Stability of switching circuits using complete-cycle solution matrices

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***Abstract*— The appearance of nonlinear phenomena like bifurcations and chaos in dc-dc converters are mainly studied by using the Poincaré map of the system. This paper presents an alternative method based on the eigenvalues of the state transition matrix over one full cycle which provides better insight of the system and its stability properties. The paper shows how the state transition matrix for a full cycle can be applied to a wide class of power electronic circuits to investigate the stability of various limit cycles and offers considerable advantages over other convectional methods without increasing the complexity of the analysis. Another advantage of this method is its ability to explain and predict the length of intermittent subharmonic phenomena which occur when these converters are coupled with spurious signals.**

I. Introduction

DURING the last 15 years the scientific community has turned its attention to the study of the nonlinear behavior of piecewise smooth systems and more specifically the analysis of the bifurcation structure of dc-dc converters. The main and desired behavior of the system is a stable periodic motion around a predetermined/predefined value. The period of that motion being equal the period of the external clock that is used to control the system. There are cases, however, where a small change of a system parameter may cause the limit cycle to lose its stability and another periodic orbit maybe born with double period. By far the most common method of studying this period doubling bifurcation is the use of the Poincaré map, [1]–[3], which samples the states of the system at regular intervals with the stability properties of the system being described by the eigenvalues of this map. Other maps have also been proposed like the A and S-switching maps [4]. Methods like the trajectory sensitivity analysis [5] have also been used to study the stability. Most of these methods, apart from answering the fundamental question of determining the stability of the limit cycle, offer little knowledge of how and why this loss of stability occurs. In [6], [7] we have shown that using the monodromy matrix it is possible to study the stability of period-1 limit cycles of the step down dc-dc converter. Based on this knowledge we were able to propose and optimally design a period doubling bifurcation controller [6] which can greatly extend the region of the stable period-1 motion.

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This paper extends the use of the monodromy matrix to period two limit cycles and proves that the state transition matrix during the switching can be the identity matrix (when there is no sliding mode [8]). Initially the stability of the period one and two limit cycles of the buck converter is studied before the analysis is extended to another dc-dc converter, namely the boost converter. Finally we use the eigenvalues of the monodromy matrix to identify the length of the intermittent subharmonic windows that were reported in [9].

II. Definition of the monodromy matrix

One of the most important properties of a system is its stability. Even though the stability of equilibria is fairly simple the determination of the stability of limit cycles is rather tricky. One method of determining the stability of periodic motions is to calculate the state transition matrix of the system over one full cycle and then to determine its eigenvalues. If they lie in the unit circle then the periodic motion is stable. The state transition matrix of the system (also called fundamental solution matrix) evaluated over a full period is referred to as the monodromy matrix [10], [11]. If the system is piecewise smooth then the effects of the switching(s) must be taken into account when calculating the monodromy matrix. For example, if there is one switching in a period, then the monodromy matrix must be broken into three state transition matrices. Two for the two smooth regions before and after the switching, and one for the transition across the switching surface:

$$\mathbf{M}(T_p,0)=\mathbf{\Phi}(T_p,t_\Sigma)\times\mathbf{\Phi}(t_\Sigma,t_\Sigma)\times\mathbf{\Phi}(t_\Sigma,0) \qquad (1)$$

where $\mathbf{\Phi}(T_A,t_A)$, is the state transtion matrix from $t=t_A$ until $t=t_B$, T_p is the period of the cycle and t_Σ is the instant where the trajectory crosses the manifold. The matrix $\mathbf{S}=\mathbf{\Phi}(t_\Sigma,t_\Sigma)$ is the state transition matrix during the switching (also called the saltation matrix) and is defined as [11]:

$$\mathbf{S}=\mathbf{I}+\frac{(\lim\limits_{t\downarrow t_\Sigma}(\mathbf{f}_-(\mathbf{x}(t)))-\lim\limits_{t\uparrow t_\Sigma}(\mathbf{f}_+(\mathbf{x}(t))))\mathbf{n}^T}{\mathbf{n}^T\lim\limits_{t\uparrow t_\Sigma}(\mathbf{f}_+(\mathbf{x}(t)))+\dfrac{\partial h}{\partial t}(\mathbf{x}(t),t_\Sigma)} \qquad (2)$$

To determine the Floquet multipliers of the system i.e. the eigenvalues of the monodromy matrix we have to calculate the three state transition matrices shown in eqn. 1. If the system is linear time invariant then the state transition matrix before and after the switching can be easily calculated by using the exponential matrix. On the other hand the calculation of the saltation matrix can be rather cumbersome as the system may have transcendental equations [7].

