures of over 25 degrees. But what about a temperature of 24.99 degrees, is this hot? Obviously it is, but not so hot as the 25. This introduces the need to create sets (fuzzy sets) with elements that do not follow Boolean logic but will have a membership function that will determine how much they belong to a set. For example the temperature 25 is hot 100%, then 24 is hot 90%, 20 is 50% and so on:

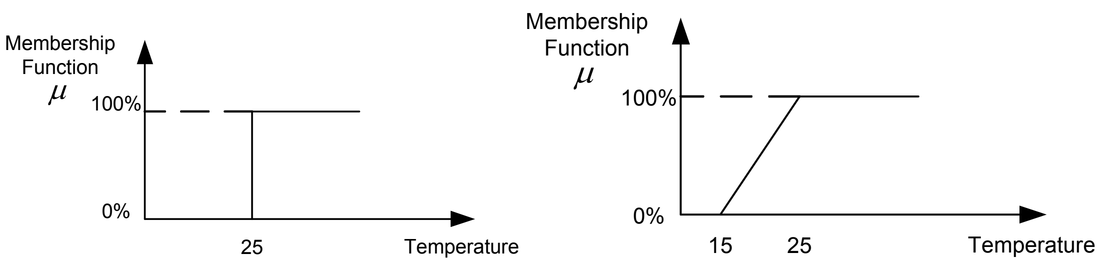

Fig. 2 & 3 *Normal and fuzzy sets*

The membership function that is shown in Figs. 3 & 4 is nothing more than a function that shows how much an element is a member or not of a fuzzy set. I.e. if "*t*" is the temperature then its membership function for a specific set is "*μ(t)*". In the previous case for *t*=25, *μ(t)*=1, for *t*=24, *μ(t)*=0.9 and for *t*=15, *μ(t)*=0.

Therefore to fully define a fuzzy set a simple mention that an element belongs to or is not in; is not enough. A pair is needed which will define the element and how much the element belongs or not, i.e. its membership function: $A = \{x, \mu(x) \mid x \in X\}$, this expression says that the set *A* has the elements *x* with the membership function *μ(x)* when *x* belongs to the universe of discourse *X*.

The first question to be answered when using fuzzy sets is the shape of the membership function, i.e. the mapping of value x to a value between 0 and 1. The main membership functions that Matlab has are:

1. Triangular form
2. Trapezoidal from
3. Gaussian forms
4. Sigmoidal forms
5. Polynomial forms

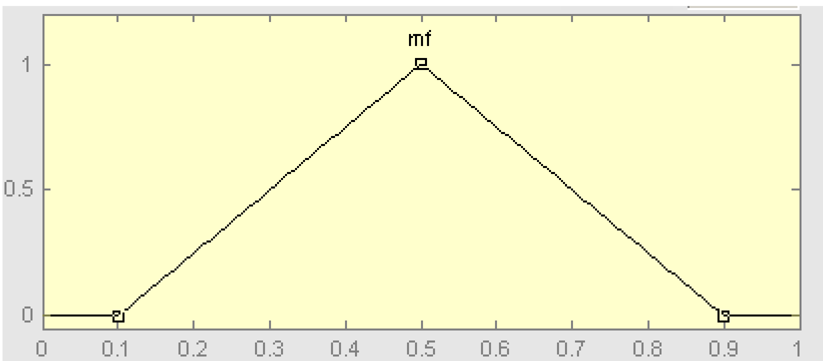

Fig. 4 *Triangular form*

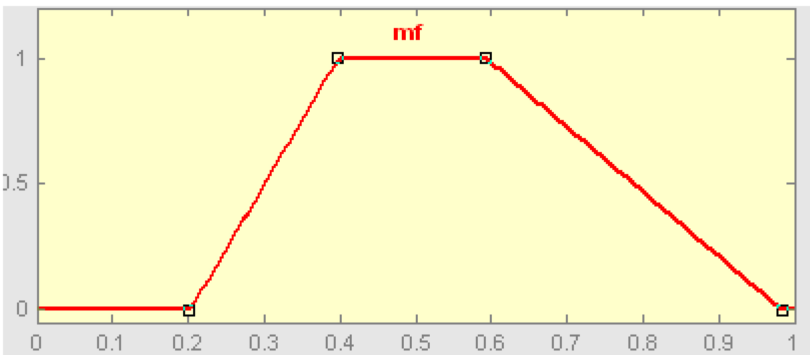

Fig. 5 *Trapezoidal form*

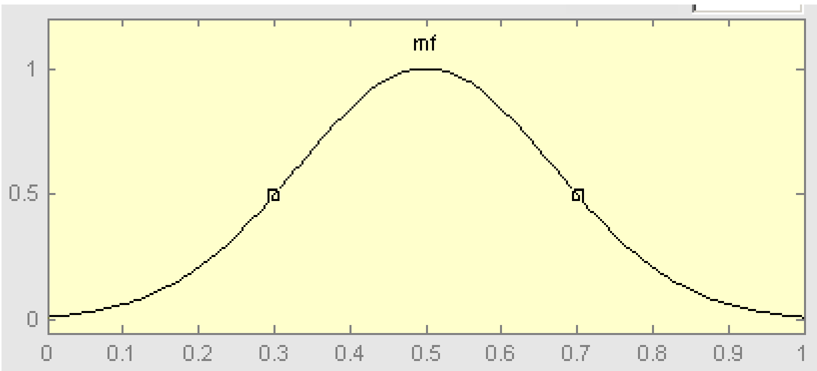

Fig. 6 *Gaussian form*

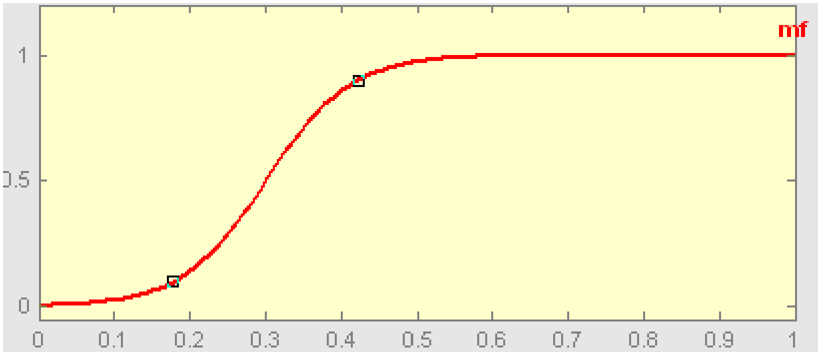

Fig. 7 *Sigmoidal form*

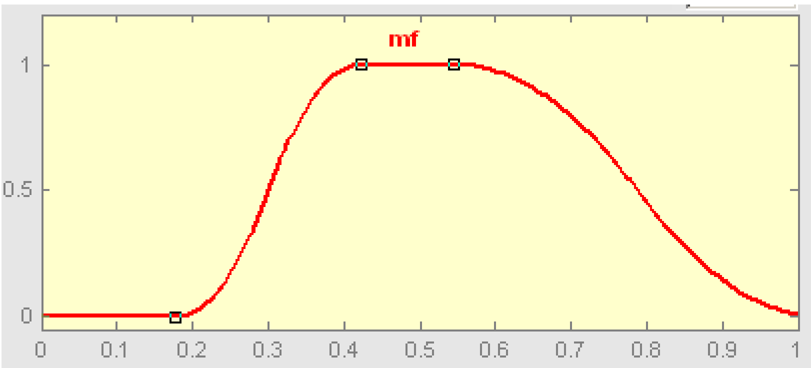

Fig. 8 *Polynomial forms*

The choice of membership function shape is not straightforward and only experience can help the designer. The most common choices are the triangular and the trapezoidal, due to their linearity.

