CAPITAL BUDGETING, VALUATION AND PERSONAL TAXES

by

Ian M Dobbs

and

Anthony D Miller*

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* The authors are, respectively, Reader in Business Economics and Finance, and Lecturer in Accounting and Financial Management in the Department of Accounting and Finance, University of Newcastle Upon Tyne.

Abstract

This paper examines the relationship between before tax and after tax valuation and uses this to examine the literature on capital budgeting and capital structure in the presence of corporate and personal taxes, a literature which features a bewildering array of valuation formulae. Some of the variation between such formulae naturally arises out of variations in underlying model assumptions; however, in several cases, it arises because there are (by no means obvious) internal inconsistencies. The potential magnitude of the errors that might arise in a capital budgeting context is then explored through sensitivity analysis.

I. Introduction and Review of the Literature

For many years the Value Additivity Principle (*VAP*) has provided the cornerstone for the valuation of complex assets within a setting of perfect capital markets. Under this principle, a portfolio can be correctly valued by breaking it into its constituent assets, independently valuing each asset, and then adding the resulting values together (Haley & Schall [1973]). Applied to the theory of capital structure[1], where the focus is on the interaction between an investment decision and its financing, the *VAP* prescribes that the value of a levered investment should be equal to the value of an otherwise-identical unlevered investment plus the value of incremental cash flows attributable to leverage. Modigliani & Miller [1963] presented a seminal model in which debt created a valuable incremental corporation tax shield. In addition to assigning a value to this tax shield, Modigliani and Miller (*MM*) derived an adjusted discount rate (*ADR*) which could be used to compute the value of a levered investment without *explicit* consideration of the incremental cash flows arising from leverage. This *ADR* subsequently found a place in conventional textbook accounts of capital budgeting procedures as the weighted average cost of capital (e.g. Brealey and Myers [1996], Buckley et al. [1998]). *MM*'s specific results, however, were based upon a number of restrictive assumptions, including (a) that there are no personal taxes, (b) that operating cash flows conform to a specific and permanent stable pattern, and (c) that the level of debt is fixed and permanent.

The work of *MM* was subsequently generalised in a number of different ways. Miller [1977] and DeAngelo and Masulis [1980] discussed the value of corporation tax shields in a world with personal taxation, whilst Miles & Ezzell [1980], retaining the assumption

1 For a useful survey of capital structure research outside the perfect capital markets setting, see Harris & Raviv [1991].

of no personal taxation, derived an *ADR* for *any* pattern of operating cash flows. Strictly speaking, the Miles-Ezzell result is *not* a full generalisation of *MM*'s earlier result, because the Miles-Ezzell financing policy, a so-called Active Debt Management Policy (*ADMP*), and the *MM* financing policy, assumption (c) above, are in general mutually conflicting.[2] Concurrent with the above developments were attempts to integrate the Capital Asset Pricing Model (*CAPM*) of Sharpe [1964], Lintner [1965] and Mossin [1966] with *MM*'s model of capital structure.[3] This raised interesting issues for multi-period capital budgeting, for as well as the *MM* restrictions outlined above, use of the 1-period *CAPM* implied further, and possibly contradictory, restrictions.[4]

Following this early work, researchers have striven to synthesize these various strands with the objective of furnishing a realistic yet practical approach to capital budgeting and, in particular, the valuation of arbitrary risky multi-period cash flows. Clubb & Doran [1991, 1992], Appleyard & Strong [1989], Strong & Appleyard [1992] and Taggart [1991] all employ the Miles-Ezzell *ADMP* to derive formulae relevant to the valuation of a levered asset with an arbitrary pattern of operating cash flows, and all authors allow for non-zero personal taxes. Yet, despite the broadly common framework adopted by these authors, inspection of their results reveals a bewildering variety of valuation formula. This paper demonstrates that discrepancies and inconsistencies can arise, and indeed, have arisen, in models which incorporate personal taxation. However, before presenting a more formal analysis, the following stylised example may help to

2. Moreover, both financing policies were assumed merely for analytical convenience. Neither has normative force.

3. See, for example, Hamada[1972].

4. For example, the Sharpe-Lintner-Mossin *CAPM* was derived under an assumption that investors have a one-period investment horizon. Fama [1970, 1977] outlined sufficient conditions for the validity of this one-period *CAPM* in a multi-period investment context. See also Merton [1973] for a discussion of the same issue in a continuous time setting.

clarify what is at issue with the conventional valuation procedures used in some of the capital budgeting and valuation literature.

Consider a risky cash flow which will be received next period. Suppose its current expected value is £100 before personal taxes (*BT*) and, say, £95 after personal taxes (*AT*). Given these figures, the implied effective tax rate is 5%.[5] Let the equilibrium *AT* discount rate for cash flows belonging to this risk class be 10%. The conventional method of valuation would compute the present value by discounting the expected *AT* cash flow at the *AT* discount rate, giving a correct value (current market price) for this cash flow of £100(1-0.05)/1.1=£95/1.1=£86.3636. An alternative method of valuation would specify an equilibrium risk-adjusted *BT* discount rate to be applied to the expected *BT* cash flow. How should this rate be determined? In much of the above literature, it is assumed that there is a well-defined relationship between the two kinds of discount rate and the tax rate - namely, following Miller [1977], that

$$r = \rho/(1-\tau^*), \tag{1}$$

where r is the equilibrium *BT* discount rate, ρ is the equilibrium *AT* discount rate and τ^* is the effective tax rate. This specification applied to the above numerical example would give the *BT* rate as $r = 0.1/(1-0.05) = 0.10526$ and a value for the cash flow of £100/1.10526 = £90.4762. The discrepancy between the *AT* valuation of £86.3636 and the *BT* valuation of £90.4762 (an error of about 5%) clearly indicates that the relationship encapsulated in equation (1) cannot be generally correct. In fact, if the market value of the cash flow was indeed £86.3636 as indicated by the *AT*

[5] For example, this would be the result if the marginal rate of income tax was 5% and the capital gains tax rate was zero. However, note that the details of the tax regime – and hence of how the effective tax on a cash flow arises - are of no importance in this paper – so long as there are *some* tax effects. The concern in this paper is purely with the problem of how to conduct consistent before- and after-tax valuation analysis.

analysis, correct valuation using a *BT* discount rate r would require that this *BT* rate be given by

$$86.3636 = \frac{100}{1+r} \Rightarrow r = 0.15789\,.$$

Thus the correct BT discount rate is a whole five percentage points above the rate calculated in (1).[6] Furthermore, if the example is modified slightly by lengthening the time before the cash flow will be received to two periods, with no other changes, then its value (based on an *AT* analysis) is $£100(1-0.05)/1.1^2 = £95/1.1^2 =$ £78.5124. If £78.5124 is in fact the market value of the risky cash flow, the implicit value for the *BT* rate r to give this answer is

$$78.5214 = \frac{100}{\left(1+r\right)^2} \Rightarrow r = 0.12858\,,$$

a rate which differs from the period-1 *BT* rate calculated above. These calculations thus demonstrate that the relationship between *BT* and *AT* rates and taxation is generally more complex than implied by the simple ‘grossing up’ procedure in equation (1) and that this relationship is affected by the time to maturity of the expected cash flow. Note that these conclusions have been reached without any assumption being made about the precise tax regime and despite the fact that the parameters of the problem are held constant when time to maturity is lengthened.

The object of the present paper is to identify the relationship between before and after personal tax discount rates and to show how the relationship is generally a non-linear

[6] And of course, valuation errors associated with individual cash flows may also be magnified in an overall calculation of net present value. To illustrate, suppose the above effective tax rate (5%) and an *AT* discount rate of 10% apply and that a project involved an initial outlay of £90 (*BT*) to generate the above period one *BT* expected cash flow of £100. The correct net present value is thus £90-£86.3636=+£3.6364 whilst the *AT* valuation would be £90-£90.4762= -£0.4762. In this example, not only would the project be rejected on this incorrect calculation, but the valuation error would be over 100%.

function of the timing of the cash flow (as illustrated in the above numerical example). Having done this, the paper then addresses the above literature to see to what extent it deals adequately with these relationships - and hence whether or not the observed differences in valuation formulae arise out of differences in assumptions - or simply out of internal incoherence in model assumptions.

Section II sets out the basic framework which is then used to investigate the above literature, generally referred to below as 'the surveyed works', which deals with personal taxes. Specifically, section III focuses on the case of the level perpetuity (as dealt with in Miller [1977]) whilst section IV deals with the literature concerned with the valuation of arbitrary finite cash flows (Clubb & Doran [1991, 1992], Appleyard & Strong [1989], Strong & Appleyard [1992] and Taggart [1991]). Section V then examines the magnitude of the error implied if the simple 'grossing up' rule is used, and section VI draws together the main conclusions.

II. Personal Taxes and Discount Factors

The standard approach adopted in the literature to valuing an asset is to (a) estimate the expected future cash flows the asset will generate, (b) specify a discount factor for each of these cash flows, and (c), invoking the value additivity principle (*VAP*), sum the discounted expected cash flows to obtain a single numerical value. However, when taxation is charged at the personal level, it is possible to pursue a 'before personal taxes' calculation of value or an 'after personal taxes' calculation of value. That is,

(i) to work with the set of expected gross cash flows i.e. before personal taxes have been deducted (*BT*), - and use *BT* discount rates in performing the valuation calculation, or

(ii) to work with the set of expected net cash flows i.e. after personal taxes have been deducted (*AT*) - and use *AT* discount rates.