III. THE BUCK CONVERTER

A. Analysis of period-1 operation

The voltage controlled buck converter is shown in Fig. 1. It consists of a dc voltage source (V_{in}), a switch (S), an low pass filter (L, C) and a resistor (R) which represents the load. The switch is controlled through a PWM signal which is obtained through a comparison of the control signal (V_{con}) with a sawtooth signal (V_{ramp}). Due to the switching action the response of the system will be a dc value with a small periodic ripple (limit cycle). In normal operating conditions the period of this cycle equals the period of the saw tooth signal T, Fig 2a. To study the stability of this cycle it suffices to study the evolution of the system in $t \in [0, T)$. In that interval it can be seen that we will have one crossing of the trajectory with the switching manifold, at $t_\Sigma = (1-d)T$, where d is the duty cycle. The two vector fields before and after the switching are:

$$\begin{aligned} \mathbf{f}_+(\mathbf{x}(t)) &= \begin{bmatrix} x_2(t)/C - x_1(t)/RC \\ -x_1(t)/L \end{bmatrix}, \\ \mathbf{f}_-(\mathbf{x}(t)) &= \begin{bmatrix} x_2(t)/C - x_1(t)/RC \\ (V_{\text{in}} - x_1(t))/L \end{bmatrix}. \end{aligned}$$

Since the system is piecewise linear $\mathbf{\Phi}(t_\Sigma, T) = e^{\mathbf{A}_s dT}$ and $\Phi(0, t_\Sigma) = e^{\mathbf{A}_s(1-d)T}$, where $\mathbf{A}_s$ is the Jacobian of the two vector fields: $\mathbf{A}_s = \begin{bmatrix} -1/RC & 1/C \\ -1/L & 0 \end{bmatrix}$. The switching hypersurface (h) is given by

$$h(\mathbf{x}(t), t) = x_1(t) - V_{\text{ref}} - \frac{v_{\text{ramp}}(t)}{A} = 0,\ A \neq 0. \quad (3)$$

$$v_{\text{ramp}}(t) = V_L + (V_U - V_L)\left(\frac{t}{T} \mod 1\right) \quad (4)$$

We numerically calculated the duty cycle by solving a nonlinear transcendental equation to derive the duty cycle and hence we determined the stability of the system. The results of this analysis which were first presented in [7] and were verified experimentally and numerically. The results are presented for convenience in Table I.

