# INTERTEMPORAL PRICE CAP REGULATION UNDER UNCERTAINTY

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## ADDITIONAL NOTES AND DERIVATIONS

The published paper omits detailed derivations. These additional notes give full derivations for all the reported results. The following are very much 'step by step' derivations, so each step should be easy and quick to follow.

## A1. The Price Process

The price process, on time intervals on which there is no entry/investment is given by the demand function:

$$p_t = Q_t^{\eta} A_t^{-\eta} \tag{A.1}$$

Write

$$f(A_t, Q_t) \equiv Q_t^{\eta} A_t^{-\eta} \quad \text{so} \quad p_t = f(A_t, Q_t) \tag{A.2}$$

Applying Itô's lemma,

$$dp_t = f_A.dA_t + f_Q.dQ_t + \tfrac{1}{2} f_{AA} dA_t^2 + \tfrac{1}{2} f_{QQ} dQ_t^2 + f_{AQ} dA_t dQ_t \tag{A.3}$$

where

$$f_{A_t}(A_t, Q_t) = -\eta p_t / A_t \tag{A.4}$$

$$f_{A_t A_t}(A_t, Q_t) = \eta(\eta+1) p_t / A_t^2 \tag{A.5}$$

$$f_{Q_t}(A_t, Q_t) = \eta p_t / Q_t \tag{A.6}$$

$$f_{Q_t Q_t}(A_t, Q_t) = \eta(\eta-1) p_t / Q_t^2 \tag{A.7}$$

$$f_{A_t Q_t}(A_t, Q_t) = -\eta^2 p_t /(Q_t A_t) \tag{A.8}$$

whilst (using the Itô rules), we get

$$dA_t = \alpha A_t dt + \sigma A_t d\varpi_t \tag{A.9}$$

$$dA_t^2 = \left(\alpha A_t dt + \sigma A_t d\varpi_t\right)^2 = \sigma^2 A_t^2 dt \tag{A.10}$$

$$dQ_t = -\theta Q_t dt \tag{A.11}$$

$$dQ_t^2 = \theta^2 Q_t^2 dt^2 = 0 \tag{A.12}$$

and

$$dA_t dQ_t = 0. \tag{A.13}$$

Hence

$$\begin{aligned} dp_t &= f_A.dA_t + f_Q.dQ_t + \tfrac{1}{2} f_{AA} dA_t^2 + \tfrac{1}{2} f_{QQ} dQ_t^2 + f_{AQ} dA_t dQ_t \\ &= (-\eta p_t / A_t) dA_t + (\eta p_t / Q_t) dQ_t + \tfrac{1}{2}\eta(\eta+1)\left(p_t / A_t^2\right) dA_t^2 \\ &\quad + \tfrac{1}{2}\eta(\eta-1)\left(p_t / Q_t^2\right) dQ_t^2 + \left(-\eta^2 p_t /(Q_t A_t)\right) dA_t dQ_t \\ &= (-\eta p_t / A_t)\left(\alpha A_t dt + \sigma A_t d\varpi_t\right) + (\eta p_t / Q_t)\left(-\theta Q_t dt\right) \\ &\quad + \tfrac{1}{2}\eta(\eta+1)\left(p_t / A_t^2\right)\sigma^2 A_t^2 dt \end{aligned} \tag{A.14}$$

$\Rightarrow$

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$$dp_t = -\eta p_t\left(\alpha dt + \sigma d\varpi_t\right) - \theta\eta p_t dt + \tfrac{1}{2}\eta(\eta+1)p_t\sigma^2 dt \qquad (A.15)$$

$\Rightarrow$

$$dp_t = -\eta\left(\alpha + \theta - \tfrac{1}{2}(\eta+1)\sigma^2\right)p_t dt - \eta\sigma p_t d\varpi_t \qquad (A.16)$$

Writing

$$\mu_p = -\eta\left(\alpha + \theta - \tfrac{1}{2}(\eta+1)\sigma^2\right) \qquad (A.17)$$

then this becomes

$$dp_t = \mu_p p_t dt - \eta\sigma p_t d\varpi_t \qquad (A.18)$$

which is equation (5) in the paper.

**A2 <u>Derivation of the fundamental differential equation for value:</u>**

The arbitrage equation was (reproduced here for convenience):

$$\left(\theta + r\right)v dt = p dt + E(dv) \qquad (A.19)$$

Itô's lemma gives

$$dv = v_p dp + \tfrac{1}{2}v_{pp}dp^2 + v_K dK + \tfrac{1}{2}v_{KK}dK^2 + v_{pK}dpdK \qquad (A.20)$$

Now,

$$v \equiv \psi(x)K \quad \text{where } x = p/K \qquad (A.21)$$

so

$$v_p = \psi'\left(p/K\right)K/K = \psi'\left(p/K\right) \qquad (A.22)$$

$$v_{pp} = \frac{\psi''\left(p/K\right)}{K} \qquad (A.23)$$

$$v_K = \psi\left(p/K\right) - K\psi'\left(p/K\right)\frac{p}{K^2} = \psi\left(p/K\right) - \psi'\left(p/K\right)\frac{p}{K} \qquad (A.24)$$

$$\begin{aligned} v_{KK} &= -\psi'\left(p/K\right)\frac{p}{K^2} + \psi'\left(p/K\right)\frac{p}{K^2} \\ &\quad + \psi''\left(p/K\right)\frac{p^2}{K^3} = \psi''\left(p/K\right)\frac{p^2}{K^3} \end{aligned} \qquad (A.25)$$

$$v_{pK} = \left(-p/K^2\right)\psi''\left(p/K\right) \qquad (A.26)$$

From (A.18)

$$dp = \mu_p p dt - \eta\sigma p d\varpi \qquad (A.27)$$

$$dp^2 = \left(\mu_p p dt - \eta\sigma p d\varpi_t\right)^2 = \eta^2 p^2\sigma^2 dt \qquad (A.28)$$

Since $dK = -\delta K dt$ it follows that $dK^2 = 0$. Hence the term $dv$ is given as

$$\begin{aligned} dv &= v_t dt + v_p dp + \tfrac{1}{2}v_{pp}dp^2 + v_K dK + \tfrac{1}{2}v_{KK}dK^2 + v_{pK}dpdK \\ &= [0]dt + \left[\psi'\left(p/K\right)\right]dp + \tfrac{1}{2}\left[\frac{\psi''\left(p/K\right)}{K}\right]dp^2 \\ &\quad + \left[\psi\left(p/K\right) - \psi'\left(p/K\right)\frac{p}{K}\right]dK + \tfrac{1}{2}\left[\psi''\left(p/K\right)\frac{p^2}{K^3}\right]dK^2 \\ &\quad + \left[\left(-p/K^2\right)\psi''\left(p/K\right)\right]dpdK \end{aligned} \qquad (A.29)$$

and substituting further (and abbreviating the notation a little)

$$dv = \psi' dp + \tfrac{1}{2}\left[\frac{\psi''}{K}\right]dp^2 + \left[\psi - \psi'\frac{p}{K}\right]dK + \tfrac{1}{2}\left[\psi''\frac{p^2}{K^3}\right]dK^2 + \left[\left(-p/K^2\right)\psi''\right]dpdK$$

$$= \psi'\left[\mu_p pdt - \eta\sigma pd\varpi\right] + \tfrac{1}{2}\left[\frac{\psi''}{K}\right]\left(\eta^2 p^2\sigma^2 dt\right) + \left[\psi - \psi'\frac{p}{K}\right](-\delta K dt)$$