Since the market value of any given investment is a unique number, it follows that, in a given model, it should make no difference which approach is adopted. That is, starting with a given set of *BT* (*AT*) project cash flows, calculating value using *BT* (*AT*) discount rates should give the same value as that from first computing *AT* (*BT*) cash flows and then computing value using the *AT* (*BT*) discount rates. Indeed this is such an important a property, it is worth stating more formally.

> **Lemma 1**: A necessary condition for the validity of using *both* the *BT* and *AT* valuation approaches within a model is that the two approaches must assign an identical market value to any given asset.

As will be seen shortly, Lemma 1 implies the existence of definite relationships between the discount factors used in the *BT* and *AT* approaches to valuation.

Table 1 about here (or earlier – but not later)

To proceed, focus upon the valuation of a single risky *BT* cash flow x_T [7] payable after T periods. For convenience, table 1 gathers together a list of the principle notation used in what follows. The analysis is more conveniently presented in terms of discount *factors* rather than discount *rates*. However, given the results obtained for factors, corresponding results in terms of discount rates can be obtained by using the following definitional relationships:

$$p\left(\beta,0,T\right)\equiv\frac{1}{\left(1+r(\beta,T)\right)^{T}}\Rightarrow r(\beta,T)=p\left(\beta,0,T\right)^{-1/T}-1, \qquad (2)$$

7 Focusing upon a *single* cash flow involves no loss of generality for, given the *VAP*, it is permissible to value each individual cash flow out of a set of cash flows independently of the others in the set. Proposition 1, which follows, thus holds for each cash flow associated with a complex asset. Furthermore, analogous propositions might be developed for multiple cash flows considered jointly. See, for example, Proposition 2 below.

$$\pi_{\beta t} \equiv \frac{1}{1+\rho_{\beta t}} \;\Rightarrow\; \rho_{\beta t} = \left(1/\pi_{\beta t}\right) - 1. \tag{3}$$

The following assumptions are made:-

(A1) The rate of personal tax levied on x_T is τ and this is a constant over time and is payable without time lag.

(A2) Nominal capital gains/losses, whether realised or unrealised, are taxed at the rate of τ_g, again constant over time and payable without time lag.

(A3) The per-period risk of $E_t(x_T)$, $0 \le s < t \le T$, is known for certain and constant over time (i.e. β is a constant).

(A4) $E_s(x_T)$, ($0 \le s \le T$) is known for certain at time *s*, as are all tax rates and discount factors.

(A5) $\pi_{\beta t} = \pi_\beta$ ($\forall\beta, \forall t \ge 0$).

(A6) $0 \le \tau, \tau_g < 1$ and $0 < \pi_\beta < \pi_f < 1$.

These assumptions provide an analytical framework consistent with the surveyed literature which deals with valuation in the presence of personal taxes. One can, of course, debate whether the above assumptions provide a realistic or useful basis for valuing assets. However, this lies outside the scope of the present paper, which is concerned solely with capital budgeting and valuation within the framework already established in the literature - and in particular with clarifying the extent to which the above literature properly accounts for the implied relationship between *BT* and *AT* discount factors. Nevertheless, some brief remarks concerning A1-A6 may be of interest.

Assumptions A1, A3-A6 are explicit or trivially implicit in all of the surveyed works. Assumption A4 is one of a set of sufficient conditions permitting use of the single-period *CAPM* in a multi-period valuation context.[8] Assumptions A1, A3 and A5 are standard and widely used simplifying assumptions (see Fama [1977] and, especially, Myers &

8. For the other sufficient conditions, see Fama [1977].

Turnbull [1977]). A6 merely imposes that taxes are non-negative and less than 100%, and that the risky discount rate is greater than a riskless one and that both are positive.

The main assumption used here which is less than obvious in the literature is A2. This assumption entails that, for a single positive cash flow receivable at time *T*, there will be a stream of CGT payments in each period prior to *T*, followed by a reclaim of CGT at time *T* (since on payment of the cash flow, the market value of the asset falls to zero, so there is a capital loss). Clubb and Doran [1992] explicitly make this assumption, and in their [1991] paper, set $\tau_g =0$, so this is also consistent with A2 as a special case. The remaining literature contains little discussion of taxation bases, but in all cases there is an explicit assumption that there is an average or overall 'equity tax rate' $\tau*$ which is constant over time for all assets (whatever their risk). It is possible to show that a necessary and sufficient condition for this to be the case within these models is that $\tau = \tau_g$; that is, the assumption of a constant effective equity tax rate requires that dividend and CGT rates are equal and constant over time in these models (the proof is given in appendix *A1*). Since the rate $\tau*>0$ is applied to each and every cash flow in a multi-period cash flow, this also entails our assumption A2, namely that capital gains tax is payable on all changes in market value whether or not capital gains are realised.[9]

[9] As well as being necessary for modelling the surveyed work, this assumption, A2, is of interest in its own right because of its non-distortionary properties. By contrast, it is well known that, if CGT is payable only on realisation, this leads to 'lock in' effects (the desire to hold appreciating assets to defer and so reduce the present value of CGT payments). Although tax authorities often limit the associated tax arbitrage opportunities by imposing loss-offset limits, these in turn distort investment choices away from more risky investments (Stiglitz[1969]). It is for these reasons that there are now arguments for introducing a mark-to-market form of CGT system (Shakow [1986]), and in a recent article, Auerbach [1991] develops an operational form for this.

What values the personal tax rates might take is naturally an empirical question, although from a theoretical perspective, the relevant rates are those associated with the 'marginal investor' (Miller [1977]).[10] Given such rates can only be estimated, it is often useful to study the sensitivity of valuation results to variation in such tax parameters; this kind of analysis is conducted in section 5 below.

Under assumptions A1-A6, the relationship between *BT* and *AT* discount factors is established in the following proposition:

> **Proposition 1**: Assumptions A1-A6 imply the following necessary and sufficient condition for internally consistent valuation of any given *BT* risky cash flow x_T at any given time *T*; that the *BT* and *AT* discount factors must be related by the formula
>
> $$p(\beta,0,T)=\left(\frac{1-\tau}{1-\tau_g}\right)\left(\pi_\beta\frac{(1-\tau_g)}{(1-\pi_f\tau_g)}\right)^T.$$
>
> **Proof:** See appendix A2

Writing

$$k\left(\tau,\tau_g\right)\equiv(1-\tau)\big/\left(1-\tau_g\right),\tag{4}$$

and

$$a\left(\pi_f,\tau_g\right)\equiv\frac{(1-\tau_g)}{(1-\pi_f\tau_g)}\tag{5}$$

(and suppressing arguments for the functions *k* and *a* in what follows), the result can be written more compactly as

$$p(\beta,0,T)=k\left(a\pi_\beta\right)^T.\tag{6}$$

The full proof for Proposition 1 is completed in appendix A2. However, to get an understanding for the processes involved, the first steps are detailed here. Given an

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[10] For the complications induced by tax clientele effects, see for example Elton and Gruber [1970], Miller and Scholes [1978], Litzenberger and Ramaswamy [1982].

arbitrary risky *BT* cash flow, x_T, this can be valued directly, using the *BT* discount factor, or by first converting to *AT* cash flows and then applying appropriate *AT* discount factors. By Lemma 1, the *BT* and *AT* approaches are mutually consistent only if they give the same market valuation - thus equating the market valuations by these alternative approaches establishes the above relationship between *BT* and *AT* discount factors.

***The BT Approach*:**

Under the *BT* approach there is just one expected cash flow, $E_0\left(x_T\right)$. Invoking the *VAP* and using the discount factor $p(\beta,0,T)$, the present value is simply

$$V_0 = p(\beta,0,T)E_0(x_T)\,. \tag{7}$$

***The AT Approach*:**

Given assumptions A1 and A2, payment of x_T gives rise to a stream of *AT* cash flows χ_t from period 1 all the way through to period T as illustrated in Table 2.

Table 2: *AT* cash flows generated by a single *BT* cash flow at time *T*.

Time t	*AT* cash flow, χ_t
0	$\chi_0 = 0$
1	$\chi_1 = -\tau_g \left[V_1 - V_0 \right]$
2	$\chi_2 = -\tau_g \left[V_2 - V_1 \right]$
….	….
….	….
T-2	$\chi_{T-2} = -\tau_g \left[V_{T-2} - V_{T-3} \right]$.
T-1	$\chi_{T-1} = -\tau_g \left[V_{T-1} - V_{T-2} \right]$
T	$\chi_T = x_T \left(1 - \tau \right) - \tau_g \left[V_T - V_{T-1} \right]$
T+1	0
T+2	0
…	…

The cash flows arise here because there is a capital gain/loss which is taxed at the rate τ_g whenever the market value of the time *T* cash flow changes (its market value naturally changes as *T* is approached). In the final period *T*, the cash flow x_T is itself taxed, at the rate τ .

Invoking the *VAP*, each element of this set of *AT* cash flows is now valued, with V_0 being given by the sum of these valuations. Notice that, with $\tau_g \neq 0$, each *AT* cash flow χ_t also involves valuations (namely V_t and V_{t-1}). Such values may be computed recursively, working backwards from s=*T* through *s*=0, as follows.

<u>Derivation of V_s at s=T</u>

Since $\chi_t = 0$ for all $t>T$, it follows trivially that $V_T = 0$.