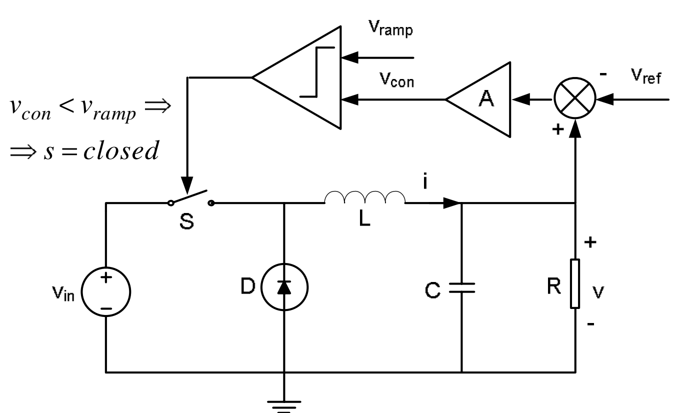
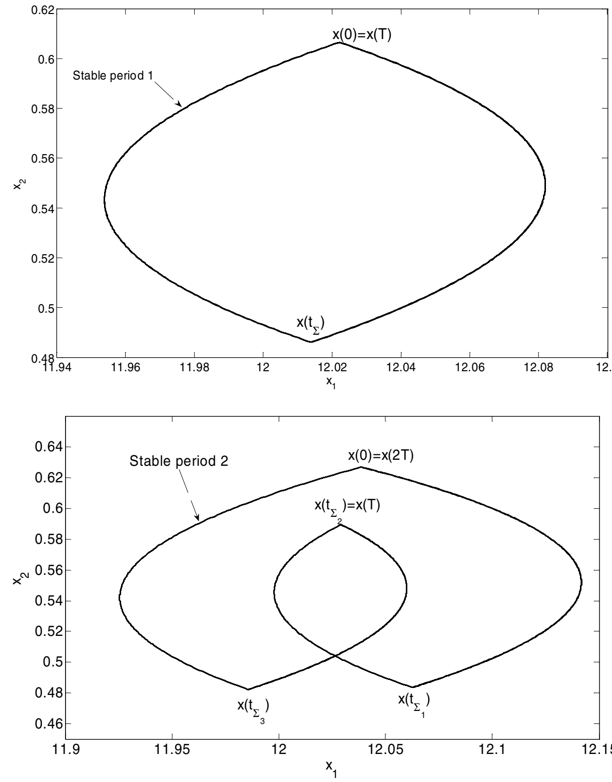


Fig. 1. The voltage mode controlled buck dc-dc converter.



(a)

(b)

Fig. 2. Period 1 and 2 of the buck converter.

TABLE I

FLOQUET MULTIPLIERS FOR VARIOUS INPUT VOLTAGES, PERIOD 1

V_{in}, V	Floquet multipliers
14	$-0.6265 \pm 0.5354j$
20	$-0.6919 \pm 0.4477j$
24	$-0.8211 \pm 0.0708j$
25	$\begin{Bmatrix} -0.6214 \\ -1.0929 \end{Bmatrix}$

B. Period two

Fig. 2b also shows a stable period two limit cycle of the buck converter and it is obvious from this, that the trajectory will cross the switching manifold 3 times (assuming no border collision or other discontinuous bifurcations) in $t \in [0, 2T_p)$. Hence the overall state transition matrix is $\mathbf{M}(2T, 0) = \mathbf{\Phi}(2T, t_{\Sigma_3}) \times \mathbf{S}_3 \times \mathbf{\Phi}(t_{\Sigma_3}, T) \times \mathbf{S}_2 \times \mathbf{\Phi}(T, t_{\Sigma_1}) \times \mathbf{S}_1 \times \mathbf{\Phi}(t_{\Sigma_1}, 0)$. To calculate $\mathbf{S}_2$ we have to find the time derivative of the switching manifold at $t = T$. Since the manifold is discontinuous (with respect to time) at this point the time derivative will be either $-\infty$ or $+\infty$. Regardless of its sign the saltation matrix will be $\mathbf{S}_2 = \mathbf{I}$ hence our analysis is greatly simplified as we do not have to take into account this discontinuity. $\mathbf{S}_1$ and $\mathbf{S}_3$ are given by:

$$\mathbf{S}_1 = \begin{bmatrix} 1 & 0 \\ \dfrac{V_{\text{in}}/L}{\dfrac{x_2(t_{\Sigma_1}) - x_1(t_{\Sigma_1})/R}{C} - \dfrac{V_U - V_L}{AT}} & 1 \end{bmatrix} \quad (5)$$

$$\mathbf{S}_3 = \begin{bmatrix} 1 & 0 \\ \dfrac{V_{\text{in}}/L}{\dfrac{x_2(t_{\Sigma_3}) - x_1(t_{\Sigma_3})/R}{C} - \dfrac{V_U - V_L}{AT}} & 1 \end{bmatrix} \tag{6}$$

