Magnetic Suspension System Control Using Jacobian and Input State Linearisation

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ABSTRACT

This paper presents two different methods to control a Magnetic Suspension System (MSS). The first control method uses a Jacobian linearisation of the MSS around an Equilibrium Point (EP), with a classical Pole Placement (PP) controller. The second method uses Input State Linearisation (ISL) of the nonlinear model followed by a PP controller. Robustness tests on both methods are described. The advantages and the disadvantages of each method are indicated and methods of further improvement suggested.

INTRODUCTION

The basic characteristic of a MSS is that it can suspend objects without any contact. A classical MSS (Fig. 1) consists of an electromagnet and a steel ball. The resultant force on the ball is the difference between the electromagnetic force, which comes from the magnet, and the force of gravity. In the idealised case where the resultant force is zero the ball would balance at a distance x_e from the magnet (this is called the EP). Unfortunately because of disturbances, which cannot be modelled exactly, the ball would either move towards to the electromagnet and strike the coil or would fall to the ground; thus the system is unstable if the current is constant and there is no feedback. Additionally the electromagnetic force is a function of the square of the current in the coil. Thus the system is also nonlinear as well as being unstable.

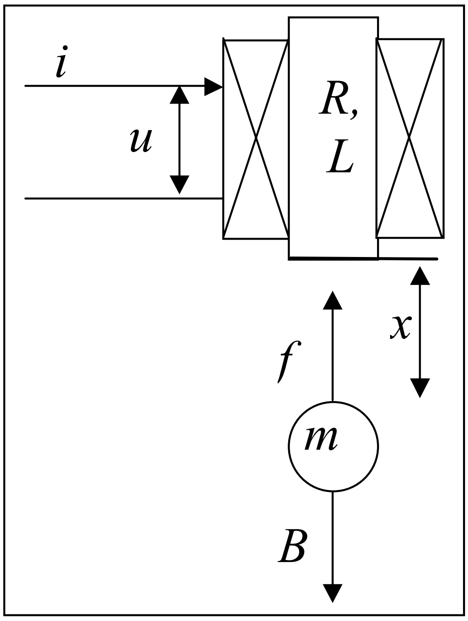


Fig. 1 Schematic of a MSS

In Fig. 1 *i* is the current at the coil, *u* is the voltage supply to the coil, *R* is the resistance of the coil, *L* is the inductance of the coil, *x* is the distance of the steel ball from the electromagnet, *m* is the mass of the steel ball, *B* is the gravitational force, *f* is the electromagnetic force applied to the steel ball. The force *f*, the current *i*, the voltage *u* and the distance *x* are all functions of time.

EQUATIONS OF THE MSS

The resultant force on the ball, assuming simple Newtonian dynamics with negligible friction, can be expressed as:

$$\sum F = f + B = m\frac{d^2x}{dt^2} \quad or \quad \frac{\partial W(i,x)}{\partial x} + mg = m\frac{d^2x}{dt^2}$$

which with a linear magnetic circuit can be written as:

$$\frac{1}{2}i^2\frac{\partial L(x)}{\partial x} + mg = m\frac{d^2x}{dt^2} \tag{1}$$

where *W* is the energy of the electromagnet and *g* is the acceleration due to gravity. According to [1] the inductance function can be modelled in the following exponential form:

$$L(x) = L_1 + L_0 e^{-x/a} \tag{2}$$

where $L(\infty) = L_1$, $L(0) = L_1 + L_0$, a is a length constant. Thus using eqn. 2 in eqn. 1 gives:

$$-\frac{L_0}{2a} i^2 e^{-x/a} + mg = m\frac{d^2x}{dt^2} \text{ or } \frac{d^2x}{dt^2} = -\frac{L_0}{2ma} i^2 e^{-x/a} + g \tag{3}$$

and: $u = iR + \dfrac{d(L(x)i))}{dt}$ (4)

Using eqn. 2 with eqn. 4 gives:

$$u = iR + \frac{d\left(\left(L_1 + L_0 e^{-\frac{x}{a}}\right)i\right)}{dt} \text{ or } \dot{i} = \frac{\left(u - iR + \frac{L_0}{a} i e^{-\frac{x}{a}} \dot{x}\right)}{L(x)} \tag{5}$$