4. Logical Operators (Boolean and Fuzzy)

In classical set theory there are logical operators that govern the sets. These are the AND, OR and NOT operators.

If the goal is to find the intersection of two sets then the AND operator is used. For example assume the sets $A = \{1,3,5,10\}$ and $B = \{5,6,7,8\}$. The intersection is the set that consists of the elements that are common in both. Hence elements that belong to a set *A* AND to a set *B*.

If the element "1" is a member of set of *A* and *B*, then it belongs to the intersection. The rest of the elements will follow the same logic. There is an associated truth or false (1 or 0) table associated with this operator which summarises the AND logic. The OR operator is associated with the union of two sets. Since the union is a set with elements either from A OR B:

A	B	AND	OR
1	0	0	1
0	1	0	1
1	1	1	1
0	0	0	0

And NOT:

A	Not
1	0
0	1

How these operators can be expanded to Fuzzy sets? Boolean sets can be considered to be a special case of Fuzzy sets where the membership functions are ones and zeros. The AND operator is replaced now by the minimum of the two membership functions and the OR by the maximum:

A	B	Min(A, B)	Max(A, B)
1	0	0	1
0	1	0	1
1	1	1	1
0	0	0	0

For example what is the intersection of the sets:

$Hot = \{temperature \mid \quad temperature > 25\}$ & $Very\ Hot = \{temperature \mid \quad temperature > 30\}$

It is clear that the intersection is:

$$Inter = \{temperature \mid \quad temperature > 25\ \&\ temperature < 30\}$$

Or the union: $Union = \{temperature \mid \quad temperature > 25\}$ since Very Hot is a subset of Hot:

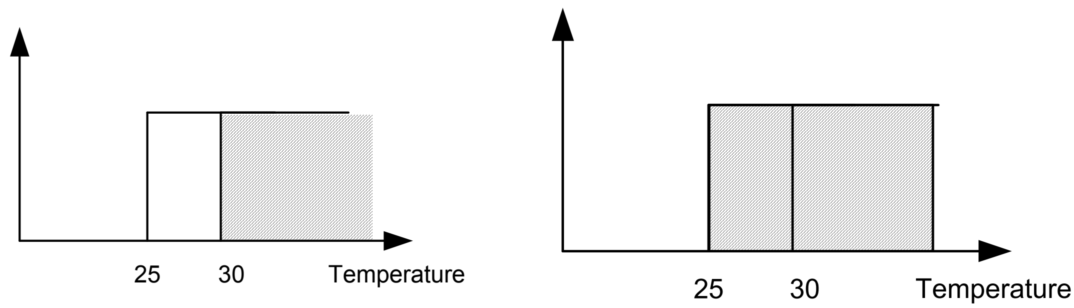

Fig. 9 & 10 *Boolean intersection and union*

The same operators with Fuzzy logic are:

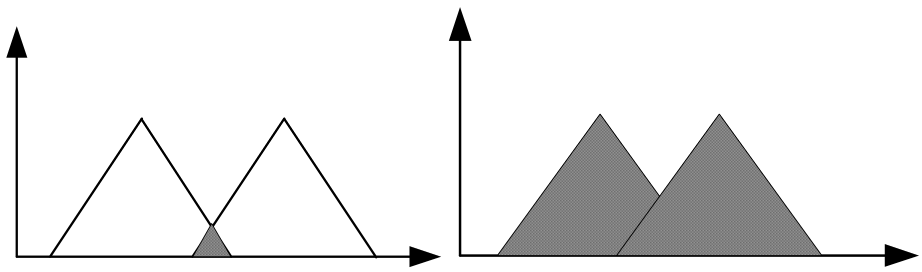

Fig. 11 & 12 *Fuzzy intersection and union*

5. Advanced Fuzzy Logic, Notation – Definitions

As it has been said a crisp set is a set that has only distinct values. If the set is Boolean and discrete then is called classical crisp set or simple crisp set while if it is fuzzy and discrete then is called fuzzy crisp set. Also in the classical Boolean logic the membership function has only two values, 0 and 1. Or it can be said that the membership function is another crisp set with only two values: $\mu(x_k) = \{0,1\}, \ \forall\ k \in N$. Hence now the definition of the Boolean set is: $A = \{(x_k, \mu(x_k)) : \mu(x_k) = 1\}, \forall\ k \in N$. Since there is only one value for the membership function it can be omitted: $A = \{x_k\}$.

On the other hand a fuzzy set has elements with membership functions from 0 to 1: $\mu(x_k) = \{[0,1]\}, \ \forall\ k \in N$. And hence a fuzzy set is defined as: $A = \{(x_k, \mu(x_k)) : \mu(x_k) = \{(0,1]\}\}$ or simply $A = \{x_k, \mu(x_k)\}, \ \forall\ k \in N$. Another popular notation that can be found in many textbooks is $A = \{x_k / \mu(x_k)\}\ \forall\ k \in N$.

Hence it can be seen that a Boolean set is effectively a subset of a Fuzzy set. If there is a set with even just one element with membership function different than 0 and 1 then this set is called **proper fuzzy set**.

Also it can be said that all the elements of any set (Fuzzy or Boolean) can be considered to belong to an overall crisp set. Hence it can be said that the elements of a fuzzy set belong to a crisp universe of discourse and therefore the fuzzy set is nothing more than one subset of the original crisp universe of discourse.

A fuzzy subset A of X is called **normal** if there exists at least one element $x \in X : \mu(x) = 1$. Note: All the crisp sets, expect the zero, are considered to be normal.

Height of a fuzzy subset A of X is the value of the maximum membership function: $Height(A) = \max \mu(x_k)\ \forall k \in N$. It can be seen that the height of a normal set is 1.

Support of a fuzzy subset A of X is a crisp set that has all the elements of A that have a membership function other than zero: $Supp(A) = \{x_k : \mu(x_k) > 0\}, \forall k \in N$

Core of a fuzzy subset A of X is another crisp set that has all the elements of the subset A that have membership function 1: $Core(A) = \{x_k : \mu(x_k) = 1\}, \forall k \in N$.