To calculate the state vector at $t = t_{\Sigma_1}$ and $t = t_{\Sigma_3}$ we will follow a procedure similar to [12] and [7]:

$$\mathbf{x}(2T) = e^{\mathbf{A}_s(2T - t_{\Sigma_3})}\mathbf{x}(t_{\Sigma_3}) + \int_{t_{\Sigma_3}}^{2T} e^{\mathbf{A}_s(2T-\tau)} \begin{bmatrix} 0 \\ \dfrac{V_{\text{in}}}{L} \end{bmatrix} d\tau \tag{7}$$

$$\mathbf{x}(t_{\Sigma_3}) = e^{\mathbf{A}_s(t_{\Sigma_3} - T)}\mathbf{x}(T) \tag{8}$$

$$\mathbf{x}(T) = e^{\mathbf{A}_s(T - t_{\Sigma_1})}\mathbf{x}(t_{\Sigma_1}) + \int_{t_{\Sigma_1}}^{T} e^{\mathbf{A}_s(T-\tau)} \begin{bmatrix} 0 \\ \dfrac{V_{\text{in}}}{L} \end{bmatrix} d\tau \tag{9}$$

$$\mathbf{x}(t_{\Sigma_1}) = e^{\mathbf{A}_s(t_{\Sigma_1})}\mathbf{x}(0) \tag{10}$$

and

$$\mathbf{x}(2T) = \mathbf{x}(0)$$

t_{Σ_1} is defined by $t_{\Sigma_1} = d'_1 T$ and hence $T - t_{\Sigma_1} = (1 - d'_1)T$. Likewise we define as $t_{\Sigma_3} - T = d'_2 T$ and hence $t_{\Sigma_3} = T + d'_2 T$ so $2T - t_{\Sigma_3} = 2T - T + d'_2 T = (1 - d'_2)T$. From the circuit topology it is obvious that $d'_1 = 1 - d_1$ and $d'_2 = 1 - d_2$, where d_1 and d_2 are the two duty cycles for each clock cycle respectively.

The two switchings occur when:

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t_{\Sigma_1}) = V_{\text{ref}} + \frac{V_L + (V_U - V_L)d'_1}{A} \tag{11}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t_{\Sigma_3}) = V_{\text{ref}} + \frac{V_L + (V_U - V_L)d'_2}{A} \tag{12}$$

These two eqns. will give a system of two nonlinear transcendental equations with two unknowns which was solved numerically. The solutions of that are shown in Fig. 3 and a sample of the corresponding state vector in Table II.

TABLE II
STATE VECTORS AT SWITCHING INSTANTS, PERIOD 2

V_{in}, V	$\mathbf{x}(t_{\Sigma_1})$	$\mathbf{x}(t_{\Sigma_3})$
25	$\begin{bmatrix} 11.986 & 0.48225 \end{bmatrix}^T$	$\begin{bmatrix} 12.062 & 0.4835 \end{bmatrix}^T$
30	$\begin{bmatrix} 11.94 & 0.4454 \end{bmatrix}^T$	$\begin{bmatrix} 12.19 & 0.4731 \end{bmatrix}^T$
30.5	$\begin{bmatrix} 11.938 & 0.4412 \end{bmatrix}^T$	$\begin{bmatrix} 12.199 & 0.47249 \end{bmatrix}^T$
31	$\begin{bmatrix} 11.937 & 0.43694 \end{bmatrix}^T$	$\begin{bmatrix} 12.207 & 0.47191 \end{bmatrix}^T$
31.5	$\begin{bmatrix} 11.935 & 0.43258 \end{bmatrix}^T$	$\begin{bmatrix} 12.214 & 0.47137 \end{bmatrix}^T$

By using the values of Table II is possible to compute the monodromy matrix of the period 2 cycle for various values of the input voltage and from that to calculate the corresponding floquet multipliers (Fig. 4 and Table III).