The states of the system are: $x_1 = x, \quad x_2 = \dot{x}, \quad x_3 = i$, the input of the system is u and the output is *x*. Also the ratio $\dfrac{L_0}{2ma}$ is constant and hence for simplicity it can be replaced by a constant gain *k*. Hence the nonlinear system is defined by:

$$X = \begin{bmatrix} x \\ \dot{x} \\ i \end{bmatrix}, \quad \dot{X} = f(X) = \begin{bmatrix} \dot{x} \\ g - ki^2 e^{-x/a} \\ \dfrac{\left(u - iR + \frac{L_0}{a} i e^{-\frac{x}{a}} \dot{x}\right)}{L(x)} \end{bmatrix}, \quad Y(t) = CX = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ i \end{bmatrix} \tag{6}$$

The parameters that define the model were taken from [1] and are: coil inductance L_1 0.349 H, coil inductance L_0 0.229 H, constant a 6.66 mm, mass of the ball *m* 0.8 kg, acceleration of gravity *g* 9.81 m/s^2 and coil resistance *R* 4.3 Ohm.

JACOBIAN LINEARISATION AND POLE PLACEMENT

The Lyapunov linearisation method allows a nonlinear model to be simplified (linearised) around an EP. The EPs of a system can be found by setting $\dot{X} = 0$ hence:

$$\dot{X}_e = \begin{bmatrix} \dot{x}_e \\ g - k i_e^2 e^{-x_e/a} \\ \dfrac{\left(u_e - i_e R + \dfrac{L_0}{a} i_e e^{-\frac{x_e}{a}} \dot{x}_e\right)}{L(x_e)} \end{bmatrix} = 0 \text{ or } \begin{matrix} \dot{x}_e = 0 \\ i_e = \pm\sqrt{\dfrac{g}{k} e^{\frac{+x_e}{a}}} = \pm 1.43 \quad A \\ u_e = i_e R = 6.16 \quad \mathrm{A} \end{matrix} \tag{7}$$

For simplicity the parameters i_e and x_e will be written as I and d respectively. So the linearised model around the EP (d,0,I) is:

$$A = \left.\frac{\partial f(X,u)}{\partial X}\right|_{EP} = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{kI^2 e^{-d/a}}{a} & 0 & -2kIe^{-d/a} \\ 0 & \dfrac{L_0 I e^{-d/a}}{aL(d)} & -\dfrac{R}{L(d)} \end{bmatrix},\ B = \left.\frac{\partial f(X,u)}{\partial u}\right|_{EP} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{L(d)} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{8}$$

The linearised system can be represented in the usual state space form by the matrices *A*, *B*, *C* (*D*=0, since there is no direct input-output coupling). Using the variables from [1] and for *d*=1cm these matrices are:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1473 & 0 & -13.71 \\ 0 & 27.41 & -10.75 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 2.5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{9}$$

This linearised system can be used to investigate several control laws. Since a state space representation is used the choice of the PP control law was obvious as the system is controllable. The closed loop system is $A_{CL} = A - BK$, $B_{CL} = BF$ and $C_{CL} = C$, where *K* will be found from the PP control law and *F* from the error criteria. The desired pole locations were chosen to be [-1 –2 –3]. With the use of Matlab the state gain vector was found to be *K*=[-258.1 -32.34 -1.9], the closed loop matrices were found to be: $A_{CL} = \begin{bmatrix} 0 & 1 & 0 \\ 1473 & 0 & -13.7 \\ 645.2 & 108.3 & -6 \end{bmatrix}$, $B_{CL} = \begin{bmatrix} 0 \\ 0 \\ -0.0044 \end{bmatrix}$. Finally the gain matrix *F*, using trial and error methods, was found to be -0.00175.

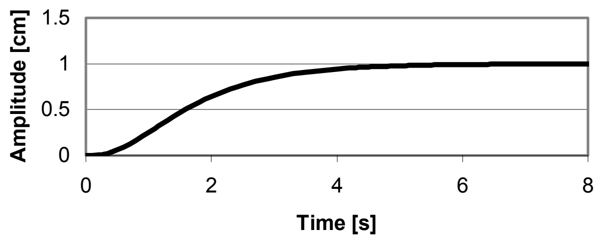


Fig. 2 Step response of the Jacobian linearised system with pole placement control law

INPUT STATE LINEARISATION

Theory Background

The previous method, even though it is straightforward, has a very big drawback. This comes from the fact that the whole operating region of the system must be suppressed around the EP. The system may not work at all outside this region since the linear controller will be unable to compensate the nonlinearities that will appear as soon as the system moves far away from the EP. There are various ways to overcome this problem. One method that has been used in many difficult problems is ISL. This powerful method is based on use of a transformation of the states and a transformation of the input; to describe the system in a linear way. Then a simple linear controller can be applied. For more detailed information about this method see [2].