Subset of a fuzzy set is defined as: $A \subset B$, if $\mu_A(x_k) \leq \mu_B(x_k), \forall k \in N$

And finally **power** of a fuzzy set the sum: $Power(A) = \sum_{k=1}^{n} \mu_A(x_k), k = \{1,2,\cdots,n\}$

Examples:

1. The set $A = \{1/1, 3/0, 4.5j/1\}$ is not a proper fuzzy set.
2. The set $B = \{1/1, 3/0, 4.5j/0.1\}$ is a proper fuzzy set, since the element x=4.5j has a membership function of 0.1.
3. The set $C = \{1/0.01, 3/0, 4.5j/0.1\}$ is not a normal fuzzy set.
4. The set $D = \{1/0.01, 3/1, 4.5j/0.1\}$ is a normal fuzzy set, since the element 3 has a membership function of 1.
5. The height of the set A is max(1,0,1)=1. The height of the set B is max(1,0,0.1)=1. The height of the set C is max(0.01,0,0.1)=0.1.
6. The support of the fuzzy set $E = \{1/0, 3/1, 4.5/0.1\}$ is the set $F = \{3, 4.5\}$
7. The core of the fuzzy set E is the set $G = \{3\}$
8. Finally the Power of the fuzzy set E is 0+1+0.1=1.1

6. Operations on Fuzzy Sets

In this section some common concept in the Boolean logic are expanded to include FS. At this point it must be noted that if the sets are reduced to normal Boolean then the meaning of these properties must be also reduced to their original meaning.

Assume the two fuzzy subsets A and B of X. Their **union** is a fuzzy subset C of X, denoted as $C = A \cup B$ such that $\forall x \in X \quad \mu_C(x) = \max(\mu_A(x), \mu_B(x))$.

Their intersection is a fuzzy subset D of X, denoted as $D = A \cap B$ such that $\forall x \in X \quad \mu D_C(x) = \min(\mu_A(x), \mu_B(x))$.

Some important properties are

- $D \subset C$

- $A \subset C$ and $B \subset C$
- $D \subset A$ and $D \subset B$

Another popular way to calculate the minimum and maximum value of two numbers is to use the following expressions:

- $\max(a,b) = \dfrac{a+b+|a-b|}{2}$
- $\min(a,b) = \dfrac{a+b-|a-b|}{2}$

Assume A and B are two fuzzy subsets of X. The relative complement of B with respect to A is another set E=A-B, subset also of X that is defined as:

$$\mu_E(x) = \max(0, \mu_A(x) - \mu_B(x))$$

Assume A a two fuzzy subset of X. The complement or negation of A is another set $\overline{A} = X - A$, subset also of X that is defined as:

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

Some properties of the negation are:

- $\overline{\overline{A}} = A$
- $\left.\begin{array}{l}\overline{(A \cup B)} = \overline{A} \cap \overline{B} \\ \overline{(A \cap B)} = \overline{A} \cup \overline{B}\end{array}\right\}$ De Morgan' s Law
- $\begin{array}{l}\overline{\varnothing} = X \\ \overline{X} = \varnothing\end{array}$
- If $A \subset B$ then $\overline{A} \supset \overline{B}$ and $A - b = \varnothing$

Assume A, a fuzzy subset of X and a>0. Another fuzzy subset B=A^a of X can be defined as: $\mu_B(x) = (\mu_B(x))^a$

It can be proved that if a>1 then $A^a \subset A$ and if a<1 $A^a \supset A$. If a>1 then the above operation is called concentration operation and is called dilation if a<1.

An operation closely related to the above is that of contrast intensification:

$$B = \left\{ x, \mu_B(x) : \begin{cases} \mu_B(x) = (\mu_A(x))^2, & \mu_A(x) \in [0, 0.5] \\ \mu_B(x) = (\mu_A(x))^{1/2}, & \mu_A(x) \in [0.5, 1] \end{cases} \right\}$$

Bounded sum of two fuzzy subsets A and B of X is another subset $D = A \oplus B$ of X such that $\mu_D(x) = \min(1, \mu_A(x) + \mu_B(x))$.

Assume a subset A of X and a number $a \in [0,1]$ then a new set can be defined as F=aA as $\mu_F(x) = a\mu_A(x)$.

Assume a subset A of X; the a-level set of A, denoted A_a, is the crisp subset of X such that $\mu_A(x) \geq a$ or: $A_a = \{x : \mu_A \geq a, \forall x \in X\}$

Examples:

1. If $A = \{1/0.5,\ 3/0.6,\ 4/0.1\}$ and $B = \{2/0.7, 3/0.2, 4/0.8\}$ their union is $A \cup B = \{1/0.5,\ 2/0.7,\ 3/\max(0.6,0.2),\ 4/\max(0.1,0.8)\} \Leftrightarrow$ $\Leftrightarrow A \cup B = \{1/0.5,\ 2/0.7,\ 3/0.6,\ 4/0.8\}$
2. If $A = \{1/0.5,\ 3/0.6,\ 4/0.1\}$ and $B = \{2/0.7, 3/0.2, 4/0.8\}$ their intersection is $\{3/\min(0.6,0.2),\ 4/\min(0.1,0.8)\} = \{3/0.2,\ 4/0.1\}$
3. If $D = A \cap B$ and $C = A \cup B$ then $C = \{1/0.5,\ 2/0.7,\ 3/0.6,\ 4/0.8\}$ and $D = \{3/0.2,\ 4/0.1\}$. The set D can also be written as $D = \{1/0,\ 2/0, 3/0.2,\ 4/0.1\}$. It can be seen that for all the elements of D and C $\mu_D(x) < \mu_C(x)$ this implies that $D \subset C$.
4. Also $A = \{1/0.5,\ 2/0,\ 3/0.6,\ 4/0.1\}$, $B = \{1/0,\ 2/0.7,\ 3/0.2,\ 4/0.8\}$ and $C = \{1/0.5,\ 2/0.7,\ 3/0.6,\ 4/0.8\}$. It can be seen that for all the elements of A, B and C $\mu_A(x) < \mu_C(x)$ and $\mu_B(x) < \mu_C(x)$ this implies that $A \subset C$ and $B \subset C$.
5. Also $D = \{1/0,\ 2/0, 3/0.2,\ 4/0.1\}$, hence $\forall x \in X$ $\mu_A(x) > \mu_D(x)$ and $\mu_B(x) > \mu_D(x)$, therefore $D \subset A$ and $D \subset B$.
6. Calculate the min and max of $(0.1,0.5)$, $(0.5,0.5)$ by using $\max(a,b) = \frac{a+b+|a-b|}{2}$ and $\min(a,b) = \frac{a+b-|a-b|}{2}$.
   a. $\max(0.1,0.5) = \frac{0.1+0.5+|0.1-0.5|}{2} = \frac{1}{2} = 0.5$
   b. $\min(0.1,0.5) = \frac{0.1+0.5-|0.1-0.5|}{2} = \frac{0.6-0.4}{2} = 0.1$
   c. $\max(0.5,0.5) = \frac{0.5+0.5+|0.5-0.5|}{2} = 0.5$
   d. $\min(0.5,0.5) = \frac{0.5+0.5-|0.5-0.5|}{2} = 0.5$
7. Find the relevant complement of B with respect to A. The sets A and B are $A = \{1/0.5,\ 3/0.6,\ 4/0.1\}$, $B = \{2/0.7, 3/0.2, 4/0.8\}$ $\mu_E(x) = \max(0, \mu_A(x) - \mu_B(x))$, hence $E = \{1/0.5,\ 2/0,\ 3/0.4,\ 4/0\}$.
8. Find the complement of A: $\overline{A} = \{1/0.5,\ 3/0.4,\ 4/0.9\}$.