The presented results of period 1 and 2 were validated by numerical and experiential results and by the bifurcation diagram shown in Fig. 5

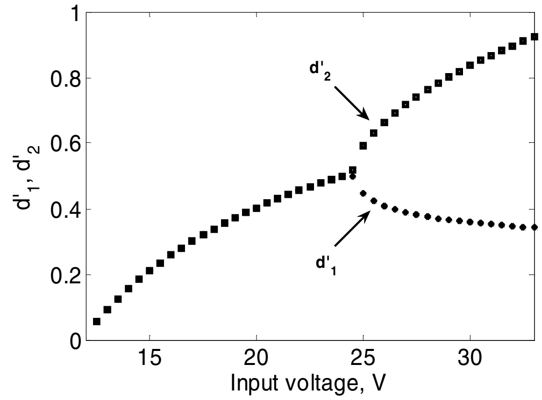


Fig. 3. Evaluation of the duty cycles, period 2

TABLE III
FLOQUET MULTIPLIERS FOR VARIOUS INPUT VOLTAGES, PERIOD 2

V_{in}, V	Floquet multipliers
25	$0.613895 \pm 0.29059j$
30	$-0.50408 \pm 0.45521j$
31	$\begin{Bmatrix} -0.90387 \\ -0.51037 \end{Bmatrix}$
31.5	$\begin{Bmatrix} -1.241 \\ -0.37172 \end{Bmatrix}$

IV. BOOST CONVERTER

The preceding analysis can also be carried out on the boost converter. We recognize that the stability analysis of the current controlled boost converter is a lot easier than the one of the buck converter when we use the Poinacré map since it does not involve any transcendental equations. Nevertheless we present here how the previously mentioned method can be used in order to demonstrate the fact that it is universally applicable to any piecewise smooth system.

The current controlled boost converter, shown in Fig. 6 is another piecewise system whose main and desired behavior is a period one limit cycle, Fig. 7. The switching manifold is defined as $h(x(t)) = x_2 - I_{\text{ref}}$, and the system before and

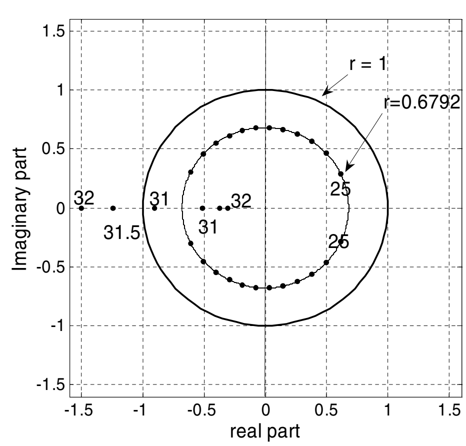


Fig. 4. Evolution of the Floquet multipliers of the period-2 orbit, as V_{in} is varied.

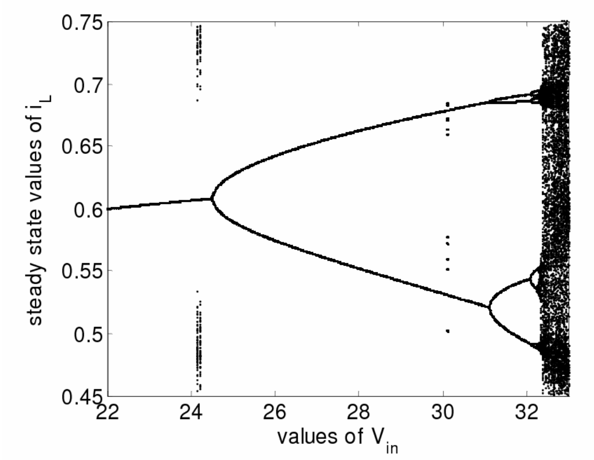


Fig. 5. Bifurcation diagram of the buck converter

after the switching is described by

$$\dot{\mathbf{x}} = \begin{bmatrix} \frac{-1}{RC} & 0 \\ 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} V_{\text{in}}, \quad \dot{\mathbf{x}} = \begin{bmatrix} \frac{-1}{RC} & \frac{1}{C} \\ \frac{-1}{L} & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} V_{\text{in}}$$

where

$$\begin{bmatrix} v_c \\ i_L \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The two smooth vector fields before and after the switching are:

$$\mathbf{f}_{-}(x) = \begin{bmatrix} -x_1/RC \\ V_{\text{in}}/L \end{bmatrix}, \quad \mathbf{f}_{+}(x) = \begin{bmatrix} \dfrac{x_2 R - x_1}{RC} \\ \dfrac{V_{\text{in}} - x_1}{L} \end{bmatrix} \tag{13}$$