$\Rightarrow$

$$dv = \psi'\left[\mu_p pdt - \eta\sigma pd\varpi\right] + \tfrac{1}{2}\left[\frac{\psi''}{K}\right]\left(\eta^2 p^2\sigma^2 dt\right) + \left[\psi - \psi'\frac{p}{K}\right](-\delta K dt)$$
$$= \left[\tfrac{1}{2}\frac{\psi''}{K}\eta^2 p^2\sigma^2 + \mu_p p\psi' - \delta K\psi + \delta K\psi'\frac{p}{K}\right]dt + \left[-\eta\sigma p\psi'\right]d\varpi \tag{A.30}$$

Thus taking expectations,

$$E(dv) = \left[\frac{\eta^2 p^2\sigma^2}{2K}\psi'' + \left(\mu_p + \delta\right)p\psi' - \delta K\psi\right]dt \tag{A.31}$$

Hence the arbitrage equation becomes

$$(\theta + r)vdt - pdt - \left[\frac{\eta^2 p^2\sigma^2}{2K}\psi'' + \left(\mu_p + \delta\right)p\psi' - \delta K\psi\right]dt = 0 \tag{A.32}$$

$\Rightarrow$

$$(\theta + r)\psi K - p - \frac{\eta^2 p^2\sigma^2}{2K}\psi'' - \left(\mu_p + \delta\right)p\psi' + \delta K\psi = 0 \tag{A.33}$$

$\Rightarrow$

$$\tfrac{1}{2}\sigma^2 x^2\psi'' + x\left(\mu_p + \delta\right)\psi' - \left(\theta + r + \delta\right)\psi + x = 0 \tag{A.34}$$

which is equation (A.2) in the appendix to the paper. The general solution to (A.34) can be written as the sum of the general solution to the homogeneous equation

$$\tfrac{1}{2}\sigma^2 x^2\psi'' + x\left(\mu_p + \delta\right)\psi' - \left(\theta + r + \delta\right)\psi = 0 \tag{A.35}$$

and a particular solution to (A.34). A particular solution, easily verified, is

$$\psi = B_0 x \tag{A.36}$$

where

$$B_0 = \frac{1}{\theta + r - \mu_p} = \frac{1}{(\theta + r) + \eta\left(\alpha + \theta - \tfrac{1}{2}(\eta + 1)\sigma^2\right)}, \tag{A.37}$$

and the general solution to the homogenous equation (A.35) can be written as

$$\psi(x) = B_1 x^{\lambda_1} + B_2 x^{\lambda_2}\,. \tag{A.38}$$

where the roots are defined as

$$\lambda_1 = \left(-R_1 + R_2\right)/\eta^2\sigma^2 \tag{A.39}$$

$$\lambda_2 = \left(-R_1 - R_2\right)/\eta^2\sigma^2\,, \tag{A.40}$$

and where

$$R_1 \equiv \left(\mu_p + \delta - \tfrac{1}{2}\eta^2\sigma^2\right). \tag{A.41}$$

$$R_2 \equiv \left( \left( \mu_p + \delta - \tfrac{1}{2}\eta^2\sigma^2 \right)^2 + 2\eta^2\sigma^2 \left( \theta + r + \delta \right) \right)^{1/2}. \tag{A.42}$$

Hence the solution to (A.34) takes the form

$$\psi(x) = B_0 x + B_1 x^{\lambda_1} + B_2 x^{\lambda_2}. \tag{A.43}$$

Notice that $2\eta^2\sigma^2 \left( \theta + r + \delta \right) > 0$ if $\sigma^2 > 0$, so the roots are real and of opposite sign when uncertainty is present. The arbitrary constants $B_1, B_2$ are determined by boundary conditions. As $x \to 0$, if value is to be finite, it must be that $B_2 = 0$ or the solution explodes (see Dixit, 1993). The other constant is determined by an analysis of smooth pasting conditions at the boundary (at which new investment is triggered).

## A3 Analysis of smooth pasting conditions

Given $B_2 = 0$, from (A.43), the solution is

$$\psi(x) = B_0 x + B_1 x^{\lambda_1} \tag{A.44}$$

where $B_0$ is given by (A.37).

*(a) The Competitive case:*

In this case it was established that the smooth pasting conditions are

$$\psi(\xi_u) = 1 \tag{A.45}$$

and

$$\psi'(\xi_u) = 0. \tag{A.46}$$

From these equations we get

$$\psi(\xi_u) = B_0 \xi_u + B_1 \xi_u^{\lambda_1} = 1, \tag{A.47}$$

$$\psi'(\xi_u) = B_0 + \lambda_1 B_1 \xi_u^{\lambda_1 - 1} = 0 \Rightarrow B_1 \xi_u^{\lambda_1} = -B_0 \xi_u / \lambda_1, \tag{A.48}$$

and hence

$$\psi(\xi_u) = B_0 \xi_u \left( 1 - \tfrac{1}{\lambda_1} \right) = 1, \tag{A.49}$$

$$\Rightarrow$$

$$\xi_u = \left( \frac{\lambda_1}{\lambda_1 - 1} \right) \left( \theta + r - \mu_p \right). \tag{A.50}$$

which is Result 2(i).

*(b) The monopoly case*

In the monopoly case, investment commences at a time $\tilde{t}$ at which price $p_{\tilde{t}}$ reaches the level $p_{\tilde{t}} = \xi_M K_{\tilde{t}}$, where $\xi_M$ is the relative price at which new capacity is added. Since $\xi_M$ is a free choice by the firm, smooth pasting involves first and second derivative conditions (see Dumas, 1991). The first derivative condition is that, with respect to the control variable, the rate of change of value should just equal the rate of change of cost;

$$\partial V(p_{\tilde{t}}, K_{\tilde{t}}, Q_{\tilde{t}}) / \partial p_{\tilde{t}} = \partial \left( K_{\tilde{t}} Q_{\tilde{t}} \right) / \partial p_{\tilde{t}} \tag{A.51}$$

where $V(p_{\tilde{t}}, K_{\tilde{t}}, Q_{\tilde{t}}) = \psi(x_{\tilde{t}}) K_{\tilde{t}} Q_{\tilde{t}}$, $Q_{\tilde{t}} = A_{\tilde{t}} p_{\tilde{t}}^{\gamma}$ and $x_{\tilde{t}} = p_{\tilde{t}} / K_{\tilde{t}}$. Substituting these into (A.51) gives

$$\partial\left\{A_{\tilde{t}}K_{\tilde{t}}\left[\psi(p_{\tilde{t}}/K_{\tilde{t}})-1\right]p_{\tilde{t}}^{\gamma}\right\}/\partial p_{\tilde{t}}=0\,, \tag{A.52}$$

which gives

$$\begin{aligned}&\gamma A_{\tilde{t}}K_{\tilde{t}}\left[\psi(p_{\tilde{t}}/K_{\tilde{t}})-1\right]p_{\tilde{t}}^{\gamma-1}+A_{\tilde{t}}K_{\tilde{t}}p_{\tilde{t}}^{\gamma}\psi'(p_{\tilde{t}}/K_{\tilde{t}})(1/K_{\tilde{t}})\\&=A_{\tilde{t}}p_{\tilde{t}}^{\gamma-1}\left\{\gamma K_{\tilde{t}}\left[\psi(p_{\tilde{t}}/K_{\tilde{t}})-1\right]+p_{\tilde{t}}\psi'(p_{\tilde{t}}/K_{\tilde{t}})\right\}=0\end{aligned} \tag{A.53}$$

$$\Rightarrow\gamma\left[\psi(x_{\tilde{t}})-1\right]+x_{\tilde{t}}\psi'(x_{\tilde{t}})=0 \tag{A.54}$$