7. Fuzzy Relationships

Assume that there is a crisp set X with elements x_1, x_2,.. x_n and another crisp set Y with elements y_1, y_2,.. y_m. The Cartesian product of these two sets is another set of all the pairs: $X \times Y = \{(x_k, y_j), \forall k, j \in N\}$. Hence this set will have m x n elements. Graphically:

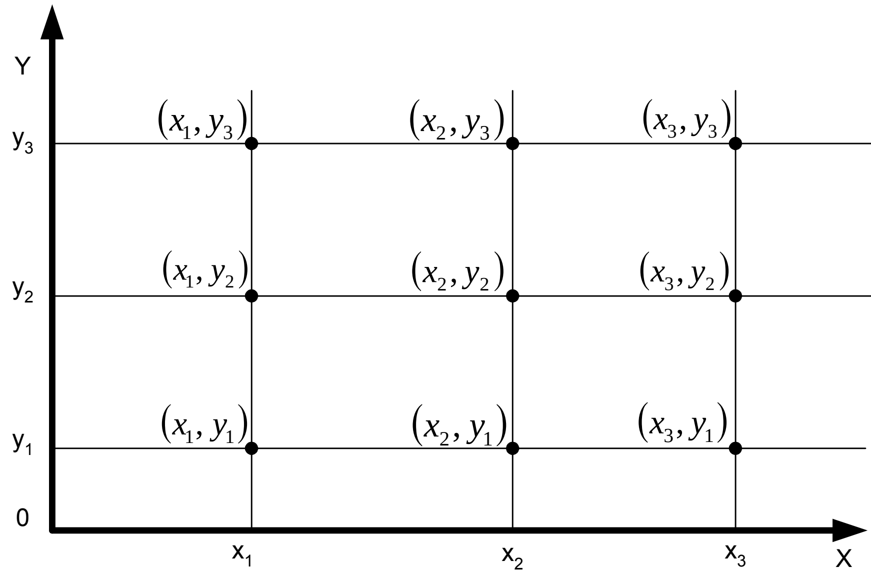

Fig. 13 *Cartesian product*

This set (X x Y) can be considered to be universe of discourse for a fuzzy set A: $A = \{(x_k, y_j)/\mu_A(x_k, y_j), \forall k, j \in N\}$. The fuzzy set A can is also called a **fuzzy relationship** over the pair X and Y.

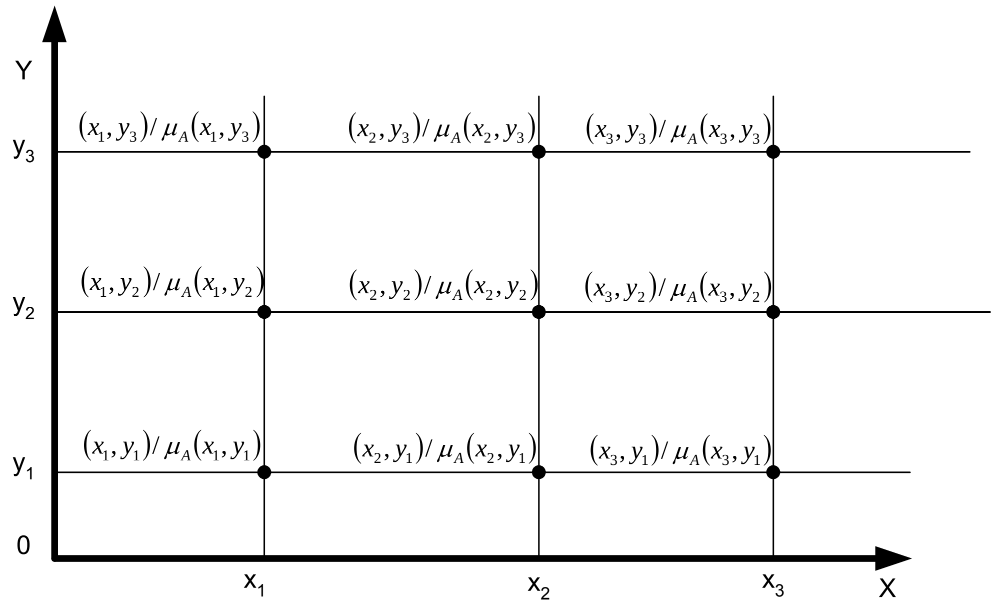

Fig. 14 *Fuzzy cartesian product*

Projection o the fuzzy subset A onto X is a fuzzy subset $A^o = Proj_X A$ and is defined as: $\mu_{A^o}(x_k) = \max(\mu_A(x_k, y_j)) \forall k, j \in N$ or

$$\mu_{A^o}(x_k) = \max(\mu_A(x_k, y_1), \mu_A(x_k, y_2), \mu_A(x_k, y_3) \cdots, \mu_A(x_k, y_m))$$

Assume, now, two fuzzy subsets A and B of X and Y respectively, their Cartesian product (A x B) is a fuzzy relationship on the set X x Y, denoted as T=A x B where $\mu_T(x_k, y_j) = \min(\mu_A(x_k), \mu_B(y_j)) \forall k, j \in N$ or:

$$T = \{(x_k, y_j), \mu_T(x_k, y_j) : \mu_T(x_k, y_j) = \min(\mu_A(x_k), \mu_B(y_j)) \forall k, j \in N\}$$