Hence $\mathbf{n} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

By using the formula of eqn (1) it is found that the saltation matrix is:

$$\mathbf{S} = \mathbf{I} + \frac{(\mathbf{f}_{+} - \mathbf{f}_{-})\mathbf{n}^T}{\mathbf{n}^T \mathbf{f}_{-}} = \begin{bmatrix} 1 & \frac{x_2(t_\Sigma)L}{CV_{\text{in}}} \\ 0 & \frac{V_{\text{in}} - x_1(t_\Sigma)}{V_{\text{in}}} \end{bmatrix} \tag{14}$$

Notice that since the switching manifold is not time dependent the calculation of the saltation matrix is simpler than its counterpart for the buck converter. To calculate the value of the state vector at (t_Σ) we can use either the same process as for the buck converter or we can obtain it analytically, [1]. Based on these values it is possible to find the Floquet multipliers of the system (Table IV) and to study its stability. These values agree perfectly with the results that were obtained from the numerical simulation of the system (Fig. 8) and they also agree with the bifurcation diagram of the system (Fig. 9).

V. Intermittent subharmonic windows

In [9] it was first reported that external spurious signals may influence the stability properties of the system. To model these signals the authors in [9] substituted V_{ref} by $V_{\text{ref}} + aV_{\text{ref}}\sin(\omega_s t)$, where a is the strength of the spurious signal and ω_s is their frequency which maybe different from $\omega = \frac{2\pi}{T}$. For convenience we reproduce these results in Fig. 10

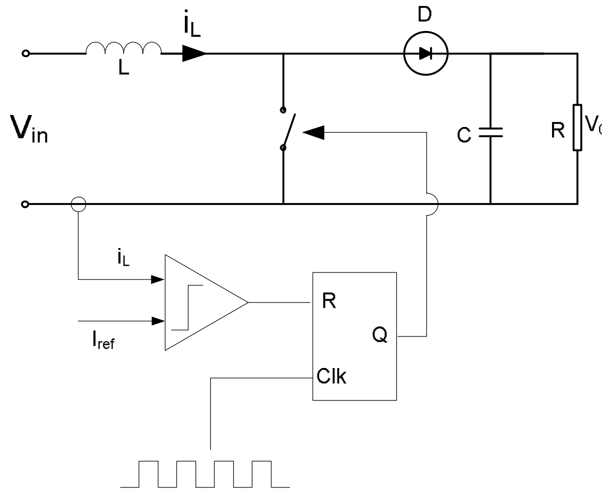


Fig. 6. Current controlled boost converter

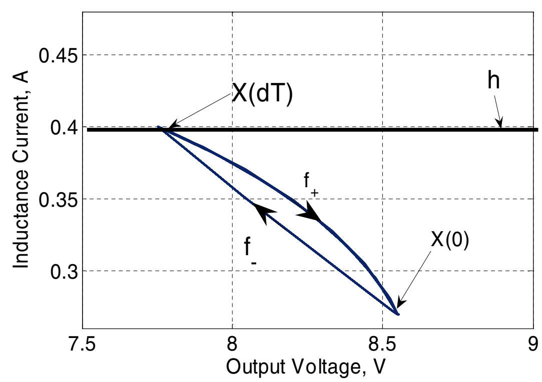


Fig. 7. Phase space for boost converter, $I_{\text{ref}} = 0.4A$

The problem with the analysis of that system is that the value of V_{ref} is time periodic with a period other than T and hence we cannot use the assumption $t \in [0, T)$. This implies that we cannot simplify the equation of the switching manifold to: $h(\mathbf{x}(t), t) = x_1(t) - V_{\text{ref}} - \dfrac{V_L + (V_U - V_L)/T}{A} = 0$ as we have for the normal uncoupled period one operation. In [7] we studied the cause for the appearances of subharmonic windows by assuming that $\omega = \omega_s$. In order to carry out the analysis the duration of the subharmonic window had to be extended from $-\infty$ to $+\infty$ instead of the experimentally