$$\Rightarrow\gamma\left[\psi(\xi_M)-1\right]+\xi_M\psi'(\xi_M)=0 \tag{A.55}$$

The second derivative condition is

$$\partial^2V(p_{\tilde{t}},K_{\tilde{t}},Q_{\tilde{t}})/\partial p_{\tilde{t}}^{\ 2}=\partial^2\left(K_{\tilde{t}}Q_{\tilde{t}}\right)/\partial p_{\tilde{t}}^{\ 2} \tag{A.56}$$

which gives

$$\begin{aligned}&(\gamma-1)\,A_{\tilde{t}}p_{\tilde{t}}^{\gamma-2}\left\{\gamma K_{\tilde{t}}\left[\psi(p_{\tilde{t}}/K_{\tilde{t}})-1\right]+p_{\tilde{t}}\psi'(p_{\tilde{t}}/K_{\tilde{t}})\right\}\\&+A_{\tilde{t}}p_{\tilde{t}}^{\gamma-1}\left\{\gamma K_{\tilde{t}}\psi'(p_{\tilde{t}}/K_{\tilde{t}})(1/K_{\tilde{t}})+\psi'(p_{\tilde{t}}/K_{\tilde{t}})+p_{\tilde{t}}\psi''(p_{\tilde{t}}/K_{\tilde{t}})(1/K_{\tilde{t}})\right\}=0\end{aligned} \tag{A.57}$$

$$\Rightarrow(\gamma-1)\left\{\gamma\left[\psi(x_{\tilde{t}})-1\right]+x_{\tilde{t}}\psi'(x_{\tilde{t}})\right\}+x_{\tilde{t}}\left\{(1+\gamma)\psi'(x_{\tilde{t}})+x_{\tilde{t}}\psi''(x_{\tilde{t}})\right\}=0 \tag{A.58}$$

$$\Rightarrow\begin{aligned}&\gamma(\gamma-1)\left[\psi(x_{\tilde{t}})-1\right]+(\gamma-1)\,x_{\tilde{t}}\psi'(x_{\tilde{t}})\\&\quad+\left\{(1+\gamma)x_{\tilde{t}}\psi'(x_{\tilde{t}})+x_{\tilde{t}}^2\psi''(x_{\tilde{t}})\right\}=0\end{aligned} \tag{A.59}$$

$$\Rightarrow$$
$$\gamma(\gamma-1)\left[\psi(\xi_M)-1\right]+(\gamma-1)\,\xi_M\psi'(\xi_M)+\left\{(1+\gamma)\xi_M\psi'(\xi_M)+\xi_M^2\psi''(\xi_M)\right\}=0 \tag{A.60}$$

$$\Rightarrow$$
$$\begin{aligned}&\gamma(\gamma-1)\left[\psi(\xi_M)-1\right]+(\gamma-1)\,\xi_M\psi'(\xi_M)+(1+\gamma)\xi_M\psi'(\xi_M)+\xi_M^2\psi''(\xi_M)=0\\&\Rightarrow\xi_M^2\psi''(\xi_M)+2\gamma\xi_M\psi'(\xi_M)+\gamma(\gamma-1)\left[\psi(\xi_M)-1\right]=0\end{aligned} \tag{A.61}$$

Now, from (A.44),

$$\psi(x)=B_0x+B_1x^{\lambda_1} \tag{A.62}$$

so

$$\psi'(x)=B_0+\lambda_1B_1x^{\lambda_1-1} \tag{A.63}$$

$$\psi''(x)=\lambda_1(\lambda_1-1)B_1x^{\lambda_1-2} \tag{A.64}$$

so, using (A.62)-(A.64), equation (A.55) gives

$$\gamma\left[\psi(\xi_M)-1\right]+\xi_M\psi'(\xi_M)=0$$

$$\Rightarrow\gamma\left[B_0\xi_M+B_1\xi_M^{\lambda_1}-1\right]+\xi_M\left(B_0+\lambda_1B_1\xi_M^{\lambda_1-1}\right)=0$$

$$\Rightarrow(1+\gamma)B_0\xi_M+B_1\xi_M^{\lambda_1}\left(\lambda_1+\gamma\right)-\gamma=0 \tag{A.65}$$

$$\Rightarrow B_1\xi_M^{\lambda_1}=\frac{\gamma-(1+\gamma)B_0\xi_M}{(\lambda_1+\gamma)} \tag{A.66}$$

and, from (A.61), again using (A.62)-(A.64),

$$\begin{aligned}&\xi_M^2\lambda_1(\lambda_1-1)B_1\xi_M^{\lambda-2}+2\gamma\xi_M\left[B_0+\lambda_1B_1\xi_M^{\lambda-1}\right]+\gamma(\gamma-1)\left[B_0\xi_M+B_1\xi_M^{\lambda}-1\right]=0\\&\Rightarrow\lambda_1(\lambda_1-1)B_1\xi_M^{\lambda}+\left[2\gamma B_0\xi_M+2\gamma\lambda_1B_1\xi_M^{\lambda}\right]+\gamma(\gamma-1)\left[B_0\xi_M+B_1\xi_M^{\lambda}-1\right]=0\end{aligned} \tag{A.67}$$

$$\Rightarrow B_1\xi_M^{\lambda}\left\{\lambda_1(\lambda_1-1)+2\gamma\lambda_1+\gamma(\gamma-1)\right\}+\left\{2\gamma+\gamma(\gamma-1)\right\}B_0\xi_M=\gamma(\gamma-1) \tag{A.68}$$

Substituting for $B_1\xi_M^{\lambda_1}$ using (A.66), this gives

$$\left(\frac{\gamma-(1+\gamma)B_0\xi_M}{(\lambda_1+\gamma)}\right)\begin{Bmatrix}\lambda_1(\lambda_1-1)+2\gamma\lambda_1\\ +\gamma(\gamma-1)\end{Bmatrix}+\left\{2\gamma+\gamma(\gamma-1)\right\}B_0\xi_M=\gamma(\gamma-1) \tag{A.69}$$

$\Rightarrow$

$$\begin{aligned}&(\gamma-(1+\gamma)B_0\xi_M)\left\{\lambda_1(\lambda_1-1)+2\gamma\lambda_1+\gamma(\gamma-1)\right\}\\ &+\left\{2\gamma+\gamma(\gamma-1)\right\}(\lambda_1+\gamma)B_0\xi_M=\gamma(\gamma-1)(\lambda_1+\gamma)\end{aligned} \tag{A.70}$$

$\Rightarrow$

$$B_0\xi_M\begin{Bmatrix}-(1+\gamma)\begin{Bmatrix}\lambda_1(\lambda_1-1)\\ +2\gamma\lambda_1+\gamma(\gamma-1)\end{Bmatrix}\\ +\left\{2\gamma+\gamma(\gamma-1)\right\}(\lambda_1+\gamma)\end{Bmatrix}=\begin{Bmatrix}-\gamma\begin{Bmatrix}\lambda_1(\lambda_1-1)+2\gamma\lambda_1\\ +\gamma(\gamma-1)\end{Bmatrix}\\ +\gamma(\gamma-1)(\lambda_1+\gamma)\end{Bmatrix} \tag{A.71}$$

$\Rightarrow$

$$(1+\gamma)B_0\xi_M\begin{Bmatrix}\gamma(\lambda_1+\gamma)-\lambda_1(\lambda_1-1)\\ -2\gamma\lambda_1-\gamma(\gamma-1)\end{Bmatrix}=\gamma\begin{Bmatrix}-\lambda_1(\lambda_1-1)-2\gamma\lambda_1-\gamma(\gamma-1)\\ +\lambda_1\gamma+\gamma^2-\lambda_1-\gamma\end{Bmatrix} \tag{A.72}$$

$\Rightarrow$

$$(1+\gamma)B_0\xi_M\begin{Bmatrix}\gamma\lambda_1+\gamma^2-\lambda_1(\lambda_1-1)\\ -2\gamma\lambda_1-\gamma^2+\gamma\end{Bmatrix}=\gamma\begin{Bmatrix}-\lambda_1(\lambda_1-1)-2\gamma\lambda_1\\ +\lambda_1\gamma-\lambda_1\end{Bmatrix} \tag{A.73}$$

$\Rightarrow$

$$(\lambda_1+\gamma)(\lambda_1-1)(1+\gamma)B_0\xi_M=\gamma\lambda_1(\lambda_1+\gamma) \tag{A.74}$$

$\Rightarrow$

$$(\lambda_1-1)(1+\gamma)B_0\xi_M=\gamma\lambda_1 \tag{A.75}$$

Substituting for $B_0$ using (A.37), this gives

$$\xi_M=\left(\frac{\gamma}{1+\gamma}\right)\left(\frac{\lambda_1}{\lambda_1-1}\right)(\theta+r-\mu_p) \tag{A.76}$$

which is Result 2(ii).