Derivation of V_s at s=T-1

The market value V_{T-1} is the sum of (discounted) future cash flows expected at time T-1. In valuing the cash flows in Table 2, it is important to distinguish the risk associated with each element, and to value each using the appropriate discount factor (the risky or riskless discount factor). Row *T* of Table 2 gives the (period *T*) *AT* cash flow as $\chi_T = x_T(1-\tau) - \tau_g[V_T - V_{T-1}]$. As previously noted, $V_T = 0$. Viewed from period *s*=*T*-1, $x_T(1-\tau)$ is a risky cash flow, whilst $\tau_g V_{T-1}$ is known for certain (from A3, A4). The *s*=*T*-1 expected value of the former is thus discounted using the one-period risky discount factor, π_β, whilst the latter is valued using the one-period riskless discount factor π_f (by A5). Hence,

$$V_{T-1} = \pi_f \tau_g V_{T-1} + \pi_\beta (1-\tau) E_{T-1}(x_T)\,. \tag{8}$$

Solving for V_{T-1} gives

$$V_{T-1} = \left[\pi_\beta (1-\tau) E_{T-1}(x_T)\right] \Big/ \left[1 - \pi_f \tau_g\right]. \tag{9}$$

Using (4) and (5), this becomes simply

$$V_{T-1} = ka\pi_\beta E_{T-1}(x_T)\,, \tag{10}$$

where *k*, *a* and π_β are known for certain (by A4).

Derivation of V_s at *s*=*T*-2

V_{T-2} equals the sum of (discounted) future cash flows (from periods *T*-1 and *T*) expected at s=*T*-2. At time t=*T*-1 there are two cash flows; -$\tau_g V_{T-1}$ and $\tau_g V_{T-2}$. Viewed from time *s*=*T*-2, the term $\tau_g V_{T-2}$ is riskless, so is valued at time *T*-1 using the riskless factor π_f. The cash flow $-\tau_g V_{T-1}$ viewed from time *T*-1 is risky; from (10), V_{T-1} is simply a constant multiplied by $E_{T-1}(x_T)$, so, by assumptions A3 and A5, the appropriate one-period discount factor for valuing this term is π_β.

There are also two non-zero cash flows arising at time *T*, namely $\tau_g V_{T-1}$ and $\left(1-\tau\right)x_T$. To obtain the present values for these, their *s*=*T*-2 expected values must be discounted two periods. The first cash flow, $\tau_g V_{T-1}$, is a random variable which from (10) is a scalar multiple of $E_{T-1}(x_T)$ up until time *T*-1, and thereafter is known for certain. Hence the two-period discount factor for $\tau_g E_{T-2}(V_{T-1})$ is $\pi_\beta \pi_f$. The second cash flow, $\left(1-\tau\right)x_T$, is a random variable, with risk β in each period (assumption A3). The two-period discount factor for $\left(1-\tau\right)E_{T-2}(x_T)$ is therefore π_β^2.

Adding the values for *t*=*T*–1 and *t*=*T* cash flows,

$$V_{T-2} = \pi_f \tau_g V_{T-2} - \pi_\beta \tau_g E_{T-2}(V_{T-1}) + \pi_\beta \pi_f \tau_g E_{T-2}(V_{T-1}) + \pi_\beta^2 (1-\tau) E_{T-2}(x_T), \quad (11)$$

so, solving for V_{T-2} gives

$$V_{T-2} = \left[(\pi_f - 1)\pi_\beta \tau_g E_{T-2}(V_{T-1}) + \pi_\beta^2 (1-\tau) E_{T-2}(x_T)\right] \Big/ \left[1-\pi_f \tau_g\right]. \quad (12)$$

To simplify equation (12) further, note that, by the law of iterated expectations (see e.g. Hamilton [1994 p. 742]), for any arbitrarily chosen cash flow, x_T,

$E_s\left(E_t\left(x_T\right)\right) = E_s\left(x_T\right)$ for all s,t,T such that $0 \le s \le t \le T$. Now, using (9),

$$\begin{aligned} E_{T-2}(V_{T-1}) &= E_{T-2}\left(\left[\pi_\beta(1-\tau)E_{T-1}(x_T)\right] \Big/ \left[1-\pi_f \tau_g\right]\right) \\ &= \left[\pi_\beta (1-\tau) E_{T-2}(x_T)\right] \Big/ \left[1-\pi_f \tau_g\right] \end{aligned}. \quad (13)$$

Substituting into (12) then gives

$$V_{T-2} = \frac{\left[(\pi_f - 1)\pi_\beta \tau_g \left\{ \frac{\pi_\beta (1-\tau) E_{T-2}(x_T)}{\left[1 - \pi_f \tau_g\right]} \right\} + \pi_\beta^2 (1-\tau) E_{T-2}(x_T) \right]}{1 - \pi_f \tau_g}, \tag{14}$$

which simplifies to give

$$V_{T-2} = k\left(a\pi_\beta\right)^2 E_{T-2}(x_T)\,. \tag{15}$$

<u>Derivation of V_s at s=0</u>

Equations (10) and (15) suggest a pattern to the value equation of the form

$$V_{T-i} = k\left(a\pi_\beta\right)^i E_{T-i}(x_T) \text{ for } i = 1, 2, .., T\,, \tag{16}$$

and this is formally established in appendix A2. Setting $i = T$, this implies

$$V_0 = k\left(a\pi_\beta\right)^T E_0(x_T)\,. \tag{17}$$

By Lemma 1, the right hand sides of equations (7) and (17) must be equal, hence, cancelling through by $E_0(x_T)$ gives the proposition 1 result that

$$p(\beta, 0, T) = k\left(a\pi_\beta\right)^T . \tag{18}$$

Proposition 1 shows that A5, the assumption of time-invariant one-period *AT* discount factors (respectively, time invariant *AT* discount rates), is not in general compatible with a similar assumption concerning one-period *BT* discount factors (respectively, time invariant *BT* discount rates). That is, it is not in general possible to write $p(\beta,0,T) = p(\beta,0,1)^T \quad \forall T > 0, \forall \beta$; this is possible only in the special case where $\tau = \tau_g$:

Corollary: Under assumptions A1-A6, a necessary and sufficient condition for $p(\beta,0,T)=p(\beta,0,1)^T$, $\forall T>0,\forall\beta$, is that $\tau=\tau_g$.

Proof:

Given $p(\beta,0,T)=k\left(a\pi_\beta\right)^T$, then $p(\beta,0,T)=p(\beta,0,1)^T$, $\forall T>0,\forall\beta$, if and only if $p(\beta,0,1)=k^{1/T}a\pi_\beta$ which is true for all *T*>0 if only if *k*=1. However $k\equiv(1-\tau)/(1-\tau_g)$, and *k*=1 $\Leftrightarrow$ $\tau_g=\tau$. ■

Thus, when dealing with arbitrary finite risky cash flow profiles, a valuation procedure within the scope of assumptions A1-A6 cannot additionally assume *both* $\tau\neq\tau_g$ *and* $p(\beta,0,T)=p(\beta,0,1)^T$, ($\forall T>0$, $\forall\beta$) without violating Proposition 1. A time-invariant one-period discount factor is equivalent to a flat term structure in one-period discount rates. The above analysis therefore shows that, when $\tau\neq\tau_g$, assuming flat term structures for both *BT* and *AT* rates is, within the framework (A1-A6), logically inconsistent.[11] This point is taken up again in section IV below.

Many papers focus on rates of return rather than discount factors, so it is worth translating (18) into this format, using equations (2) and (3). This gives

$$\frac{1}{\left(1+r\left(\beta,T\right)\right)^T}=\left(\frac{1-\tau}{1-\tau_g}\right)\left\{\frac{(1-\tau_g)(1+\rho_f)}{(1+\rho_\beta)(1+\rho_f-\tau_g)}\right\}^T. \qquad (19)$$

As in the Corollary to Proposition 1, clearly $r\left(\beta,T\right)$ is a constant for all $T\geq 1$ if and only if $\tau=\tau_g$.

11. To the best of our knowledge, there has been no explicit discussion in the literature on equity valuation of the complex relationships between *AT* and *BT* term structures. For work on the term structure of interest rates in bond markets, see Livingston [1979], Kim [1990] and Kryzanowski, Xu and Zhang [1995].

Equation (18) illustrates the precise relationship between (i) the multi-period *BT* discount factor, $p(\beta,0,T)$, (ii) the one-period *AT* discount factor for any given risk class β, (iii) the discount factor for the riskless asset, (iv) the timing of receipt of the *BT* cash flow, and (v) rates of personal taxation. The first order partial derivatives of $p(\beta,0,T)$ with respect to τ, τ_g are (appendix A3 gives derivations)

$$\partial p(\beta,0,T)/\partial\tau = -p(\beta,0,T)/\left(1-\tau\right), \tag{20}$$

$$\partial p(\beta,0,T)/\partial\tau_g = \frac{p(\beta,0,T)\left([1-\pi_f\tau_g]-T(1-\pi_f)\right)}{(1-\tau_g)(1-\pi_f\tau_g)}. \tag{21}$$

Intuitively, one might expect that the higher the tax rate, the greater is the personal tax burden associated with each unit of *BT* cash flow, x_T, and the lower is the unit present value, $p(\beta,0,T)$. Given A6, (20) is indeed strictly negative. By contrast, (21) is strictly negative if and only if

$$T > (1-\pi_f\tau_g)/(1-\pi_f). \tag{22}$$

That is, when $T < (1-\pi_f\tau_g)/(1-\pi_f)$, CGT actually raises the time zero market value of a positive cash flow received at time *T*. The intuition for this is perhaps easiest seen if we consider the case of a single positive riskless cash flow. In this case, the market value necessarily increases as T is approached, so there are CGT payments to be made in each period until the last, in which the value falls to zero, and CGT can be reclaimed. In the absence of discounting, the sum of the capital gains would in fact be a capital loss equal to the initial value of the asset. Thus, in the absence of discounting, the overall impact of CGT would necessarily be to increase the market value of the asset. Given there is discounting, and given the CGT claimed back occurs at time *T*, it follows that the larger *T* is, the more heavily this benefit is discounted, and so the more likely it becomes that the impact of CGT is no longer beneficial.[12] [13]

Having spent some time discussing the role of CGT, it is worth emphasising that the non-linearity in the transformation from AT to BT discount factors does not disappear when CGT is zero. This point has already been made in our numerical example in section 1. More formally, it can be seen by setting τ_g =0 in (19). This then simplifies to give

$$\frac{1}{\left(1+r\left(\beta,T\right)\right)^{T}}=\frac{\left(1-\tau\right)}{\left(1+\rho_{\beta}\right)^{T}}\Rightarrow 1+r\left(\beta,T\right)=\left(1+\rho_{\beta}\right)\left(1-\tau\right)^{-1/T} \qquad (23)$$

That is, if $\tau > 0$ whilst $\tau_g = 0$, $r\left(\beta,T\right)$ continues to be a non-linear function of *T*, and so, even if $\tau_g = 0$, it is not possible to assume that both *BT* and *AT* term structures are flat.