TABLE IV

Floquet multipliers for various I_{ref}, boost converter

I_{ref}, A	Floquet multipliers
0.46	$\{0.5560,\ -0.9240\}$
0.49	$\{0.56011,\ -0.990\}$
0.5	$\{0.5613,\ -1.012\}$
0.51	$\{0.5624,\ -1.034\}$

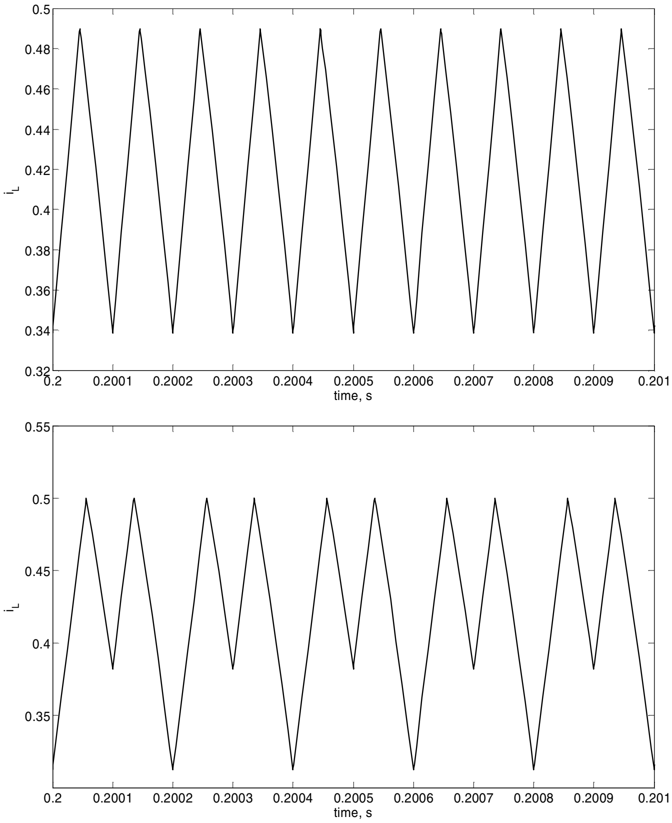


Fig. 8. Numerical results for $I_{\mathrm{ref}} = 0.49A$ and $0.5A$

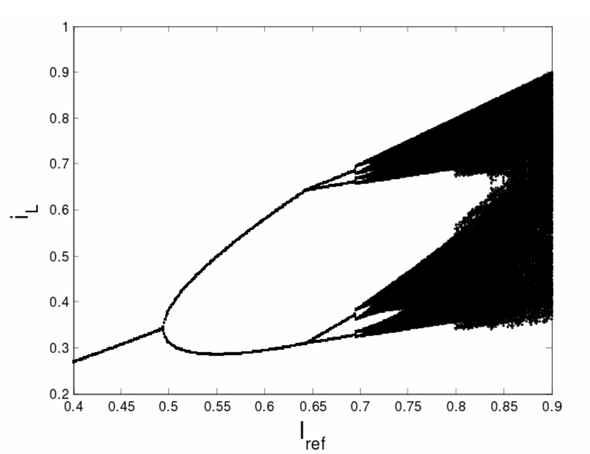


Fig. 9. Bifurcation diagram of boost converter

observed period $2\pi/|\omega - \omega_s|$ [9]. In this part of the paper the analysis is extended to semi-analytically predict the length of the subharmonic window. To do this we break the frequency of the spurious signal to $\omega_s = \omega + \Delta\omega$ and hence $\sin(\omega_s t) = \sin((\omega + \Delta\omega)t)$. Since $\Delta\omega$ is a lot smaller than ω it can be assumed to be constant and hence we can repeat the analysis [7] for various values of $\Delta\omega$. This implies that the switching manifold can be considered to be:

$$h(x(t), t, \tau) = x_1 - V_{\mathrm{ref}} - aV_{\mathrm{ref}} \sin(\omega t + \Delta\omega\tau) - V_{\mathrm{ramp}} \quad (15)$$