## A4 <u>Relationship between alternative option value multipliers (competitive case)</u>

Start with the original result (A.50) that

$$\xi_u=\left(\frac{\lambda_1}{\lambda_1-1}\right)(r+\theta-\mu_p) \tag{A.77}$$

and note that, from (A.39)-(A.42) that

$$\begin{aligned}\lambda_1\lambda_2&=\left(\frac{-R_1+R_2}{\sigma^2}\right)\left(\frac{-R_1-R_2}{\sigma^2}\right)=\left(\frac{R_1^2-R_2^2}{\sigma^4}\right)\\ &=\left(\frac{-2(\theta+r+\delta)}{\sigma^2}\right)\end{aligned} \tag{A.78}$$

$\Rightarrow$

$$\theta+r+\delta=-\sigma^2\lambda_1\lambda_2/2 \tag{A.79}$$

and

$$\lambda_1+\lambda_1=\frac{-R_1+R_2}{\sigma^2}+\frac{-R_1-R_2}{\sigma^2}=\frac{-2R_1}{\sigma^2}$$
$$=\frac{-2\left(\mu_p+\delta-\frac{1}{2}\sigma^2\right)}{\sigma^2} \qquad \text{(A.80)}$$

$\Rightarrow$

$$\left(\mu_p+\delta\right)=\tfrac{1}{2}\sigma^2\left(1-\left(\lambda_1+\lambda_2\right)\right). \qquad \text{(A.81)}$$

Write

$$\xi_u=\left(\frac{\lambda_1}{\lambda_1-1}\right)\left(r+\theta-\mu_p\right)=\left(\frac{\lambda_1}{\lambda_1-1}\right)\left(r+\theta+\delta-\left(\delta+\mu_p\right)\right) \qquad \text{(A.82)}$$

so, from (A.79) and (A.81)

$$\xi_u=\left(\frac{\lambda_1}{\lambda_1-1}\right)\left(r+\theta+\delta-\left(\delta+\mu_p\right)\right)$$
$$=\left(\frac{\lambda_1}{\lambda_1-1}\right)\left(-\tfrac{1}{2}\sigma^2\lambda_1\lambda_2-\tfrac{1}{2}\sigma^2\left(1-\left(\lambda_1+\lambda_2\right)\right)\right) \qquad \text{(A.83)}$$

$\Rightarrow$

$$\xi_u=-\tfrac{1}{2}\sigma^2\left(\frac{\lambda_1}{\lambda_1-1}\right)\left(\lambda_1-1\right)\left(\lambda_2-1\right)$$
$$=-\tfrac{1}{2}\sigma^2\lambda_1\left(\lambda_2-1\right) \qquad \text{(A.84)}$$

From (A.79)

$$-\tfrac{1}{2}\sigma^2\lambda_1=\left(\theta+r+\delta\right)/\lambda_2 \qquad \text{(A.85)}$$

so

$$\xi_u=\left(\frac{\lambda_2-1}{\lambda_2}\right)\left(\theta+r+\delta\right) \qquad \text{(A.86)}$$

That is,

$$\xi_u=\left(\frac{\lambda_2-1}{\lambda_2}\right)\left(\theta+r+\delta\right)=\left(\frac{\lambda_1}{\lambda_1-1}\right)\left(r+\theta-\mu_p\right) \qquad \text{(A.87)}$$

which is reported in the paper in equations (22), (23) and result 2(i).
Hence also, from (A.76)**,**

$$\xi_M=\left(\frac{\gamma}{1+\gamma}\right)\left(\frac{\lambda_2-1}{\lambda_2}\right)\left(\theta+r+\delta\right). \qquad \text{(A.88)}$$

**A5 <u>The limiting case:</u>**

Under certainty, the certainty relative price at which entry takes place is $\xi_c=\theta+r+Max\left[\left(\eta(\alpha+\theta)\right),\delta\right]$ or equivalently, that

(a) if $\delta<\eta(\alpha+\theta)$, $\xi_c=\left(\theta+r+\eta(\alpha+\theta)\right)$.

(b) if $\delta\geq\eta(\alpha+\theta)$, then $\xi_c=\theta+r+\delta$.

This section shows that $\xi_u\rightarrow\xi_c$ as $\sigma^2\rightarrow 0$. That is, writing $Lim\equiv Lim_{\sigma^2\rightarrow 0}$, that

$$Lim\,\xi_u=Lim\frac{\lambda_1}{(\lambda_1-1)}\left[\theta+r+\eta\left(\alpha+\theta-\tfrac{1}{2}(\eta+1)\sigma^2\right)\right]$$
$$=\theta+r+Max\left[\eta(\alpha+\theta),\delta\right] \qquad \text{(A.89)}$$

Define

$$\xi\left(\sigma^2\right) \equiv \frac{\lambda_1\left(\sigma^2\right)}{(\lambda_1\left(\sigma^2\right)-1)}\phi\left(\sigma\ \right) \tag{A.90}$$

where

$$\phi\left(\sigma^2\right) \equiv \theta + r + \eta\left(\alpha + \theta - \tfrac{1}{2}(\eta+1)\sigma^2\right) \tag{A.91}$$

Clearly, from (A.91),

$$Lim\ \phi\left(\sigma^2\right) = \theta + r + \eta\left(\alpha + \theta\right) \tag{A.92}$$

Case (a) $\delta < \eta(\alpha+\theta)$

First, examine $Lim\ \lambda_1$ and $Lim\ \frac{\lambda_1}{(\lambda_1 - 1)}$. From (A.39)-(A.42), substituting for $\mu_p$ using (A.17):

$$\lambda_1 = \frac{-\begin{pmatrix}\mu_p + \delta \\ -\tfrac{1}{2}\eta^2\sigma^2\end{pmatrix} + \left(\begin{pmatrix}\mu_p + \delta \\ -\tfrac{1}{2}\eta^2\sigma^2\end{pmatrix}^2 + 2\eta^2\sigma^2\left(\theta + r + \delta\right)\right)^{1/2}}{\eta^2\sigma^2}$$

$$= \frac{-\begin{pmatrix}-\eta(\theta+\alpha) + \tfrac{1}{2}\eta(\eta+1)\sigma^2 \\ +\delta - \tfrac{1}{2}\eta^2\sigma^2\end{pmatrix} + \left(\begin{pmatrix}-\eta(\theta+\alpha) + \tfrac{1}{2}\eta(\eta+1)\sigma^2 \\ +\delta - \tfrac{1}{2}\eta^2\sigma^2\end{pmatrix}^2 + 2\eta^2\sigma^2\left(\theta + r + \delta\right)\right)^{1/2}}{\eta^2\sigma^2} \tag{A.93}$$