III. Level perpetuities and the Miller [1977] model

The (risky) level perpetuity is an important special case where a simpler relationship between *BT* and *AT* discount factors exists. This perpetuity offers risky *BT* cash payments, x_T for *T*=1,...,∞. It is characterised by the condition $E_0\left(x_T\right)=\bar{x}$, a constant, for all *T*>0. The level perpetuity is assumed to have a constant level of

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[12.] More formally, inspection of Table 2 indicates that the capital gains tax saving at time *T*, ($\tau_g(V_{T\text{-}1}\text{-}V_T) = \tau_g V_{T\text{-}1}$), is greater in absolute magnitude than the *total* of capital gains tax payments from t=1 through T-1, ($\Sigma_t \tau_g(V_t\text{-}V_{t\text{-}1}) = \tau_g(V_{T\text{-}1}\text{-}V_0)$). But when receipt of this tax saving is sufficiently distant (large *T*) and/or the time discounts are sufficiently large, the total present value of the stream of expected capital gains tax cash flows will be negative. Then, since the payment of capital gains tax reduces present value, the higher is τ_g, the lower is $p(\beta,0,T)$.

[13] Focusing on a single cash flow thus seems to suggest that investors would want to lobby to increase the CGT rate in this framework. However that is not the case, because many of the assets that concern investors are perpetual and growing assets for which CGT is indeed a burden. This is explained in detail in section 4 below.

'riskiness' (or 'homogenous' risk) in the sense that the time-invariant per-period risk of each expected cash flow is a constant β across all expected cash flows making up the perpetuity. However, note that a risky level perpetuity is level only in the sense that time zero expectations of the risky future cash flows are constants. Its value will actually fluctuate randomly as time passes. The procedure for valuing the risky level perpetuity is exactly the same as for any arbitrary set of cash flows; each cash flow is valued separately, and then the values are summed. It is worth emphasising that every cash flow in the risky perpetuity gives rise to a stream of capital gains/losses as per Table 2 – so it follows, *a fortiori*, that there is a stream of capital gains tax cash flows associated with such a perpetuity.[14]

The perpetuity valuation formula

Let $p_{perp}(\beta)$ denote the present value of this level perpetuity per unit of $\bar{x}$ as a function of its risk and $r_{perp}(\beta)$ be the 'quasi-discount rate' that correctly values this perpetuity. Thus, $p_{perp}(\beta)$ and $r_{perp}(\beta)$ are defined by the equation

$$V_0 \equiv p_{perp}(\beta)\bar{x} \equiv \bar{x} / r_{perp}(\beta). \qquad (24)$$

Then it follows that

[14] That is, whilst $E_0\left(x_T\right) = \bar{x}$ for all T>0, $E_t\left(x_T\right)$, for $0 < t \leq T$ will generally differ from $\bar{x}$. The same is true of the value of the risky perpetuity. Here the value at time zero is calculated. The value of the perpetuity at time 1 will generally differ from that at time 0; indeed, if value at time 1 did not, then it would not be a risky perpetuity. Hence as a matter of logic, the cash flows associated with the risky perpetuity, and the value of the perpetuity must fluctuate over time - the perpetuity therefore must give rise to capital gains tax cash flows. For a more complete analysis of the evolution of expectations over time, see Fama [1977].

Proposition 2: Assumptions A1-A6 imply the following necessary and sufficient condition for internally consistent valuation of a homogeneous-risk level perpetuity:-

$$p_{perp}(\beta) = \frac{k\left(a\pi_\beta\right)}{\left(1-a\pi_\beta\right)} = \left(\frac{1-\tau}{1-\tau_g}\right)\left(\frac{\pi_\beta\left(1-\tau_g\right)}{1-\pi_\beta-\tau_g(\pi_f-\pi_\beta)}\right).$$

Equivalently,

$$r_{perp}(\beta) \equiv \frac{1}{p_{perp}(\beta)} = \frac{\rho_\beta - \tau_g\left(\dfrac{\rho_\beta-\rho_f}{1+\rho_f}\right)}{1-\tau}.$$

Proof: Under the *AT* approach, the value of a homogenous risk level perpetuity is obtained by using (17) and the *VAP*. Since in this case, $E_0\left(x_T\right) = \bar{x}, \forall T > 0$, it follows that

$$V_0 = \sum_{T=1}^{\infty} k\left(a\pi_\beta\right)^T E_0(x_T) = k\bar{x}\sum_{T=1}^{\infty}\left(a\pi_\beta\right)^T = k\bar{x}\frac{a\pi_\beta}{1-a\pi_\beta}. \qquad \text{(i)}[15]$$

Under the *BT* approach, the value of the β-risk level perpetuity is

$$V_0 = p_{perp}(\beta)\bar{x}. \qquad \text{(ii)}$$

Invoking the Lemma, equating the right hand sides of (i) and (ii), and solving for $p_{perp}(\beta)$ gives

$$p_{perp}(\beta) = k\left(a\pi_\beta\right)/\left(1-a\pi_\beta\right). \qquad \text{(iii)}$$

Expanding, using the definitions for *a, k* and (24) (and using (2), (3) to obtain discount rates) then gives the above results. ■

Notice, from (iii) and (18), that if *k*=1 (i.e. if $\tau = \tau_g$), then

$$p_{perp}(\beta) = p(\beta,0,1)/\left[1-p(\beta,0,1)\right]. \qquad (25)$$

In other words; the unit present value of a β-risk level perpetuity will be a simple function of the nearest *one-period* β-risk *BT* discount factor. In terms of rates of return, since $p(\beta,0,1) = 1/\left[1+r(\beta,1)\right]$, this gives the familiar discounting rule - that when

[15] Note, from (5) and *A6*, that the term $a\pi_\beta$ satisfies $0 < a\pi_\beta < 1$, so the geometric sum converges.

$\tau = \tau_g$, the value of the risky level perpetuity is simply the *BT* expected cash flow divided by the *BT* one period discount rate; that is, the *BT* valuation rule is simply

$$V_0 = \bar{x}/r(\beta,1). \qquad (26)$$

However, it must be stressed that this holds only if *k*=1; that is, only if $\tau = \tau_g$.

Notice also, from Proposition 2, for a riskless perpetuity, since $\beta = f$, it follows that

$$r_{perp}(f) = \rho_f/(1-\tau). \qquad (27)$$

That is, the 'quasi-discount rate' to correctly value a *BT* riskless level perpetuity is simply the 'grossed up' *AT* discount rate. Also, for a risky level perpetuity, if the capital gains tax rate is zero, then, from Proposition 2,

$$r_{perp}(\beta) = \rho_\beta/(1-\tau), \qquad \text{(if } \tau_g = 0\text{)}. \qquad (28)$$

That is, the value of a level risky *BT* perpetuity can be calculated using the simple 'grossed up' after tax discount rate. A simple 'grossed up' *AT* discount rate works in this case because the level perpetuity is a special case. One might surmise that when tax rates are equal ($\tau = \tau_g$), a similar result might be had. This is ***not*** the case however; when $\tau = \tau_g$, the formula only simplifies as far as

$$r_{perp}(\beta) \equiv \frac{1}{p_{perp}(\beta)} = \frac{\rho_\beta - \tau\left(\dfrac{\rho_\beta - \rho_f}{1+\rho_f}\right)}{1-\tau}. \qquad (29)$$

The simple time-independent 'grossing up' of the *AT* discount rate cannot be carried over to the case of non-level, finite cash flow profiles, even in special cases where $\beta = f$ and/or $\tau_g = 0$. To see this, note that, for the riskless case, from (19) with $\beta = f$,

$$\frac{1}{\left(1+r\left(f,T\right)\right)^{T}}=\left(\frac{1-\tau}{1-\tau_{g}}\right)\left(\frac{1-\tau_{g}}{1+\rho_{f}-\tau_{g}}\right)^{T}, \tag{30}$$

whilst the case with $\tau_g = 0$, from (19), gives

$$\frac{1}{\left(1+r\left(\beta,T\right)\right)^{T}}=\left(1-\tau\right)\left(\frac{1}{1+\rho_{\beta}}\right)^{T}. \tag{31}$$

With both $\beta = f$ and $\tau_g = 0$, (19) gives

$$\frac{1}{\left(1+r\left(f,T\right)\right)^{T}}=\left(1-\tau\right)\left(\frac{1}{1+\rho_{f}}\right)^{T}. \tag{32}$$

Thus in all these cases, the transformation from *AT* to *BT* is a non-linear function of the time to receipt of the cash flow.