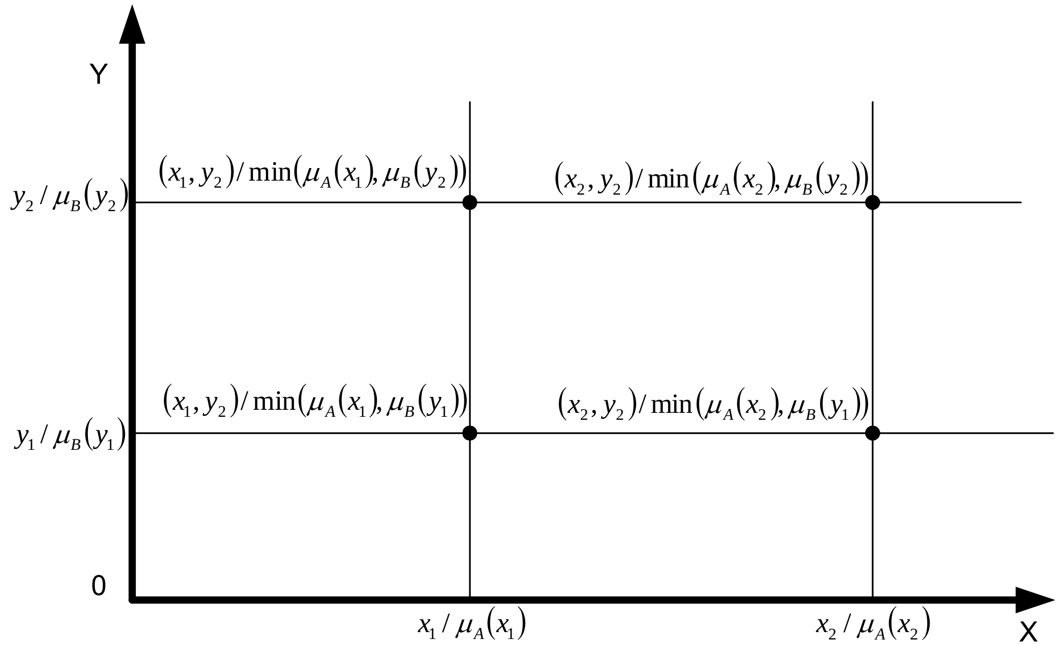

Fig. 15 *Projection of the fuzzy cartesian product*

8. Extension principle !!!

A function is a relation, which uniquely associates members of one set with members of another set. More formally, a function from X to Y is an object *f* such that $\forall x \in X$ is uniquely associated with an object $y \in Y$: $f : X \to Y$. The set X is called domain and the set Y is called range or image of X under f.

The same now can be applied on fuzzy sets: Assume a mapping f, from X to Y (X and Y are crisp sets) such that $\forall x \in X \; \exists \, y \in Y$. Also assume a fuzzy subset A of X: $A = \{x / \mu_A(x)\}$. The function f, is going to map every element of A in Y as f(x) and with a membership function of $\mu_A(x)$: $f(A) = \cup_X \{f(x) / \mu_A(x)\}$. If B=f(A) then $\mu_B(y_j) = \max_{\forall x_k : f(x_k) = y_j} (\mu_A(x_k))$

The extension principle can be used for arithmetic operations with fuzzy sets. Assume three crisp sets: $X = \{x_1, x_2, \cdots, x_n\}$, $Y = \{y_1, y_2, \cdots, y_m\}$ and $Z = \{z_1, z_2, \cdots, z_k\}$. The addition of elements from X and Y can be considered to be a mapping of the Cartesian product X x Y into Z: $f : X \times Y \to Z$ when $f(x_i, y_j) = x_i + y_j = z_l, \forall i, j, l \in N$ hence $Z = \{z_l : x_i + y_j = z_l\}, \forall i, j, l \in N$

Now assume three fuzzy subsets A, B, C of X, Y and Z respectively, then by using the extension principle C=f(A+B) and $\mu_C(z_l) = \max\limits_{\forall x_k : f(x_i, y_j) = x_i + y_j = z_l} \left(\min\left(\mu_A(x_i), \mu_B(y_j)\right)\right)$. The same can be applied to any arithmetic operation.

9. Measures of Fuzziness

It has been said that a fuzzy set has elements with membership function from 0 to 1 while the Boolean crisp sets have membership functions 0 or 1. Hence the closer the distribution of the membership functions of a fuzzy set to 0 and to 1 the "less" fuzzy the set. The less fuzzy a fuzzy set is; the bigger is the differences between the fuzzy set and its complement. Therefore it can be said that the difference between a fuzzy set and its complement can describe the fuzziness of this fuzzy set.

Assume a universe of discourse $X = \{x_1, x_2, \cdots, x_n\}$ where $n < \infty$. This set can be written in a form of a column vector as: $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

Also the values of the membership functions of a fuzzy subset A of X can be written in a form of a vector: $\mu_A(x) = \begin{bmatrix} \mu_A(x_1) \\ \mu_A(x_2) \\ \vdots \\ \mu_A(x_n) \end{bmatrix}$. Hence the membership function of the complement of A is: $\mu_{\bar{A}}(x) = \begin{bmatrix} 1-\mu_A(x_1) \\ 1-\mu_A(x_2) \\ \vdots \\ 1-\mu_A(x_n) \end{bmatrix}$. The difference now between this two vectors is an indication of the fuzziness of a fuzzy set:

$D(A,\overline{A}) = \mu_A(x) - \mu_{\overline{A}}(x) = \begin{bmatrix} \mu_A(x_1) \\ \mu_A(x_2) \\ \vdots \\ \mu_A(x_n) \end{bmatrix} - \begin{bmatrix} 1-\mu_A(x_1) \\ 1-\mu_A(x_2) \\ \vdots \\ 1-\mu_A(x_n) \end{bmatrix} = \begin{bmatrix} 2\mu_A(x_1)-1 \\ 2\mu_A(x_2)-1 \\ \vdots \\ 2\mu_A(x_n)-1 \end{bmatrix}$. Thus the length of this vector can describe the fuzziness. The length of a vector can be found by its norm: $\left\| D(A,\overline{A}) \right\|_P = \left[\sum_{i=1}^{n} \left| 2\mu_A(x_i) - 1 \right|^P \right]^{1/P}$ for P=1, 2, 3…

Notice that if A is not fuzzy then $\left\| D(A,\overline{A}) \right\|_P = \left[\sum_{i=1}^{n} (2-1)^P \right]^{1/P} = \left[\sum_{i=1}^{n} 1 \right]^{1/P} = n^{1/P}$ and this is the maximum distance between the two sets.

Now the fuzziness of a fuzzy set can be described as: $\mathbf{FUZ}_P = \dfrac{n^{1/P} - \left\| D(A,\overline{A}) \right\|_P}{n^{1/P}}$, which will give 0 if the set is Boolean.

To have the maximum $\mathbf{FUZ}_P$ the $\left\| D(A,\overline{A}) \right\|_P$ must be zero. Hence $\mu_A(x_i) = 0.5,\ i = 1,2 \ldots n$

10. Linguistic variables and Fuzzification

Human reasoning does not use numerical values, i.e. it does not say the temperature here is 15.5 degrees and hence lets switch on the heater. Rather human reasoning says that it feels cold here, lets switch on the heater. This is a linguistic variable instead of a numerical variable. For this reason FL uses linguistic variables to name the fuzzy sets.