$\Rightarrow$

$$\lambda_1 = \frac{-\left(\delta - \eta(\theta+\alpha) + \tfrac{1}{2}\eta\sigma^2\right) + \left(\left(\delta - \eta(\theta+\alpha) + \tfrac{1}{2}\eta\sigma^2\right)^2 + 2\eta^2\sigma^2\left(\theta + r + \delta\right)\right)^{1/2}}{\eta^2\sigma^2}$$

$$= \frac{-\left(\delta - \eta(\theta+\alpha)\right)}{\eta^2\sigma^2} - \frac{1}{2\eta} + \frac{1}{\sigma^2}\left(\left(\delta - \eta(\theta+\alpha) + \tfrac{1}{2}\eta\sigma^2\right)^2 + 2\eta^2\sigma^2\left(\theta + r + \delta\right)\right)^{1/2} \tag{A.94}$$

Now, if $\delta < \eta(\alpha+\theta)$ then

$$-\left(\delta - \eta\left(\theta + \alpha\right)\right) > 0 \tag{A.95}$$

and, taking the limit in (A.94), as

$$\sigma^2 \to 0 \text{ so } \lambda_1 \to +\infty$$

(whatever happens to the other terms, they are non-negative, and the first term $\to +\infty$). Thus, if

$$\delta - \eta\left(\theta + \alpha\right) > 0 \text{ then } Lim\ \frac{\lambda_1}{(\lambda_1 - 1)} = Lim\ \frac{1}{(1 - 1/\lambda_1)} = 1 \tag{A.96}$$

and hence

$$Lim\ \xi_u = Lim\left(\frac{\lambda_1}{\lambda_1 - 1}\right) Lim\,\phi = \theta + r + \eta(\alpha + \theta) \tag{A.97}$$

Case (b) $\delta > \eta(\alpha+\theta)$

Then

$$-\left(\delta - \eta\left(\theta + \alpha\right)\right) < 0, \tag{A.98}$$

and $\lambda_1$ behaves rather differently in the limit. To see this, define

$$f(\sigma^2) \equiv -\left(\delta - \eta(\theta+\alpha) + \tfrac{1}{2}\eta\sigma^2\right) + \left(\left(\delta - \eta(\theta+\alpha) + \tfrac{1}{2}\eta\sigma^2\right)^2 + 2\eta^2\sigma^2(\theta + r + \delta)\right)^{1/2} \tag{A.99}$$

and

$$g(\sigma^2) \equiv \eta^2\sigma^2 \tag{A.100}$$

so that

$$\lambda(\sigma^2) = f(\sigma^2)/g(\sigma^2) \tag{A.101}$$

Now, when (A.98) holds, then $f(0) = 0$ and $g(0) = 0$ so the limit for $\lambda(\sigma^2)$ is not immediate from (A.101). However, it can be obtained by applying l'Hopital's rule; thus

$$g'(\sigma^2) = \eta^2$$

and

$$\begin{aligned} f'(\sigma^2) = -\tfrac{1}{2}\eta + \tfrac{1}{2}\left(\left(\delta - \eta(\theta+\alpha) + \tfrac{1}{2}\eta\sigma^2\right)^2 + 2\eta^2\sigma^2(\theta + r + \delta)\right)^{-1/2} \times \\ \left(2\left(\delta - \eta(\theta+\alpha) + \tfrac{1}{2}\eta\sigma^2\right)\tfrac{1}{2}\eta + 2\eta^2(\theta + r + \delta)\right) \end{aligned} \tag{A.102}$$

so that

$$Lim\ g'(\sigma^2) = \eta^2 \tag{A.103}$$

and

$$\begin{aligned} Lim\ f'(\sigma^2) &= -\tfrac{1}{2}\eta + \tfrac{1}{2}\frac{\left(\eta\left(\delta - \eta(\theta+\alpha)\right) + 2\eta^2(\theta + r + \delta)\right)}{\left(\delta - \eta(\theta+\alpha)\right)} \\ &= \frac{-\tfrac{1}{2}\eta\left(\delta - \eta(\theta+\alpha)\right) + \tfrac{1}{2}\left(\eta\left(\delta - \eta(\theta+\alpha)\right) + 2\eta^2(\theta + r + \delta)\right)}{\left(\delta - \eta(\theta+\alpha)\right)} \\ &= \frac{-\tfrac{1}{2}\eta\delta + \tfrac{1}{2}\eta^2(\theta+\alpha) + \tfrac{1}{2}\eta\delta - \tfrac{1}{2}\eta^2(\theta+\alpha) + \eta^2(\theta + r + \delta)}{\left(\delta - \eta(\theta+\alpha)\right)} \\ &= \frac{\eta^2(\theta + r + \delta)}{\left(\delta - \eta(\theta+\alpha)\right)} \end{aligned} \tag{A.104}$$

Hence, by l'Hopital's rule

$$\begin{aligned} Lim\ \lambda_1\left(\sigma^2\right) &= Lim\ \frac{f'\left(\sigma^2\right)}{g'\left(\sigma^2\right)} = \frac{Lim\ f'\left(\sigma^2\right)}{Lim\ g'\left(\sigma^2\right)} \\ &= \frac{\eta^2(\theta + r + \delta)}{\left(\delta - \eta(\theta+\alpha)\right)} \times \frac{1}{\eta^2} = \frac{\theta + r + \delta}{\delta - \eta(\theta+\alpha)} \end{aligned} \tag{A.105}$$

Thus given $Lim\ \lambda_1\left(\sigma^2\right)$ exists, it follows that

$$\begin{aligned} Lim\ \frac{\lambda_1}{(\lambda_1 - 1)} &= \frac{Lim\ \lambda_1}{(Lim\ \lambda_1 - 1)} \\ &= \frac{\dfrac{\theta + r + \delta}{\delta - \eta(\theta+\alpha)}}{\dfrac{\theta + r + \delta}{\delta - \eta(\theta+\alpha)} - 1} = \frac{\theta + r + \delta}{\theta + r + \eta(\theta+\alpha)} \end{aligned} \tag{A.106}$$

Hence

$$Lim\ \xi_u\left(\sigma^2\right) = Lim\ \frac{\lambda_1}{(\lambda_1 - 1)} Lim\ \phi$$

$$= \frac{\theta + r + \delta}{\theta + r + \eta(\theta+\alpha)} \times (\theta + r) + \eta(\theta+\alpha) \qquad \text{(A.107)}$$

$$= (\theta + r + \delta)$$

This completes the derivation for (A.89) and hence result 1 in the paper.

## A6. <u>Monopoly firm subject to intertemporal price cap.</u>

Here $p_t$ denotes the demand price (the price which clears the market); that is, such that

$$p_t = Q_t^{\eta} A_t^{-\eta} \qquad \text{(A.108)}$$

However, the firm is required to set a price which satisfies the price cap $\overline{p}_t = \overline{\xi} K_t$; denoting the set price as $p_t^s$, then this requires that $p_t^s \le \overline{p}_t$. Thus

$$p_t^s = Min[p_t, \overline{p}_t] = Min[p_t, \overline{\xi} K_t]. \qquad \text{(A.109)}$$

where $\overline{\xi}$ is a constant chosen by the regulator. For low values of the relative price $x_t$, the monopolist utilises existing capacity, whilst the price is unconstrained. Once $x_t$ reaches a certain level, the price cap binds, but investment may still be deferred (so there is quantity rationing). Finally, if $x_t$ increases sufficiently, it will be optimal for the firm to add to capacity. There is thus a transition boundary between the 2 no-investment regimes, and a further boundary at which investment commences; at each of these boundaries, smooth pasting conditions apply.