$$t \in (0, T_p), \tau \in \left(0, \frac{2\pi}{\Delta\omega}\right)$$

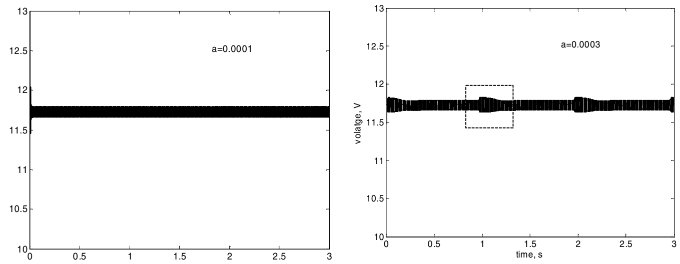


Fig. 10. Voltage response for $a = 0.0001$ and $a = 0.0003$

The interval $\left(0, \frac{2\pi}{\Delta\omega}\right)$ is broken into 100 points and for these we calculate the points where the period one orbit becomes unstable. Fig. 11 shows the numerical simulation of the buck converter with the spurious signal and Fig. 12 shows the duration of the window as it was calculated by using the previous method for $a = 0.0003$ and $V_{\mathrm{in}} = 24$V. Table V summarizes the results.

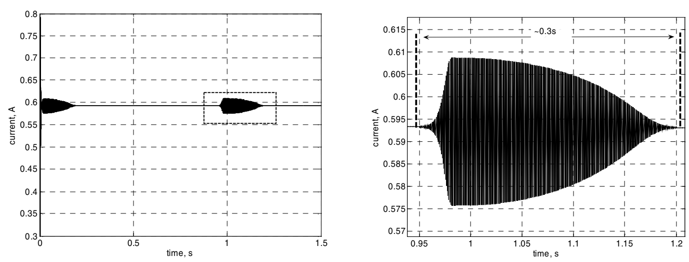


Fig. 11. Subharmonic window for a=0.0003

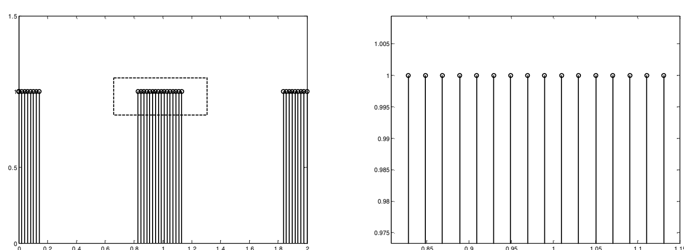


Fig. 12. Duration of the subharmonic window as it was semi-analytically calculated

TABLE V

LENGTH OF SUBHARMONIC WINDOW

Strength of spurious signal	Duration
0.0003	0.303
0.0004	0.363
0.0006	0.404
0.0009	0.424

VI. CONCLUSION

The monodromy matrix has been derived for various limit cycles for the dc-dc buck and boost converters. This matrix which defines the state transition matrix over one full cycle has been used to determine the stability of various limit cycles of both circuits. This method offers an alternative to other existing methods based on the Poincaré map and also offers a deeper insight of how and the limit cycles lose their stability. The proposed method has been shown to be universally applicable to any piecewise smooth system. This was further demonstrated by its use to determine semi-analytically, the length of subharmonic windows experienced by the system. This knowledge can be used to design appropriate controllers to ensure period one operation over a wider range in the presence of severe disturbances by external spurious signals. Currently the authors are working on the application of the new method to more complicated AC-DC converters.

APPENDIX

Parameters of buck converter: $V_{\mathrm{in}} = 24V$, $V_{\mathrm{ref}} = 11.3V$, $L = 20m$H, $R = 22\Omega$, $C = 47\mu$F, $A = 8.4$, $T = 1/2500$s,$\omega_s = 2505 \times 2\pi$s, the ramp signal varies from 3.8V to 8.2V.

Parameters of boost converter: $C = 10\mu F$,$L = 1.5mH$, $R = 40\Omega$, $T = 100\mu s$ and $V_{\mathrm{in}} = 5V$.

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