According to [2] the basic steps of the ISL are:

1. The Single Input Single Output (SISO) system must be represented by the state equations: $\dot{X} = f(X) + g(X)u$ i.e. to be in a companion form. This means that the input signal appears only in its natural form and not via its derivatives, also *f* and *g* represent smooth vector fields, i.e. are continuous and their high derivatives exist. If the system is not in this form then some kind of transformation must be applied.
2. Construct the vector fields: $g, ad_f g, \dots ad_f^{\,n-1} g$, where $ad_f g$ is the Lie bracket of *f* and *g*: $[f, g] = ad_f g = \nabla g f - \nabla f g$ and $\nabla f(X) = \partial f(X)/\partial X$.
3. Check if the controllability and involutivity conditions are satisfied. The term involutivity means: that from a set of vector fields if the lie bracket of two is taken then the resultant vector field can be expressed as a linear combination of the original set of the vector fields.
4. Find the first state z_1, by $\begin{cases} \nabla z_1 ad_f^{\,i} g = 0, & i = 1, \dots, n-2 \\ \nabla z_1 ad_f^{\,n-1} g \neq 0 \end{cases}$
5. Compute the state transformation $z(X) = \begin{bmatrix} z_1 & L_f z_1 & \dots & L_f^{\,n-1} z_1 \end{bmatrix}^T$, where $L_f z_1$ is the Lie derivative of z_1 with respect to *f*, i.e. $(\nabla z_1(X)) f(X)$.
6. Compute the input transformation $u = a(X) + \beta(X)v$, with:

   $a(X) = -\dfrac{L_f^{\,n} z_1}{L_g L_f^{\,n-1} z_1}, \quad \beta(X) = \dfrac{1}{L_g L_f^{\,n-1} z_1}$ and
7. Calculate the new linear control law *v*.

ISL and MSS

The vector fields *f*, *g* are:

$$f(X)=\begin{bmatrix} \dot{x} \\ g-ki^2e^{-x/a} \\ \dfrac{\left(-iR+\dfrac{L_0}{a}ie^{-\frac{x}{a}}\dot{x}\right)}{L(x)} \end{bmatrix},\quad g(X)=\begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{L(x)} \end{bmatrix} \tag{10}$$

According to [3] the system can be input state linearised if a<<x.

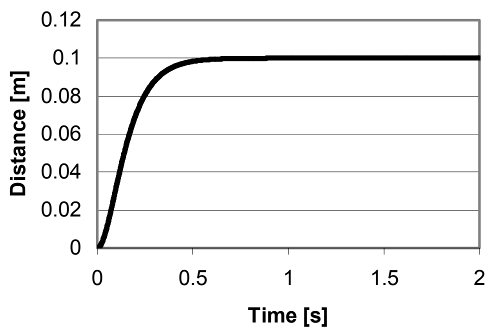
Hence: $z_1 = x_1 = x,\quad z_2 = \dfrac{\partial z_1}{\partial X} f(X) = x_2 = Dx,\quad z_3 = \dfrac{\partial z_2}{\partial X} f(X) = g - ki^2 e^{-x/a}$ (11)

$$u=\frac{v-\dfrac{\partial z_3}{\partial X}f(X)}{\dfrac{\partial z_3}{\partial X}g(X)}=\frac{v-\begin{bmatrix}\dfrac{k}{a}i^2e^{-x/a} & 0 & -2kie^{-x/a}\end{bmatrix}\begin{bmatrix} Dx \\ g-ki^2(t)e^{-x/a} \\ \dfrac{1}{L(x)}\left(\dfrac{L_0}{a}ie^{-x/a}Dx-iR\right)\end{bmatrix}}{\begin{bmatrix}\dfrac{k}{a}i^2e^{-x/a} & 0 & -2kie^{-x/a}\end{bmatrix}\begin{bmatrix}0\\0\\\dfrac{1}{L(x)}\end{bmatrix}}$$

$$u=\frac{v-\left(\dfrac{k}{a}i^2e^{-x/a}Dx\right)+\left(-2kie^{-x/a}\dfrac{1}{L(x)}\left(\dfrac{L_0}{a}ie^{-x/a}Dx-iR\right)\right)}{-2kie^{-x/a}\dfrac{1}{L(x)}} \tag{12}$$

where *v* is the new linear control law. Since the system is controllable (a requirement for the ISL) a pole placement control law was chosen.

The system now can be described as $\dot{z} = Az + Bv$ where $A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$, B= $\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$. The PP control law placed the closed loop poles at [-1 -2 -3]. Typical step responses are shown in Fig. 3 and 4.