Let $\psi$ denote the solution when there is no investment and no price constraint and let $\psi_2$ denote the solution when the price constraint applies, but there is no investment. The first task is to characterise the process in each of the regimes. Following this, smooth pasting conditions are studied.

**Regime 1: Unconstrained price, no investment.**

The solution here is identical to that already established for the unconstrained monopoly firm. That is, from (A.34), repeated for convenience,

$$\tfrac{1}{2}\sigma^2 x^2 \psi'' + \left(\mu_p + \delta\right) x\psi' - (\delta + \theta + r)\psi + x = 0 \qquad \text{(A.110)}$$

and the solution is

$$\psi(x) = B_0 x + B_1 x^{\lambda_1} + B_2 x^{\lambda_2} \qquad \text{(A.111)}$$

where $B_0$ is given by (A.37).

As before, note that, as $x \to 0$, if value is to be finite, it must be that $B_2 = 0$. Dixit (1993) discusses this sort of boundary condition in more detail. Hence the solution becomes

$$\psi(x) = B_0 x + B_1 x^{\lambda_1} \qquad \text{(A.112)}$$

**Regime 2: Unconstrained price, no investment.**
The arbitrage condition in this case is, since in this regime $p_t = \overline{\xi} K_t$, that

$$\left(\theta + r\right) v dt = \overline{\xi} K dt + E(dv) \quad \text{(A.113)}$$

where[1] $v \equiv \psi(p/K)K$. Applying Ito's lemma, taking expectations and simplifying,

$$\tfrac{1}{2}\sigma^2 x^2 \psi'' + \left(\mu_p + \delta\right) x\psi' - (\delta + \theta + r)\psi + \overline{\xi} = 0 \quad \text{(A.114)}$$

A particular solution is clearly

$$\psi = \overline{\xi} \big/ (\delta + \theta + r) \quad \text{(A.115)}$$

and so defining

$$C_0 = 1/(\theta + \delta + r), \quad \text{(A.116)}$$

the general solution in this regime to takes the form

$$\psi_2(x) = C_0 \overline{\xi} + C_1 x^{\lambda_1} + C_2 x^{\lambda_2} \quad \text{(A.117)}$$

where $\lambda_1, \lambda_2$ are as before. The arbitrary constants $C_1, C_2$ are determined by a consideration of boundary conditions (see below).

**Analysis of boundary conditions:**
Let $\tilde{t}_1$ denote a time at which there is a transition from a regime of unconstrained prices and no investment to a regime where prices are constrained but there is still no investment. Let $\tilde{t}_2$ denote a time where there is a transition from price constrained non-investment to a regime in which there is price-constrained positive investment.

**Regime 1/2 boundary:**
At the time $\tilde{t}_1$, smooth pasting conditions must apply. Since the problem is ***not*** one in which there is a free choice for the level of the boundary (it is set by the regulator), the smooth pasting conditions involve equality only of value and the first derivative of the value functions (Dumas, 1991). Also, by definition, the price cap binds, so $x_{\tilde{t}_1} = \overline{\xi}$. Thus $v(\overline{\xi}) = v_2(\overline{\xi})$ and $v'(\overline{\xi}) = v_2'(\overline{\xi})$ must hold, and so

$$\psi(\overline{\xi}) = \psi_2(\overline{\xi}), \quad \text{(A.118)}$$

$$\psi'(\overline{\xi}) = \psi_2'(\overline{\xi}) \quad \text{(A.119)}$$

where

$$\psi(\overline{\xi}) = B_0 \overline{\xi} + B_1 \overline{\xi}^{\lambda_1} \quad \text{(A.120)}$$

$$\psi'(\overline{\xi}) = B_0 + \lambda_1 B_1 \overline{\xi}^{\lambda_1 - 1} \quad \text{(A.121)}$$

$$\psi_2(\overline{\xi}) = C_0 \overline{\xi} + C_1 \overline{\xi}^{\lambda_1} + C_2 \overline{\xi}^{\lambda_2} \quad \text{(A.122)}$$

$$\psi_2'(\overline{\xi}) = \lambda_1 C_1 \overline{\xi}^{\lambda_1 - 1} + \lambda_2 C_2 \overline{\xi}^{\lambda_2 - 1} \quad \text{(A.123)}$$

The analysis of these conditions is deferred until the other conditions have been identified.

**The positive Investment boundary:**
At the hitting time $\tilde{t}_2$, smooth pasting conditions again apply. These are identical with those for the unconstrained monopoly problem except that they are now evaluated at the relative price $\xi$. The first derivative condition is that,

$$\gamma\left[\psi_2(\xi) - 1\right] + \xi\psi_2'(\xi) = 0 \quad \text{(A.124)}$$

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[1] The value function can still be written in this form in the price constrained region, simply because the price constraint is also homogenous of degree 1 in *p,K*.

- compare with (A.55) – and the second derivative condition is that.

$$\xi^2\psi_2''(\xi)+2\gamma\xi\psi_2'(\xi)+\gamma(\gamma-1)\left[\psi_2(\xi)-1\right]=0 \quad \text{(A.125)}$$

as in (A.61). Here

$$\psi_2(\xi)=C_0\xi+C_1\xi^{\lambda_1}+C_2\xi^{\lambda_2} \quad \text{(A.126)}$$

$$\psi_2'(\xi)=\lambda_1C_1\xi^{\lambda_1-1}+\lambda_2C_2\xi^{\lambda_2-1} \quad \text{(A.127)}$$

$$\psi_2''(\xi)=\lambda_1(\lambda_1-1)C_1\xi^{\lambda_1-2}+\lambda_2(\lambda_2-1)C_2\xi^{\lambda_2-2} \quad \text{(A.128)}$$

**Analysis of smooth pasting conditions:**

From (A.118), using (A.120) and (A.122),

$$B_0\bar{\xi}+B_1\bar{\xi}^{\lambda_1}=C_0\bar{\xi}+C_1\bar{\xi}^{\lambda_1}+C_2\bar{\xi}^{\lambda_2} \quad \text{(A.129)}$$

Unknowns are $B_1, C_1, C_2$.

From (A.119), using (A.121) and (A.123), and multiplying by $\bar{\xi}$,

$$B_0\bar{\xi}+\lambda_1B_1\bar{\xi}^{\lambda_1}=\lambda_1C_1\bar{\xi}^{\lambda_1}+\lambda_2C_2\bar{\xi}^{\lambda_2} \quad \text{(A.130)}$$

Unknowns are $B_1, C_1, C_2$.

From (A.124), using (A.126) and (A.127),

$$\gamma\left[C_0\underline{\xi}+C_1\underline{\xi}^{\lambda_1}+C_2\underline{\xi}^{\lambda_2}-1\right]+\underline{\xi}\left[\lambda_1C_1\underline{\xi}^{\lambda_1-1}+\lambda_2C_2\underline{\xi}^{\lambda_2-1}\right]=0 \quad \text{(A.131)}$$

Unknowns are $C_1, C_2, \underline{\xi}$.

From (A.125), using (A.126), (A.127) and (A.128),

$$\begin{aligned}&\underline{\xi}^2\left[\lambda_1(\lambda_1-1)C_1\underline{\xi}^{\lambda_1-2}+\lambda_2(\lambda_2-1)C_2\underline{\xi}^{\lambda_2-2}\right]\\&+2\gamma\underline{\xi}\left[\lambda_1C_1\underline{\xi}^{\lambda_1-1}+\lambda_2C_2\underline{\xi}^{\lambda_2-1}\right]+\gamma(\gamma-1)\left[C_0\underline{\xi}+C_1\underline{\xi}^{\lambda_1}+C_2\underline{\xi}^{\lambda_2}-1\right]=0\end{aligned} \quad \text{(A.132)}$$

Unknowns are $C_1, C_2, \underline{\xi}$.