Why CGT can 'add value'

In the special case of the level perpetuity, Proposition 2 shows that there is a strictly positive relationship between the capital gains tax rate and market value when the perpetuity is risky, $\rho_\beta > \rho_f$: that is, from (29),

$$p_{perp}(\beta)=\frac{1-\tau}{\rho_{\beta}-\tau_{g}\left(\dfrac{\rho_{\beta}-\rho_{f}}{1+\rho_{f}}\right)} \quad \Rightarrow \quad \frac{\partial p_{perp}(\beta)}{\partial \tau_{g}}>0 \tag{33}$$

Readers of an earlier version of the paper found this result rather puzzling (and 'counter-intuitive'). For this reason it is worth examining it in more detail. A useful way to do so is to examine the case where the perpetuity features a constant expected growth rate *g* (which could be zero as a special case). Consider the result of buying such a risky perpetuity and selling it after one period. The cash flows that arise are as follows:

Table 3 Cash flows for a one period buy and sell strategy

Time period:	Cash flow
0	$-V_0$ (initial payment)
1	$\tilde{V}_1$ (sale at market value, a random variable)
1	$-(\tilde{V}_1 - V_0)\tau_g$ (CGT payment)
1	$\tilde{x}_1(1-\tau)$ (receipt of $\tilde{x}_t$ and payment of tax on it)

The claim against initial value in the CGT payment, $V_0\tau_g$, is riskless and so must be discounted at the riskless rate; all the other elements of the return are risky and so are discounted at ρ_β. Hence, defining $\bar{x}_1 \equiv E_0(\tilde{x}_1)$ and $\bar{V}_1 \equiv E_0(\tilde{V}_1)$, then in equilibrium

$$V_0 = \frac{\bar{V}_1 + \bar{x}_1(1-\tau) - \bar{V}_1\tau_g}{1+\rho_\beta} + \frac{V_0\tau_g}{1+\rho_f} \tag{34}$$

The expected cash flows, and hence expected values, grow at the rate g (as in Gordon's dividend growth valuation model), so

$$\bar{x}_t \equiv E_0(\tilde{x}_t) = (1+g)^{t-1}\bar{x}_1 \qquad (\Rightarrow \bar{V}_t \equiv E_0(\tilde{V}_t) = (1+g)^t V_0) \tag{35}$$

Using this to substitute for $\bar{V}_1$ in (34) (and rearranging) gives the valuation formula

$$V_0 = \frac{(1-\tau)\bar{x}_1}{\rho_\beta - g - \tau_g\left(\dfrac{\rho_\beta - \rho_f}{1+\rho_f} - g\right)}. \tag{36}$$

As before (Proposition 2), when g=0, increases in CGT add value. The intuition for this can be seen by inspection of the cash flows at time period 1 in Table 3. From (35), notice that $E_0(\tilde{V}_1 - V_0) = \bar{V}_1 - V_0 = gV_0$; thus if g=0, the expected value at time 0 of CGT payments at time 1 is zero. However, if g=0, the *present value* of this CGT tax payment is negative; this arises because the allowance against the opening balance V_0 is riskless and so is discounted less heavily than the risky CGT payment on $\tilde{V}_1$. To spell this out,

$$PV(CGT) = \frac{E_0(\tilde{V}_1)\tau_g}{1+\rho_\beta} - \frac{V_0\tau_g}{1+\rho_f} = \frac{(1+g)V_0\tau_g}{1+\rho_\beta} - \frac{V_0\tau_g}{1+\rho_f} \tag{37}$$

which is clearly negative when g=0. Hence CGT raises value for the level risky perpetuity case.

Of course, the more usual case is that of values which are expected to increase over time. The above analysis demonstrates that if g is sufficiently positive, increasing CGT will reduce value. From (36),

$$g \gtreqless \frac{\rho_\beta - \rho_f}{1+\rho_f} \Rightarrow \partial p_{perp}(\beta)/\partial\tau_g \lesseqgtr 0 \tag{38}$$

Thus, as one would expect, when capital gains are anticipated and g is sufficiently positive, CGT reduces value. By contrast, if growth is negative, then capital losses are anticipated, and in this case, increasing CGT increases the amount clawed back in tax, and hence increases initial market value. These observations explain why, in practice, most investors would prefer a lower (zero) rather than a higher CGT rate. Most equity investments are expected to be growing, albeit risky, perpetuities (rather than finite sets of cash flow), and this is precisely the case where there is a CGT burden reducing market value.

Perpetuities in the Miller Model

In the Miller [1977] model, whether or not gearing adds value is assessed in the context of a market equilibrium for corporate debt. Equilibrium in the market for debt involves the rate of interest on debt being bid up to the point where the marginal investor, who has a particular tax-paying status, is indifferent as to whether she holds identical risk securities after all tax obligations are met (the firm’s stocks or bonds, in this case). This accords with the analysis of this paper (as in Propositions 1 and 2).

The Miller [1977] debt and taxes paper deals with a special case for simplicity - specifically, it features riskless and permanent debt along with a risky operating cash flow perpetuity. There are no taxes on equity, and a single time invariant investor specific tax rate is used for 'income from bonds' (Miller [1977, p. 267]: denoted τ_{PB}^{α} in that paper). The rate r_0 in that paper denoted the 'equilibrium rate of interest on fully tax exempt bonds' (p. 268). It is therefore not only the *BT* discount rate for the latter, but also the *AT* discount rate for all investors and for all riskless securities. In equilibrium, the *BT* price of a taxable riskless bond must have adjusted to the point where the *AT* riskless return for the marginal investor (the "α-investor" paying tax at the rate τ_{PB}^{α}) equals r_0. That is, in equilibrium,

$$r_d(B)\left(1-\tau_{PB}^{\alpha}\right)=r_0\,, \tag{39}$$

where $r_d(B)$ denotes the inverse demand function for debt. Thus, (39) implies the rate of interest on debt is bid up to the point where, in equilibrium,

$$r_d(B)=\frac{r_0}{1-\tau_{PB}^{\alpha}}\,. \tag{40}$$

This formulation corresponds to the result established in Proposition 2 above for a level riskless perpetuity. It does not, of course, hold for arbitrary finite cash flow profiles, where the *BT* rates which ensure *AT* equilibrium are more complex (as indicated in Proposition 1).

Miller [1977] does also briefly consider the risky perpetuity case, but says little about how the relationship between *BT* and *AT* rates is changed for this case, commenting "Default risk can be accommodated… by merely reinterpreting all the before-tax interest rates as risk adjusted or certainty equivalent rates." (p. 271). This appears to suggest that the simple 'grossing-up' procedure above remains valid for moving from *AT* to *BT* interest rates when dealing with risky bonds. However, the above analysis shows that this is not correct - except, of course, for the special case of the ***level*** risky perpetuity when the rate of taxation on capital gains is zero (as in equation (28)).

Recall that in the Miller [1977] model, τ_{PB}^{α} is used as the rate of taxation for 'income from bonds'. Income includes cash disbursements and capital gains. For the level riskless perpetuity, there are no capital gains, hence income is equal to cash disbursements. However, for most other cases, including that of risky level perpetuities, income is not equal to cash disbursements. Given the debt marginal rate τ_{PB}^{α} in Miller [1977] applies to income, it would appear that, implicitly, debt capital gains and cash disbursements (the two components of income) are taxed at the same rate, ruling out a zero capital gains tax rate for debt. If so, the simple 'grossing up' formula of type (40) cannot be applied in such a case. Logically therefore, whilst the general thrust of the Miller model obviously makes sense, the formal 'model' as sketched in the 1977 paper works in the way described there only if it is restricted to the case where debt is perpetual, constant and riskless, the equity cash flow is a level risky perpetuity, *and* the tax rate on equity capital gains is zero.

IV. *BT* and *AT* discount rates in the literature

This section examines the surveyed works which focus on the valuation of ***non-level*** and ***finite*** risky cash flow profiles in the presence of personal taxes (Clubb & Doran [1991, 1992], Appleyard & Strong [1989], Strong & Appleyard [1992] and Taggart [1991]). There are, within the framework of assumptions A1-A6, generally five determinants of multi-period *BT* discount factors, and four in the special case of a level perpetuity. Two determinants are asset-specific: risk, reflected in π_{β}, and the number of periods, T (which is not relevant, of course, in the case of the perpetuity). The remainder are general parameters: tax rates, τ and τ_g, and the riskless one-period *AT* discount factor π_f. This latter determinant arises from the tax deductibility of capital investments in calculating capital gains tax liability; such investments are always known for certain one period before discharge of the liability.

Turning now to the above literature, first note that all five surveyed models employ time-invariant one-period *BT* discount factors. In accordance with the Corollary to Proposition 1, within the specified framework (assumptions A1-A6), time-invariant one-period *BT* discount factors are incompatible with the assumption of time-invariant *AT* discount factors *unless* the rates of taxation, τ and τ_g, are equal. By the corollary to proposition 1, for these models to be internally consistent, the condition $\tau = \tau_g$ (i.e. k=1) must therefore hold as an explicit or implicit assumption. However, the model proposed in Clubb & Doran [1991], in explicitly assuming a zero level of equity capital gains tax (thus setting $\tau > \tau_g = 0$ for these equity tax rates), violates the Corollary (which establishes that, when $\tau > \tau_g$, it is inadmissible to assume that *both BT* and *AT* term structures for discount rates are flat.[16]

Of the remaining four models, Clubb and Doran [1992] explicitly assume $\tau = \tau_g$, whilst the other papers assume a constant overall or effective rate of tax on equity cash flows.