The first process of the fuzzy logic controller is to map all the possible inputs to fuzzy sets and to assign values of "how much" they belong to these. This process is called fuzzification. For example assume the next fuzzy set:

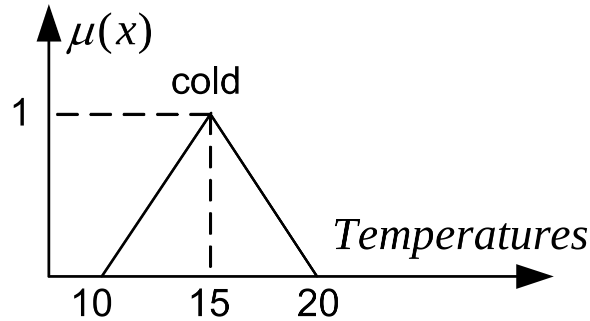

Fig. 16 *Fuzzy set "cold"*

Then a temperature of 11^{o} C is going to be assigned to the value of 0.1 for the set cold.

The universe of discourse must be fully covered by fuzzy sets. These fuzzy sets may overlap or not and may have different shapes:

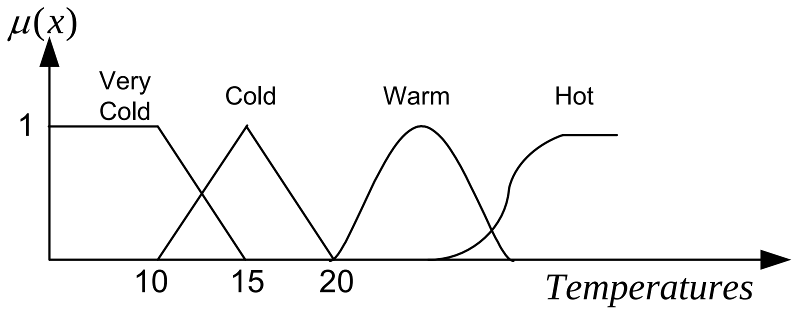

Fig. 17 *Fuzzy set "cold"*

Hence now it can be said that the temperature 11 belongs to the fuzzy set "Cold" for 10% and to the fuzzy set "Very Cold" 80%.

In this way every real input (in this case the temperature) is mapped to a specific fuzzy set and has a specific value to describe how much it belongs or not to this set. This value will be directly depended upon the shape of the membership function.

11. Fuzzy Inference

The fuzzy inference system (**FIS**) is the part of the FLC that connects fuzzified inputs to output fuzzy sets. This is the key point of the FL, which is to mimic human reasoning. This reasoning can be described by IF and THEN rules. For example "IF the room is cold THEN switch on the heater to maximum", or "IF the speed is high THEN press the brake to maximum". The IF part is called a "premise", and the THEN part is called a "conclusion". This exactly the structure of a FL Controller (FLC). The IF part will be the input to the controller and the THEN part the output. Both inputs and outputs will be described by fuzzy sets.

For example assume that the controller must determine the braking level depending on the speed of a car. Then the input will be the speed and the braking level the output. Assume also that the universes of discourse for the input and output contain only one fuzzy set.

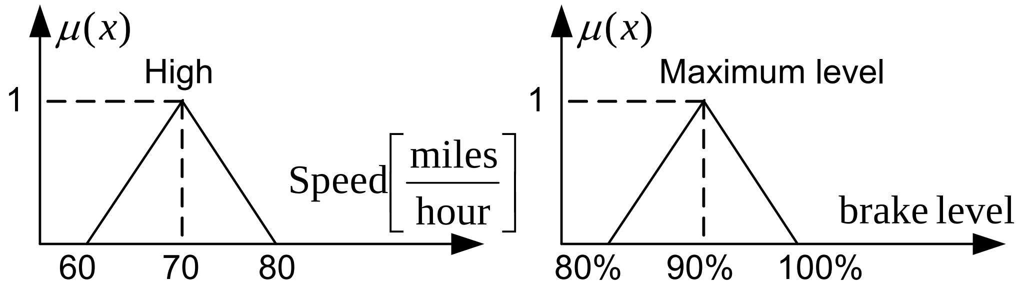

Fig. 18 & 19 *Input and Output for a fuzzy controller"*

Assume also that the speed is 65 miles/hour. This speed is a member of the fuzzy set "High" for 50% or 0.5. This value is also called degree of support for that rule. Assume also that the IF – THEN rule that is used is: If speed is high then the braking level is maximum. This means that the speed of 65 miles per hour is partially true for the set "High". But the set "High" is connected with the output set "Maximum" through the previous rule. Since now the input was partially true, the output set is also partially true with the same degree as the input, i.e. 50%:

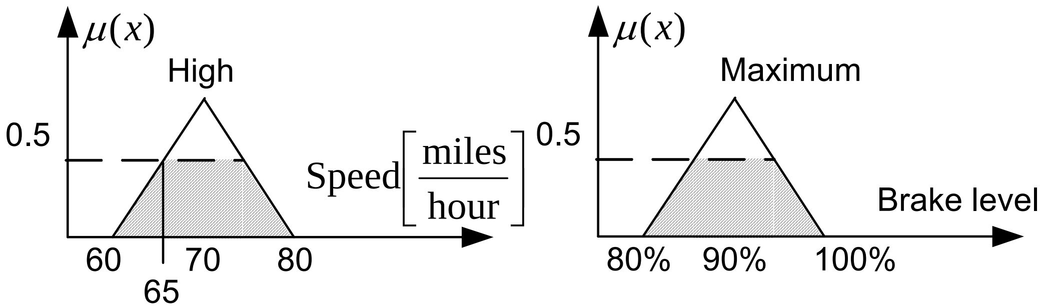

Fig. 20 & 21 *Input and Output for a fuzzy controller using the min method*

Since the fuzzy set of the output was truncated up to the value of 0.5, this method is called "min" method. Another popular approach is the so called prod or squash method where the output fuzzy set is scaled:

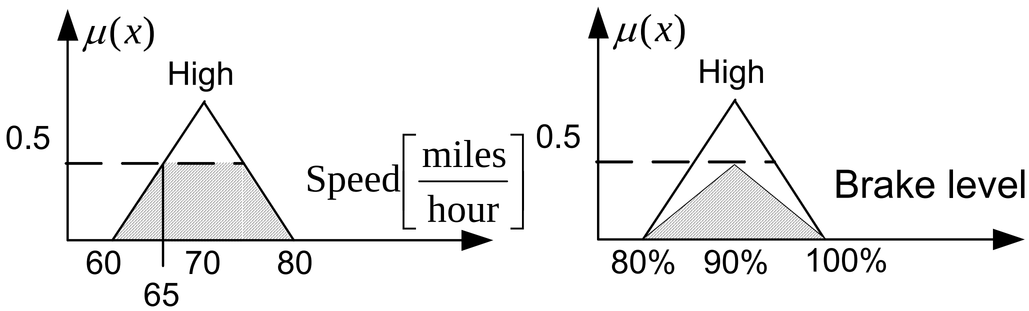

Fig. 22 & 23 *Input and Output for a fuzzy controller using the prod method*

The value now of the output will be determined by the de-fuzzification method that will be used.