Total unknowns are $B_1, C_1, C_2, \underline{\xi}$. However the variable of interest is $\underline{\xi}$. To find this, first eliminate $B_1$. Equation (A.129) implies

$$B_1\bar{\xi}^{\lambda_1}=\left(C_0-B_0\right)\bar{\xi}+C_1\bar{\xi}^{\lambda_1}+C_2\bar{\xi}^{\lambda_2} \quad \text{(A.133)}$$

whilst from (A.130)

$$B_1\bar{\xi}^{\lambda_1}=\frac{\lambda_1C_1\bar{\xi}^{\lambda_1}+\lambda_2C_2\bar{\xi}^{\lambda_2}-B_0\bar{\xi}}{\lambda_1} \quad \text{(A.134)}$$

$$\Rightarrow$$

$$B_1\bar{\xi}^{\lambda_1}=\frac{\lambda_1C_1\bar{\xi}^{\lambda_1}+\lambda_2C_2\bar{\xi}^{\lambda_2}-B_0\bar{\xi}}{\lambda_1}=\left(C_0-B_0\right)\bar{\xi}+C_1\bar{\xi}^{\lambda_1}+C_2\bar{\xi}^{\lambda_2}$$

$$\Rightarrow \lambda_1C_1\bar{\xi}^{\lambda_1}+\lambda_2C_2\bar{\xi}^{\lambda_2}-B_0\bar{\xi}=\lambda_1\left[\left(C_0-B_0\right)\bar{\xi}+C_1\bar{\xi}^{\lambda_1}+C_2\bar{\xi}^{\lambda_2}\right]$$

$$\Rightarrow$$

$$\lambda_1C_1\bar{\xi}^{\lambda_1}+\lambda_2C_2\bar{\xi}^{\lambda_2}-B_0\bar{\xi}=\lambda_1\left(C_0-B_0\right)\bar{\xi}+\lambda_1C_1\bar{\xi}^{\lambda_1}+\lambda_1C_2\bar{\xi}^{\lambda_2}$$

$$\Rightarrow$$

$$C_2\bar{\xi}^{\lambda_2}=\frac{\left[\lambda_1\left(C_0-B_0\right)+B_0\right]\bar{\xi}}{\left(\lambda_2-\lambda_1\right)} \quad \text{(A.135)}$$

This gives $C_2$. It does not determine $B_1, C_1$ but links them because

$$\left(B_1 - C_1\right)\lambda_1 \overline{\xi}^{\lambda_1} = \lambda_2 C_2 \overline{\xi}^{\lambda_2} - B_0 \overline{\xi}$$

$$\Rightarrow$$

$$\left(B_1 - C_1\right)\lambda_1 \overline{\xi}^{\lambda_1} = \frac{\lambda_2\left[\lambda_1\left(C_0 - B_0\right) + B_0\right]\overline{\xi}}{\left(\lambda_2 - \lambda_1\right)} - B_0 \overline{\xi}$$

$$\Rightarrow$$

$$\left(B_1 - C_1\right)\lambda_1 \overline{\xi}^{\lambda_1} = \frac{\lambda_2\left[\lambda_1\left(C_0 - B_0\right) + B_0\right]\overline{\xi} - \left(\lambda_2 - \lambda_1\right)B_0 \overline{\xi}}{\left(\lambda_2 - \lambda_1\right)}$$

$$\Rightarrow$$

$$\left(B_1 - C_1\right)\lambda_1 \overline{\xi}^{\lambda_1} = \frac{\left[\lambda_2\lambda_1\left(C_0 - B_0\right) + \lambda_1 B_0\right]\overline{\xi}}{\left(\lambda_2 - \lambda_1\right)} \tag{A.136}$$

That is, once $C_1$ is established, so too is $B_1$. Now use the positive investment boundary conditions: from (A.124)-(A.128),

$$\gamma\left[C_0\overline{\xi} + C_1\xi^{\lambda_1} + C_2\xi^{\lambda_2} - 1\right] + \xi\left[\lambda_1 C_1 \xi^{\lambda_1 - 1} + \lambda_2 C_2 \xi^{\lambda_2 - 1}\right] = 0 \tag{A.137}$$

and

$$\begin{aligned} &\lambda_1(\lambda_1 - 1)C_1\xi^{\lambda_1} + \lambda_2(\lambda_2 - 1)C_2\xi^{\lambda_2} \\ &+2\gamma\left[\lambda_1 C_1\xi^{\lambda_1} + \lambda_2 C_2 \xi^{\lambda_2}\right] + \gamma\left(\gamma - 1\right)\left[C_0\overline{\xi} + C_1\xi^{\lambda_1} + C_2\xi^{\lambda_2} - 1\right] = 0 \end{aligned} \tag{A.138}$$

First find $C_1$ from (A.137):

$$C_1\xi^{\lambda_1} = \frac{\gamma(1 - C_0\overline{\xi}) - \left(\gamma + \lambda_2\right)C_2\xi^{\lambda_2}}{\left(\gamma + \lambda_1\right)} \tag{A.139}$$

Then use this in (A.138):

$$\begin{aligned} &\lambda_1(\lambda_1 - 1)C_1\xi^{\lambda_1} + \lambda_2(\lambda_2 - 1)C_2\xi^{\lambda_2} \\ &+2\gamma\lambda_1 C_1\xi^{\lambda_1} + 2\gamma\lambda_2 C_2\xi^{\lambda_2} \\ &+\gamma\left(\gamma - 1\right)C_0\overline{\xi} + \gamma\left(\gamma - 1\right)C_1\xi^{\lambda_1} + \gamma\left(\gamma - 1\right)C_2\xi^{\lambda_2} - \gamma\left(\gamma - 1\right) = 0 \end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned} &\left[\lambda_1(\lambda_1 - 1) + 2\gamma\lambda_1 + \gamma\left(\gamma - 1\right)\right]C_1\xi^{\lambda_1} \\ &+\left[\lambda_2(\lambda_2 - 1) + 2\gamma\lambda_2 + \gamma\left(\gamma - 1\right)\right]C_2\xi^{\lambda_2} \\ &+\gamma\left(\gamma - 1\right)\left[C_0\overline{\xi} - 1\right] = 0 \end{aligned} \tag{A.140}$$

Substituting for $C_1\xi^{\lambda_1}$ using (A.139),

$$\begin{aligned} &\left[\lambda_1(\lambda_1 - 1) + 2\gamma\lambda_1 + \gamma\left(\gamma - 1\right)\right]\left\{\frac{\gamma(1 - C_0\overline{\xi}) - \left(\gamma + \lambda_2\right)C_2\xi^{\lambda_2}}{\left(\gamma + \lambda_1\right)}\right\} \\ &+\left[\lambda_2(\lambda_2 - 1) + 2\gamma\lambda_2 + \gamma\left(\gamma - 1\right)\right]C_2\xi^{\lambda_2} \\ &+\gamma\left(\gamma - 1\right)\left[C_0\overline{\xi} - 1\right] = 0 \end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned} &\left[\lambda_1(\lambda_1 - 1) + 2\gamma\lambda_1 + \gamma\left(\gamma - 1\right)\right]\left\{\gamma(1 - C_0\overline{\xi}) - \left(\gamma + \lambda_2\right)C_2\xi^{\lambda_2}\right\} \\ &+\left(\gamma + \lambda_1\right)\left[\lambda_2(\lambda_2 - 1) + 2\gamma\lambda_2 + \gamma\left(\gamma - 1\right)\right]C_2\xi^{\lambda_2} \\ &+\gamma\left(\gamma - 1\right)\left(\gamma + \lambda_1\right)\left[C_0\overline{\xi} - 1\right] = 0 \end{aligned}$$