[16] The relationship between *AT* and *BT* discount rates established in proposition 1 is intrinsically non-linear when $\tau > \tau_g$. The nonlinearity in the translation between AT and BT rates needs to be recognized even in the more general case where the AT term structure has an arbitrary shape.

We have already established that this is formally equivalent to making assumption A2 for CGT and also to assuming that $\tau = \tau_g$ (proof in appendix 1). These models also assume a relationship between *BT* and *AT* discount rates of the form

$$r(\beta,1) = \frac{\rho_\beta}{1-\tau}, \tag{41}$$

which, using equations (2) and (3), gives a discount factor of the form

$$p(\beta,0,1) = \frac{\pi_\beta(1-\tau)}{1-\pi_\beta\tau}. \tag{42}$$

With $\tau = \tau_g$, from (4), k=1. However, equation (18) shows that, for k=1,

$$p(\beta,0,1) = a\pi_\beta = \frac{\pi_\beta\left(1-\tau\right)}{\left(1-\pi_f\tau\right)}. \tag{43}$$

In a world of personal taxes (τ>0), the right hand sides of equations (42) and (43) will be equal (and will therefore satisfy Propositions 1 and 2) ***if and only if*** β=f. That is, if and only if all the cash flows are ***riskless***. Since the models in Appleyard & Strong [1989] and Strong & Appleyard [1992] both assume the relationship in (42) holds for all types of cash flow whether risky or not (i.e. $\beta \neq f$), these models are internally inconsistent, failing to properly account for the effect of the non-zero capital gains tax. The remaining two models, Clubb & Doran [1992] and Taggart [1991], employ the relationship in (42) only in the case of riskless cash flows. By restricting attention to this (rather limiting) special case, these models are internally coherent, at least by the test applied in the present paper.

V. Sensitivity Analysis

Whilst the primary object was to examine the theoretical relationship between *BT* and *AT* discount rates and factors and to examine whether these had been consistently used in the literature, it is of some interest to explore numerically the non-linear relationship between these rates. Suppose then that there is an underlying *AT* stationary equilibrium in which *AT* discount rates are flat. For illustrative purposes the *AT* risk free rate is taken to be 5%, the risky *AT* discount rate is 10%. Table 3 gives *BT* spot discount rates, for effective tax rates of 5% and 20%,[17] for a range of maturities and rates of capital gains tax. Figure 1 then illustrates the term structures implied (for the $\tau = 0.05$ case).

Table 3: Implied *BT* discount rates (calculated using equation (19))

	$\tau = 0.05$ (5%)					$\tau = 0.2$ (20%)				
	Capital gains tax rate, τ_g					Capital gains tax rate, τ_g				
Year	0	0.05	0.1	0.2	0.4	0	0.05	0.1	0.2	0.4
1	0.158	0.103	0.048	-0.063	-0.283	0.375	0.310	0.244	0.113	-0.149
2	0.129	0.103	0.076	0.021	-0.098	0.230	0.202	0.173	0.113	-0.017
3	0.119	0.103	0.086	0.051	-0.026	0.185	0.168	0.150	0.113	0.031
4	0.114	0.103	0.091	0.066	0.012	0.163	0.151	0.139	0.113	0.056
5	0.111	0.103	0.094	0.075	0.035	0.150	0.141	0.132	0.113	0.071
6	0.109	0.103	0.096	0.082	0.051	0.142	0.135	0.128	0.113	0.082
7	0.108	0.103	0.097	0.086	0.063	0.136	0.130	0.125	0.113	0.089
8	0.107	0.103	0.098	0.089	0.072	0.131	0.127	0.122	0.113	0.095
9	0.106	0.103	0.099	0.092	0.078	0.128	0.124	0.120	0.113	0.099
10	0.106	0.103	0.100	0.094	0.084	0.125	0.122	0.119	0.113	0.103

Figure 1 about here.

Suppose one poses the question, given the above figures, what kind of error would be made by taking the *AT* rate grossed up by the effective tax rate (i.e. by using the rate $r(1-\tau)$) rather than the correct *BT* discount rate? That is, if, instead of calculating the correct value as, from (19),

$$r(\beta,T) = \left(\frac{1-\tau_g}{1-\tau}\right)^{1/T} \left\{\frac{(1+\rho_\beta)(1+\rho_f-\tau_g)}{(1-\tau_g)(1+\rho_f)}\right\} - 1, \tag{44}$$

the calculation

[17] As noted in the introduction, the concern here is purely with establishing a consistent before tax and after tax treatment of the valuation process. For a given cash flow (or set of cash flows), there is of course an estimation issue associated with identifying what the appropriate level is for effective tax rates on cash flows and on capital gains.

$$\hat{r}(\beta,T) = \frac{\rho_\beta}{(1-\tau)} \tag{45}$$

is used. Retaining the same parameter values ($\rho_f = 0.05$, $\rho_\beta = 0.1$, $\tau = 0.05$ or 0.2) it is then possible to calculate the percentage error in the calculation of the *BT* discount rate which arises in using (45) instead of (44). However, for valuation purposes, it is probably of more interest to compute the percentage error that would arise in using the associated discount *factor*; that is, instead of using the correct factor (19), reproduced here as

$$p(\beta,0,T) = \left(\frac{1-\tau}{1-\tau_g}\right)\left\{\frac{(1-\tau_g)(1+\rho_f)}{(1+\rho_\beta)(1+\rho_f-\tau_g)}\right\}^T, \tag{46}$$

one used the discount factor based on the (incorrect) grossed up discount rate used in (45), namely

$$\hat{p}(\beta,0,T) \equiv \left(\frac{1}{1+\frac{\rho_\beta}{1-\tau}}\right)^T = \left(\frac{1-\tau}{1+\rho_\beta-\tau}\right)^T. \tag{47}$$

The percentage error in this calculation is then given as

$$\%error = \frac{\hat{p}(\beta,0,T) - p(\beta,0,T)}{p(\beta,0,T)} \times 100\,. \tag{48}$$

Note also that, since discount factors are applied directly to expected cash flows, the percentage error in these is identical with the percentage error in the valuation of the associated cash flow. Table 4 below gives the results for this latter calculation. The percentage error naturally depends on the values assumed for τ, τ_g as well as the time the actual cash flow is received, *T*. In tables 3 and 4, τ is set at 5% or 20%. Notice that the errors involved depend significantly on the choice of τ_g. For example, if

$\tau_g = 0$, the percentage error in valuation of a period 1 cash flow is 4.76% when the effective tax rate $\tau = 0.05$, and over 22% when $\tau = 0.2$.[18]

Table 4: Percentage Error Calculations for the Discount Factor - equation (48) (= percentage error in the valuation of a cash flow)

	$\tau = 0.05$ (5%)					$\tau = 0.2$ (20%)				
	Capital gains tax rate, τ_g					Capital gains tax rate, τ_g				
Year	0	0.05	0.1	0.2	0.4	0	0.05	0.1	0.2	0.4
1	4.76	-0.23	-5.22	-15.19	-35.15	22.22	16.40	10.58	-1.06	-24.34
2	4.26	-0.45	-5.17	-14.59	-33.41	19.51	14.10	8.70	-2.11	-23.67
3	3.77	-0.68	-5.12	-13.99	-31.62	16.85	11.84	6.84	-3.14	-23.00
4	3.27	-0.90	-5.07	-13.38	-29.79	14.25	9.63	5.02	-4.17	-22.32
5	2.78	-1.13	-5.02	-12.76	-27.90	11.71	7.47	3.23	-5.18	-21.63
6	2.29	-1.35	-4.98	-12.15	-25.97	9.23	5.34	1.47	-6.18	-20.94
7	1.80	-1.58	-4.93	-11.52	-23.98	6.80	3.26	-0.26	-7.18	-20.25
8	1.32	-1.80	-4.88	-10.90	-21.94	4.43	1.22	-1.96	-8.16	-19.54
9	0.84	-2.02	-4.83	-10.26	-19.85	2.11	-0.78	-3.63	-9.13	-18.83
10	0.36	-2.24	-4.78	-9.63	-17.70	-0.16	-2.75	-5.27	-10.09	-18.12

These numerical calculations establish the fact that the discrepancies which may arise in simply grossing up *AT* discount rates in order to deduce *BT* rates can be significant. Of course, one might not start with an assumption that the *AT* rates are flat. However, any analysis which attempts to move between *BT* and *AT* analysis must recognise the non-linearity of the transformation outlined in this paper if potentially significant empirical errors are to be avoided. For example, one might wish to use data on observing *BT* market values for assets, along with estimates of their *BT* expected cash flows in order to estimate the appropriate rate at which to value the *BT* expected cash flows of some other asset. A natural approach is to postulate an underlying after tax equilibrium, to use the *BT* observed data to infer *AT* discount rates, and then to use these to reconstitute the appropriate *BT* discount rates for the new project/cash flow/asset. In this sort of analysis,

[18] And it was pointed out in footnote 5 that even small errors in the valuation of a single cash flow *may* give rise to large errors in the overall valuation of a project.

as has been established in this paper, the move from *BT* to *AT* and back again must be done with some care.