In some applications one input cannot fully describe a situation, for example in order for the human to decide to press the brake the information of the speed alone is not enough. The distance from the next car, or the maximum speed limit would also be needed. Hence in order to fully describe a situation with IF – THEN rules logical operators must be used.

The problem here is to find the degree of support for this rule. If the sentence has the "or" operator then the maximum between the two number is going to be used. Otherwise, if the rule has the "and" operator the minimum is taken:

If the speed is high or the next car is very close then press the brake to maximum.

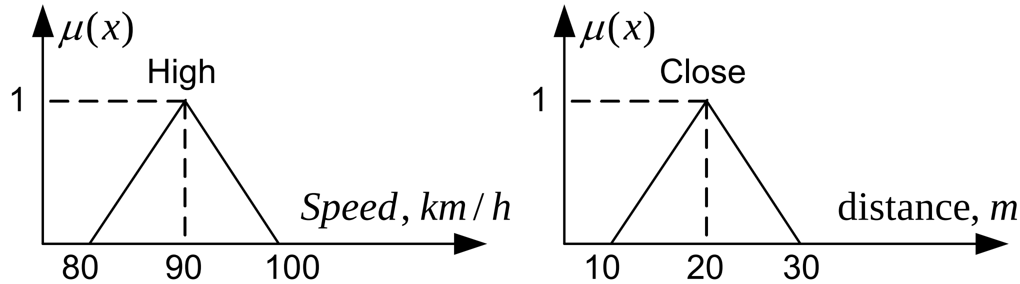

Fig. 24 & 25 *Input and Output for a fuzzy controller using the prod method*

Assume that the speed belongs to the fuzzy "High" for 0.5 and the distance to the fuzzy set "Close" for 0.6. Since the rule has "or" the max operation will be used, i.e. the degree of support is max(0.5,0.6)=0.6, hence:

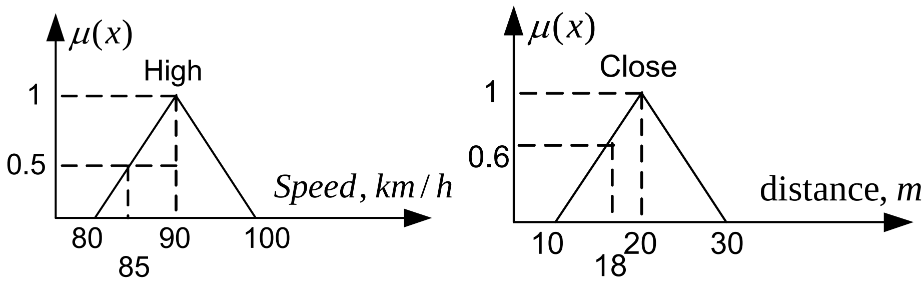

Fig. 26 & 27 *2 Fuzzy inputs*

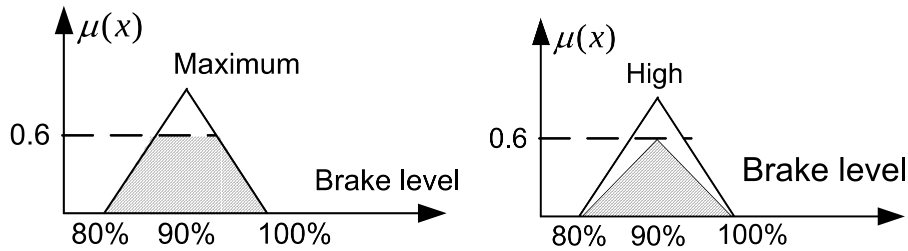

Fig. 28 & 29 *Fuzzy output of the prod and min methods*

In order to produce a satisfactory control scheme one fuzzy rule is not enough. In the example of the speed braking the compensator would not work if the speed was low or very high. General speaking the more fuzzy sets cover the input and output universes of discourse the better. For example, the previous controller would do nothing if the speed were 79 km/h. If more sets are used, then:

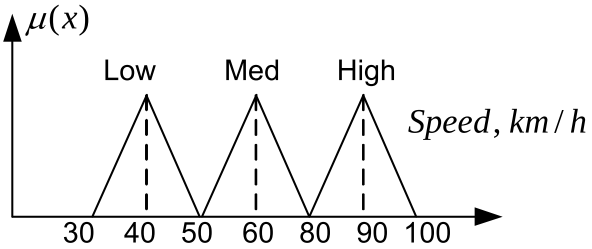

Fig. 30 *Input fuzzification by using 3 sets*

And the output:

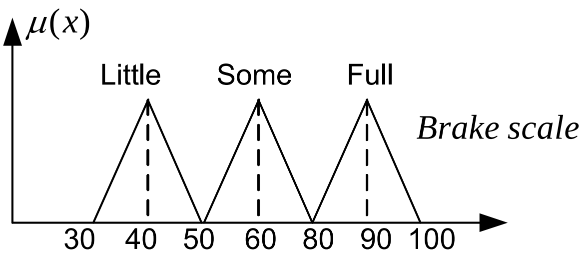

Fig. 31 *Output universe of discourse with 3 sets*

Typical If – Then rules:

1. If Speed is Low Then Brake is Little
2. If Speed is Some Then Brake is Some
3. If Speed is High Then Brake is Full

Or:

1. If Speed==Low Then Brake==Little
2. If Speed==Some Then Brake==Some
3. If Speed==High Then Brake==Full

Hence if the speed is 35 km/h then the input is:

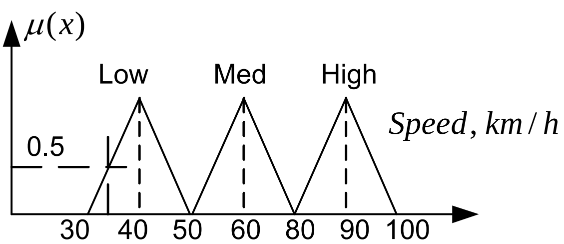

Fig. 32 *Input universe of discourse, when actual input is 35 km/h*

And hence the output by using the min method is:

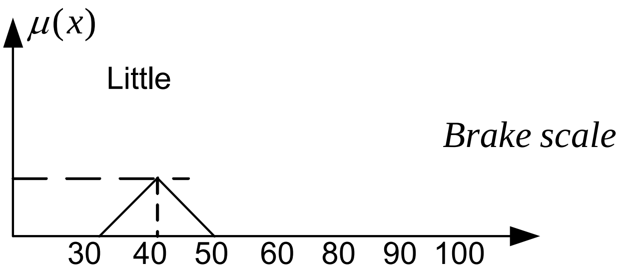

Fig. 33 *Scaled output universe of discourse, when actual input is 35 km/h*

A better coverage of the universe of discourse would include overlapping between the fuzzy sets:

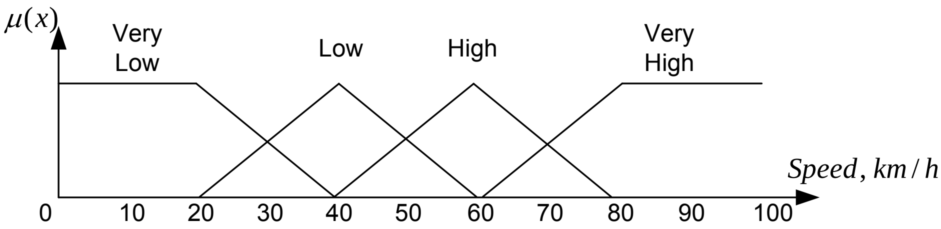

Fig. 34 *Input fuzzification by using 4 overlapping sets*

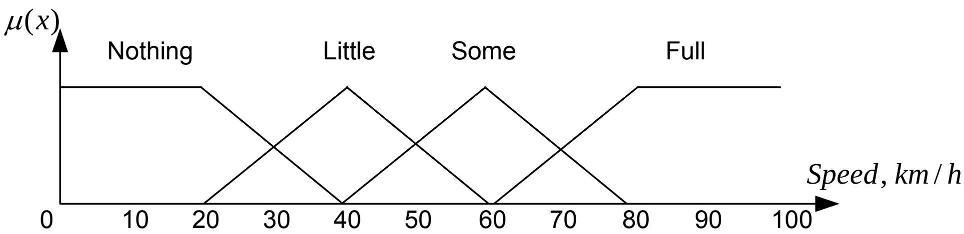

Fig. 35 *Output fuzzification by using 4 overlapping sets*

And the IF – THEN rules:

1. If Speed==Very Low Then Brake==Nothing
2. If Speed==Low Then Brake==Little
3. If Speed==High Then Brake==Some
4. If Speed==Very High Then Brake==Full

Now assume that the speed is 25 km/h. This speed corresponds to the set Very Low 0.8 and to the set Low 0.2. This implies that the degree of support for the rule 1 is 0.8 and the degree of support for the rule 2 is 0.2. Hence the two output fuzzy sets are:

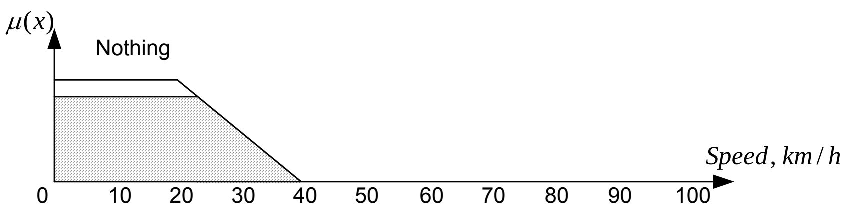

Fig. 36 *Output fuzzy set for rule 1*

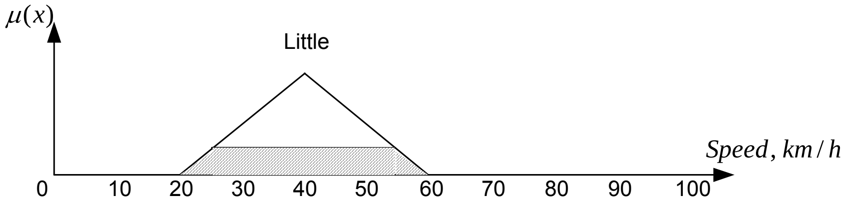

Fig. 37 *Output fuzzification by using 4 overlapping sets*

Now the problem is to find a way to extract the overall fuzzy output set.

There are many ways to aggregate all the outputs to produce an overall output fuzzy set. The three most common are:

- Max (Maximum)
- Prodor (Probabilistic Or)
- Sum

By using now the maximum operator the overall fuzzy set is:

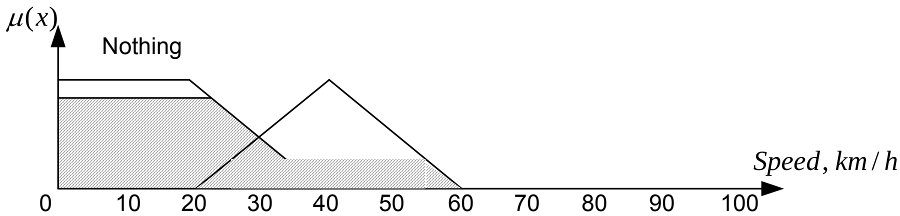

Fig. 38 *Fuzzy output set by using the max operator*

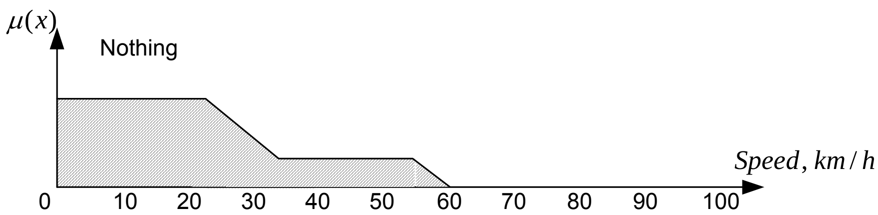

Fig. 39 *Fuzzy output set by using the max operator*

12. Defuzzification

The final step in the design of a FLC is to defuzzify the output fuzzy set to a real value, the controlling signal. There are no clear methods of how to defuzzify the output and there is certainly no theoretical base or methodology for which one is better. Everything depends on the specific application and on the experience of the designer.

12.1 Maximum

In this method the real output is the value that corresponds to the maximum value of the output fuzzy set.

$$out = y : \mu(y) = \max$$

This method is very simply but it has significant drawbacks when the output fuzzy set have multiple maxima. For example it is very difficult to use it in the fuzzy set of Fig. 36

12.2 Mean Of Maxima (MOM)

In this method the mean value of all the maxima is calculated. Hence the previous problem is overcome. Obviously if there is only one maximum then this method will give the same results as before:

$$out = \frac{1}{m}\sum_{j=1}^{m} y_j : \mu(y_j) = \max$$

12.3 Centre of area (COA)

In this method the output is calculated by the centre of the overall volume of the output fuzzy set:

$$out = \frac{\sum_{i=1}^{m} y_j \mu(y_j)}{\sum_{i=1}^{m} \mu(y_j)}$$

12.4 Centre of Gravity (COG) and others

In this method the output is calculated by the centre of the gravity of the output fuzzy set

Other methods (that are included in Matlab) are:

- Largest of maximum
- Smallest of maximum