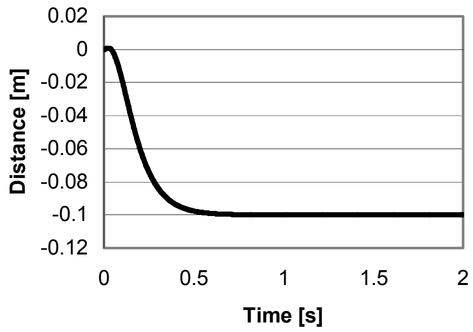


Fig. 3 Step response for input 0.1m with ISL

Fig. 4 Step response for input -0.1m with ISL

Figs. 3-4 show that when the input signal changes sign the output does also (not a practical consequence, this simply reflects the symmetry of the assumed model). So the system exhibits linear behaviour. Also the error was very small in all these cases.

ROBUSTNESS

The two compensating methods above have been checked for their robustness. Two tests have been applied for that. In the first the parameter sensitivity was checked and in the second the performance of the system in the presence of noise signals evaluated.

Jacobian Linearised System

The parameter that was changed was the mass. The variation was –30% & +30% of the nominal value. The behaviour of the system showed very good robustness against the mass change, see Fig. 5.

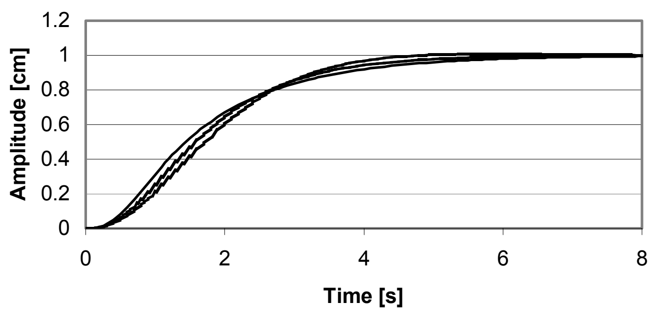


Fig. 5 Jacobian linearised system's sensitivity to mass changes

Input State Linearisation

The same tests were done for the system using ISL. The response of the system due to mass changes was not as satisfactory as before, Fig. 6. Finally the system using ISL was checked for disturbance rejection. A white noise signal was added to simulate the effect of an external unmodelled force. As can be seen in Fig. 7 the output depends a lot on the noise power. The idea of input state linearisation is based on accurate modelling of the system. Since the disturbance force is not modelled in the systems equations it is natural that the system's noise rejection is poor.

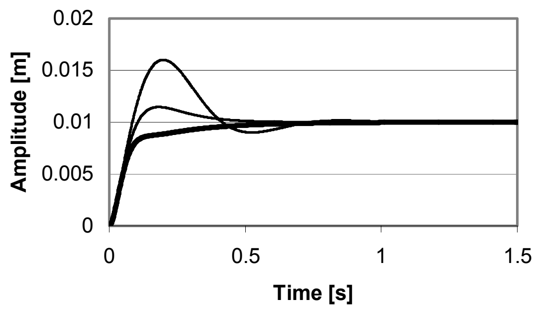


Fig. 6 System's sensitivity to mass changes with ISL

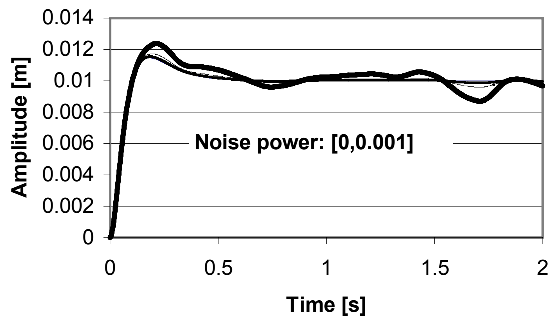


Fig. 7 Disturbance rejection with ISL

CONCLUSIONS

A system, which is both highly nonlinear and open loop unstable, was investigated using two different compensating methods. The first applied simple Jacobian linearisation and in the second an ISL scheme was used; a pole placement control law followed both linearisation techniques. The simulation results showed that the system using ISL could have a very wide operation region. Hence it overcame the classical problem that the Jacobian linearisation may have, where such a linear controller cannot satisfactorily control the system outside a small region near the EP. The problem with the ISL is that the model has to be very accurate. In the presence of noise the system with ISL can show poor behaviour. Both compensation schemes can be very good depending on the application. If the application requires a small operating region and parameter variations such as mass or electrical resistance changes are likely to be small then the Jacobian technique can be good. If the system needs to have a wide operating region then the ISL is better assuming that its model is accurate. To further improve the behaviour of the ISL scheme adaptive robust techniques can be used.

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