VI. Summary

The literature extending the Miles-Ezzell *ADMP* approach to the case of valuing arbitrary finite risky cash flow profiles in the presence of personal taxes is aimed at furnishing a realistic yet practical approach to the capital budgeting problem. A general starting point, explicit or implicit in this literature, is that the simple linear relationship between *BT* and *AT* discount rates and the effective tax rate found in Miller [1977] carries through to more general settings than the case where debt is riskless and fixed in perpetuity and where equity income evades all personal taxes. It is shown in this paper that Miller's specific model is an unproblematic special case precisely because it ***does*** assume the cash flow is a riskless perpetuity ***and*** that the capital gains tax rate is zero. Any extensions of the 'debt and taxes' model to incorporate positive equity tax rates – or risky debt – would seem to require a more careful treatment of the *BT/AT* relationship identified in this paper.

The relationships identified here between *AT* and *BT* discount factors (discount rates) have received no attention in the literature on valuation of equities in the presence of personal taxes (although related issues have been discussed in work on the valuation of bonds). Even those articles which do not suffer from the inconsistency problem identified here do not discuss, or even mention, this issue. This suggests that the relationships discussed in this paper may not be widely known or understood (as manifest by the fact that several papers in the above literature develop models based on assumptions which turn out to be internally incoherent). Specifically, the models of

Clubb & Doran [1991], Appleyard & Strong [1989] and Strong & Appleyard [1991] are found to be based upon internally contradictory assumptions; the valuation formulae derived in these papers are consequently unreliable. The models contained in Miller [1977], Clubb & Doran [1992] and Taggart [1991] are confined to special cases where it turns out the model assumptions are internally consistent (although there is no discussion in these papers of why there is a need to confine attention to these rather special cases). At an empirical level, the magnitude of the potential error that might be incurred by assuming that both rates of taxation and *BT/AT* term structures are flat has been shown to be potentially quite significant.[19] The present paper clarifies the issues involved, such that any future modelling in this area will not suffer from internal inconsistencies of the type described above.

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[19] Although this naturally depends on one’s assessment of what the effective tax rates are likely to be .

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Table 1: Basic Notation

Cash flows and Values

x_T: The *BT* actual cash flow received in period *T*.

χ_t: An *AT* cash flow received in period *t*

V_t: Market value at time *t* of x_T (equivalently, the market value of the future after tax cash flow profile $\{\chi_{t+1},\chi_{t+2},...,\chi_T\}$ which is generated by x_T).

Expectations

$E_t(x_T)$: Expectation at time *t* of x_T $(s \le t \le T)$ (note $E_T(x_T) = x_T$).

$E_t(\chi_{t+1})$: Expectation at time *t* of χ_{t+1}. $(s \le t \le T)$

$E_t(V_{t+1})$ Expectation at time *t* of V_{t+1}. $(s \le t \le T)$

$E_s(V_t)$: Expectation (at time *s*) of market value V_t at time *t* of the risky cash flow x_T to be received at time *T* $(s \le t \le T)$ ($E_t(V_t) = V_t$).

NB: *T* is used exclusively for the timing of receipt of a *BT* cash flow, subscript *s* is reserved for the 'present' time at which expectations are formed and for which closed expressions for the value of x_T will be derived (working backward from *s*=*T* through to *s*=0),whilst the subscript *t* is used as a time counter running between *s* and *T*.

Risk

β: A risk parameter measuring per-period variability of future expectations. $\beta = f$ denotes the riskless case.

Discount factors and discount rates

$\pi_{\beta t}$: A one-period *AT* discount factor; the market value (at time *t*) per unit of $E_t(\chi_{t+1})$ as a function of its risk.

$\rho_{\beta t}$: A one-period *AT* discount rate

$p(\beta, s, T)$: A multi-period *BT* discount factor; the market value at time *s* per unit of $E_s(x_T)$ as a function of its risk β.

$r(\beta, T)$: The spot *BT* discount rate appropriate for discounting a time *T BT* cash flow to time 0.

APPENDIX

A1 Proof that assuming a constant effective tax rate and constant before and after tax discount rates implies that $\tau = \tau_g$.

To establish this result, consider a simple transaction in which the marginal investor considers buying an arbitrary multi-period risky cash flow $\{\tilde{x}_t, .., \tilde{x}_T\}$ (*t*<*T*) at time *t*-1 and selling it in period *t*. The consequences are set out in the following table

Period	Cash flow
t-1	$-V_{t-1}$ (purchase of asset)
t	$\tilde{V}_t$ (sale of asset)
t	$\tilde{x}_t(1-\tau_t)$ (the receipt of dividend cash flow, net of personal income tax,)
t	$-(\tilde{V}_t - V_{t-1})\tau_{gt}$ (payment of CGT).

Notice that we allow that both τ_t, the dividend tax rate and τ_{gt}, the CGT tax rate might vary over time for the marginal investor. However, for a given time period, for any asset in the individual's portfolio on which dividends are paid, the same rate of dividend tax must apply, and equally, for any capital gains realised by such an individual, the same CG tax rate must apply. That is, for the marginal investor, personal tax rates are the same across assets at any given point in time. To simplify notation, let $\overline{V}_{t-1} = E_{t-1}(\tilde{V}_t)$ and $\overline{x}_t = E_{t-1}(\tilde{x}_t)$. The assumptions explicitly made in the literature are as follows:

1) the before tax rate of return/discount rate *r* is constant over time and over assets in the same risk class,
2) the after tax rate of return/discount rate ρ is constant over time and over assets in the same risk class and finally,
3) $\rho = r(1-\tau^*)$ where τ^* is a effective tax rate constant over time and across cash flows at the same point in time (τ^* is thus a 'weighted average' of the marginal investor's dividend and capital gains tax rates).

From assumptions 1) and 2), clearly

$$r = \frac{\overline{x}_t + \left(\overline{V}_t - V_{t-1}\right)}{V_{t-1}} \text{ or equivalently } V_{t-1} = \frac{\overline{x}_t + \overline{V}_t}{1+r} \tag{A1.1}$$

$$\rho = \frac{\left(\overline{V}_t - V_{t-1}\right)\left(1-\tau_{gt}\right) + \overline{x}_t(1-\tau_t)}{V_{t-1}} \tag{A1.2}$$

Connecting (A1.1) and (A1.2) using assumption $\rho = r(1-\tau^*)$ gives

$$\frac{\left(\overline{V}_t - V_{t-1}\right)\left(1-\tau_{gt}\right) + \overline{x}_t(1-\tau_t)}{V_{t-1}} = (1-\tau^*)\frac{\overline{x}_t + \left(\overline{V}_t - V_{t-1}\right)}{V_{t-1}} \tag{A1.3}$$

which simplifies to give

$$\bar{V}_t\left(\tau^*-\tau_{gt}\right)=\bar{x}_t(\tau_t-\tau^*)+V_{t-1}\left(\tau^*-\tau_{gt}\right) \tag{A1.4}$$

First, note that a sufficient condition for this to hold is that $\tau_t=\tau_{gt}=\tau^*$. It is also necessary. To see this, suppose that $\tau^*-\tau_{gt}\neq 0$. In this case we can write (A1.4) as

$$\bar{V}_t=\frac{\bar{x}_t(\tau_t-\tau^*)+V_{t-1}\left(\tau^*-\tau_{gt}\right)}{\tau^*-\tau_{gt}} \tag{A1.5}$$

so that, substituting this back into (A1.1), we have

$$(1+r)V_{t-1}=\bar{x}_t+\frac{\bar{x}_t(\tau_t-\tau^*)+V_{t-1}\left(\tau^*-\tau_{gt}\right)}{\tau^*-\tau_{gt}} \tag{A1.6}$$

or
$$V_{t-1}=\frac{\bar{x}_t}{r}\left(\frac{\tau_t-\tau_{gt}}{\tau^*-\tau_{gt}}\right) \tag{A1.7}$$

Thus, the implication of assuming 1), 2) and 3) above, when $\tau^*-\tau_{gt}\neq 0$, is that one can write the value at time *t*-1 of an asset with arbitrary cash flow purely as a function of the expected cash flow in the next period (and tax and discount rates). That is, the value of the multi-period cash flow $\{\tilde{x}_t,..,\tilde{x}_T\}$ would have to be independent of any cash flows happening after time *t*, a clear nonsense.[20] Given that the model is intended to deal with the multi-period and general case, it follows that we must impose $\tau^*=\tau_{gt}$. But then, by (A1.4), this also entails $\tau^*=\tau_t$. Thus conditions 1)-3) above are mutually consistent if and only if $\tau_t=\tau_{gt}=\tau^*$.

A2 Proof for Proposition 1.

In the paper it was established that, from equation (7),

$$V_0=p(\beta,0,T)E_0(x_T)\,, \tag{A2.1}$$

and, from (10) and (15), that

$$V_{T-i}=E_{T-i}(x_T)k(a\pi_\beta)^i \quad \text{for} \quad i=1,2\,. \tag{A2.2}$$

We now show that, for any $i,\ (0<i<T,\ T\geq 2)$, if

$$V_{T-j}=E_{T-j}(x_T)k(a\pi_\beta)^j \qquad \text{holds} \quad j=1,..,i \tag{A2.3}$$

holds, then it is also true that

$$V_{T-(i+1)}=E_{T-(i+1)}(x_T)k(a\pi_\beta)^{i+1}. \tag{A2.4}$$

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[20] Any and all cash flow profiles of the form $\{\tilde{x}_t,..,\tilde{x}_T\}$ which had the same expected cash flow at time *t* must have the same value; that is, for a constant effective tax rate to hold and for $\tau^*-\tau_{gt}\neq 0$, the only type of cash flow that is admissible is the one period cash flow, the rather special and uninteresting case where $\{\tilde{x}_t,..,\tilde{x}_T\}=\{\tilde{x}_t,0,0,...,0\}$.

Once this is established the rest of the proof is immediate; by induction, if (A2.2)-(A2.4) hold, this implies

$$V_0 = E_0(x_T)k(a\pi_\beta)^T . \qquad \text{(A2.5)}$$

Hence, equating (A2.1) and (A2.5) gives the relationship between *BPT* and *AT* discount factors specified in proposition 1, namely that

$$p(\beta,0,T) = k(a\pi_\beta)^T . \qquad \text{(A2.6)}$$

Returning to the proof of (A2.4), let $\Upsilon_s^t(\chi)$ denote the value at time *s* of an *AT* cash flow χ which arises at time $t \;\; (t > s)$. Now, from the value additivity principle,

$$\begin{aligned} V_{T-(i+1)} &= \Upsilon_{T-(i+1)}^{T}\left(\chi_T\right) + \Upsilon_{T-(i+1)}^{T-1}\left(\chi_{T-1}\right) + .. + \Upsilon_{T-(i+1)}^{T-i}\left(\chi_{T-i}\right) \\ &= \Upsilon_{T-(i+1)}^{T}\left(x_T\left(1-\tau\right) - \tau_g V_T + \tau_g V_{T-1}\right) + \Upsilon_{T-(i+1)}^{T-1}\left(-\tau_g V_{T-1} + \tau_g V_{T-2}\right) \\ &\quad + \Upsilon_{T-(i+1)}^{T-2}\left(-\tau_g V_{T-2} + \tau_g V_{T-3}\right) + \ldots\ldots\ldots\ldots. \\ &\quad + \Upsilon_{T-(i+1)}^{T-(i-1)}\left(-\tau_g V_{T-(i-1)} + \tau_g V_{T-i}\right) + \Upsilon_{T-(i+1)}^{T-i}\left(-\tau_g V_{T-i} + \tau_g V_{T-(i+1)}\right) \end{aligned} \qquad \text{(A2.7)}$$

Since $V_T = 0$, clearly

$$\Upsilon_{T-(i+1)}^{T}\left(\tau_g V_T\right) = 0 . \qquad \text{(A2.8)}$$

Following the principles of valuation discussed in the paper (section 2, the "*AT* approach"), the following are straightforward to establish:

$$\Upsilon_{T-(i+1)}^{T}\left(x_T(1-\tau)\right) = (1-\tau)\pi_\beta^{i+1} E_{T-(i+1)}(x_T) . \qquad \text{(A2.9)}$$

That is, the expected risky cash flow is discounted $i+1$ periods by the risky discount factor π_β.

Proceeding term by term, using (A2.3);

$$\Upsilon_{T-(i+1)}^{T}\left(\tau_g V_{T-1}\right) = \tau_g \pi_f \pi_\beta^i E_{T-(i+1)}(V_{T-1}) = \tau_g \pi_f \pi_\beta^{i+1} ka E_{T-(i+1)}(x_T) . \qquad \text{(A2.10)}$$

In (A2.10), note that the cash flow V_{T-1} occurs at time *T*, so is known for certain at time *T*-1. Prior to that it is risky, hence the discount factor is $\pi_f \pi_\beta^i$. Note also that (A2.10)makes use of the fact that

$$E_{T-(i+1)}(V_{T-j}) = E_{T-(i+1)}\left(E_{T-j}(x_T)k(a\pi_\beta)^j\right) = k(a\pi_\beta)^j E_{T-(i+1)}(x_T) \text{ for } j = 1,..,i , \qquad \text{(A2.11)}$$

by the law of iterated expectations (see e.g. Hamilton [1994, p742). This 'law' is applied repeatedly below. Thus,

$$\Upsilon_{T-(i+1)}^{T-1}\left(-\tau_g V_{T-1}\right) = -\tau_g \pi_\beta^i E_{T-(i+1)}(V_{T-1}) = -\tau_g \pi_\beta^{i+1} ka E_{T-(i+1)}(x_T) ,$$

$$\Upsilon_{T-(i+1)}^{T-1}\left(\tau_g V_{T-2}\right) = \tau_g \pi_f \pi_\beta^{i-1} E_{T-(i+1)}(V_{T-2}) = \tau_g \pi_f \pi_\beta^{i+1} ka^2 E_{T-(i+1)}(x_T) ,$$

……..

$$\Upsilon_{T-(i+1)}^{T-(i-1)}\left(-\tau_g V_{T-(i-1)}\right) = -\tau_g \pi_\beta^2 E_{T-(i+1)}(V_{T-(i-1)}) = -\tau_g \pi_\beta^{i+1} ka^{i-1} E_{T-(i+1)}(x_T) ,$$

$$\Upsilon_{T-(i+1)}^{T-(i-1)}\left(\tau_g V_{T-i}\right) = \tau_g \pi_f \pi_\beta E_{T-(i+1)}(V_{T-i}) = \tau_g \pi_f \pi_\beta^{i+1} ka^i E_{T-(i+1)}(x_T) ,$$

$$\Upsilon_{T-(i+1)}^{T-i}\left(-\tau_g V_{T-i}\right)=-\tau_g \pi_\beta E_{T-(i+1)}(V_{T-i})=-\tau_g \pi_\beta^{i+1} k a^i E_{T-(i+1)}(x_T).$$

Finally,

$$\Upsilon_{T-(i+1)}^{T-i}\left(\tau_g V_{T-(i+1)}\right)=\pi_f \tau_g V_{T-(i+1)},$$

since the cash flow $V_{T-(i+1)}$ occurring at time $T-i$ is known for certain at time $T-(i+1)$. Taking this last term over to the LHS and gathering terms gives the result

$$V_{T-(i+1)}\left(1-\pi_f \tau_g\right)=E_{T-(i+1)}(x_T)\left[\pi_\beta^{i+1}\right]\begin{Bmatrix}(1-\tau)+\tau_g\left(\pi_f-1\right)ka \\ +\tau_g\left(\pi_f-1\right)ka^2 \\ +.......+\tau_g\left(\pi_f-1\right)ka^i\end{Bmatrix} \tag{A2.12}$$

The RHS term in brackets can be written as

$$\begin{Bmatrix}(1-\tau)+\tau_g\left(\pi_f-1\right)ka \\ +\tau_g\left(\pi_f-1\right)ka^2 \\ +......+\tau_g\left(\pi_f-1\right)ka^i\end{Bmatrix}=\left\{(1-\tau)+k\tau_g\left(\pi_f-1\right)\sum_{j=1}^{j=i}a^j\right\}, \tag{A2.13}$$

where the geometric sum on the RHS can be written as $a(1-a^i)/(1-a)$.

Putting these together gives

$$V_{T-(i+1)}=\pi_\beta^{i+1}\frac{E_{T-(i+1)}(x_T)}{\left(1-\pi_f\tau_g\right)}\left\{(1-\tau)+k\tau_g\left(\pi_f-1\right)\frac{a(1-a^i)}{1-a}\right\}. \tag{A2.14}$$

Working on the last term this simplifies to give

$$\left\{(1-\tau)+\left(\frac{1-\tau}{1-\tau_g}\right)\tau_g\left(\pi_f-1\right)\frac{a(1-a^i)}{1-a}\right\}=(1-\tau)\left(\frac{1-\tau_g}{1-\pi_f\tau_g}\right)^i \tag{A2.15}$$

Hence

$$\begin{aligned}V_{T-(i+1)}&=\pi_\beta^{i+1}\frac{E_{T-(i+1)}(x_T)}{\left(1-\pi_f\tau_g\right)}(1-\tau)\left(\frac{1-\tau_g}{1-\pi_f\tau_g}\right)^i \\ &=E_{T-(i+1)}(x_T)k(a\pi_\beta)^{i+1}\end{aligned} \tag{A2.16}$$

given that $k=\left(\frac{1-\tau}{1-\tau_g}\right)$ and $a=\left(\frac{1-\tau_g}{1-\pi_f\tau_g}\right)$. This establishes (A2.4) and hence the result (A2.6) follows.

Appendix A3: Impact of τ,τ_g on the discount factor**:**

Equation (20) is trivial; this section establishes (21). From (18),

$$p(\beta,0,T)=\left(\frac{1-\tau}{1-\tau_g}\right)\left(\pi_\beta\frac{(1-\tau_g)}{(1-\pi_f\tau_g)}\right)^T. \tag{A3.1}$$

The partial derivative with respect to τ_g is given as (using the product, chain and quotient rules):

$$\frac{\partial p(\beta,0,T)}{\partial\tau_g}=\frac{p(\beta,0,T)}{(1-\tau_g)}+\left(\frac{1-\tau}{1-\tau_g}\right)\pi_\beta{}^T\left(\frac{-T(1-\tau_g)^{T-1}(1-\pi_f\tau_g)^T+\pi_f T(1-\pi_f\tau_g)^{T-1}(1-\tau_g)^T}{(1-\pi_f\tau_g)^{2T}}\right) \tag{A3.2}$$

which can be simplified to give

$$\frac{\partial p(\beta,0,T)}{\partial\tau_g}=\frac{p(\beta,0,T)\left((1-\pi_f\tau_g)-T(1-\pi_f)\right)}{(1-\tau_g)(1-\pi_f\tau_g)} \tag{A3.3}$$

which is equation (21) in the paper.