$\Rightarrow$

$$\gamma(1-C_0\bar{\xi})\left[\lambda_1(\lambda_1-1)+2\gamma\lambda_1+\gamma(\gamma-1)\right]-(\gamma+\lambda_2)C_2\xi^{\lambda_2}\left[\lambda_1(\lambda_1-1)+2\gamma\lambda_1+\gamma(\gamma-1)\right]$$
$$+(\gamma+\lambda_1)\left[\lambda_2(\lambda_2-1)+2\gamma\lambda_2+\gamma(\gamma-1)\right]C_2\xi^{\lambda_2}$$
$$+\gamma(\gamma-1)(\gamma+\lambda_1)\left[C_0\bar{\xi}-1\right]=0$$

$\Rightarrow$

$$\begin{Bmatrix}(\gamma+\lambda_1)\left[\lambda_2(\lambda_2-1)+2\gamma\lambda_2+\gamma(\gamma-1)\right]\\ -(\gamma+\lambda_2)\left[\lambda_1(\lambda_1-1)+2\gamma\lambda_1+\gamma(\gamma-1)\right]\end{Bmatrix}C_2\xi^{\lambda_2}$$
$$=\gamma(\gamma-1)(\gamma+\lambda_1)\left[1-C_0\bar{\xi}\right]-\gamma(1-C_0\bar{\xi})\left[\lambda_1(\lambda_1-1)+2\gamma\lambda_1+\gamma(\gamma-1)\right]$$

$\Rightarrow$

$$\xi^{\lambda_2}=\frac{\gamma\left(1-C_0\bar{\xi}\right)\left\{(\gamma-1)(\gamma+\lambda_1)-\lambda_1(\lambda_1-1)-2\gamma\lambda_1-\gamma(\gamma-1)\right\}}{C_2\begin{Bmatrix}(\gamma+\lambda_1)\left[\lambda_2(\lambda_2-1)+2\gamma\lambda_2+\gamma(\gamma-1)\right]\\ -(\gamma+\lambda_2)\left[\lambda_1(\lambda_1-1)+2\gamma\lambda_1+\gamma(\gamma-1)\right]\end{Bmatrix}} \tag{A.141}$$

Now, from (A.135),

$$C_2=\frac{\left[\lambda_1(C_0-B_0)+B_0\right]\bar{\xi}^{1-\lambda_2}}{(\lambda_2-\lambda_1)} \tag{A.142}$$

so

$$\xi^{\lambda_2}=\frac{\gamma\left(1-C_0\bar{\xi}\right)(\lambda_2-\lambda_1)\left\{(\gamma-1)(\gamma+\lambda_1)-\lambda_1(\lambda_1-1)-2\gamma\lambda_1-\gamma(\gamma-1)\right\}}{\bar{\xi}^{1-\lambda_2}\left[\lambda_1(C_0-B_0)+B_0\right]\begin{Bmatrix}(\gamma+\lambda_1)\left[\lambda_2(\lambda_2-1)+2\gamma\lambda_2+\gamma(\gamma-1)\right]\\ -(\gamma+\lambda_2)\left[\lambda_1(\lambda_1-1)+2\gamma\lambda_1+\gamma(\gamma-1)\right]\end{Bmatrix}} \tag{A.143}$$

where from (A.116),

$$C_0=1/(\theta+\delta+r) \tag{A.144}$$

and, from (A.37),

$$B_0=1/\left(r+\theta-\mu_p\right) \tag{A.145}$$

Simplifying (A.143),

$$\xi^{\lambda_2}=\frac{-\gamma\left(1-C_0\bar{\xi}\right)(\lambda_2-\lambda_1)\lambda_1(\lambda_1+\gamma)}{\bar{\xi}^{1-\lambda_2}\left[C_0\lambda_1+B_0(1-\lambda_1)\right]\begin{Bmatrix}(\gamma+\lambda_1)\left[\lambda_2(\lambda_2-1)+2\gamma\lambda_2+\gamma(\gamma-1)\right]\\ -(\gamma+\lambda_2)\left[\lambda_1(\lambda_1-1)+2\gamma\lambda_1+\gamma(\gamma-1)\right]\end{Bmatrix}}$$

$\Rightarrow$

$$\xi^{\lambda_2}=\frac{-\gamma\left(1-C_0\bar{\xi}\right)(\lambda_2-\lambda_1)\lambda_1(\lambda_1+\gamma)}{\bar{\xi}^{1-\lambda_2}\left[C_0\lambda_1+B_0(1-\lambda_1)\right]\begin{Bmatrix}(\gamma+\lambda_1)\lambda_2(\lambda_2-1)-(\gamma+\lambda_2)\lambda_1(\lambda_1-1)\\ +2\gamma\lambda_2(\gamma+\lambda_1)+\gamma(\gamma-1)(\lambda_1-\lambda_2)\\ -(\gamma+\lambda_2)2\gamma\lambda_1\end{Bmatrix}}$$

$\Rightarrow$

$$\xi^{\lambda_2} = \frac{-\gamma\left(1-C_0\bar{\xi}\right)\left(\lambda_2-\lambda_1\right)\lambda_1(\lambda_1+\gamma)}{\bar{\xi}^{1-\lambda_2}\left[C_0\lambda_1+B_0\left(1-\lambda_1\right)\right]\left\{\begin{array}{l}(\gamma\lambda_2{}^2-\gamma\lambda_2+\lambda_1\lambda_2{}^2-\lambda_1\lambda_2)\\ -(\gamma\lambda_1{}^2-\gamma\lambda_1+\lambda_2\lambda_1{}^2-\lambda_1\lambda_2)\\ +\left(2\gamma^2\lambda_2+2\gamma\lambda_1\lambda_2\right)+\gamma\left(\gamma-1\right)\left(\lambda_1-\lambda_2\right)\\ -\left(2\gamma\lambda_1\gamma+2\gamma\lambda_1\lambda_2\right)\end{array}\right\}}$$

$\Rightarrow$

$$\xi^{\lambda_2} = \frac{-\gamma\left(1-C_0\bar{\xi}\right)\left(\lambda_2-\lambda_1\right)\lambda_1(\lambda_1+\gamma)}{\bar{\xi}^{1-\lambda_2}\left[C_0\lambda_1+B_0\left(1-\lambda_1\right)\right]\left\{\begin{array}{l}\gamma\lambda_2{}^2-\gamma\lambda_2+\lambda_1\lambda_2{}^2-\lambda_1\lambda_2\\ -\gamma\lambda_1{}^2+\gamma\lambda_1-\lambda_2\lambda_1{}^2+\lambda_1\lambda_2\\ +2\gamma^2\lambda_2+2\gamma\lambda_1\lambda_2+\gamma^2\lambda_1-\gamma^2\lambda_2-\gamma\lambda_1+\gamma\lambda_2\\ -2\gamma\lambda_1\gamma-2\gamma\lambda_1\lambda_2\end{array}\right\}}$$

$\Rightarrow$

$$\xi^{\lambda_2} = \frac{-\gamma\left(1-C_0\bar{\xi}\right)\left(\lambda_2-\lambda_1\right)\lambda_1(\lambda_1+\gamma)}{\bar{\xi}^{1-\lambda_2}\left[C_0\lambda_1+B_0\left(1-\lambda_1\right)\right]\left\{\begin{array}{l}\gamma\lambda_2{}^2-\gamma\lambda_1{}^2+\lambda_1\lambda_2{}^2-\lambda_2\lambda_1{}^2\\ +\gamma^2\lambda_2-\gamma^2\lambda_1\end{array}\right\}}$$

$\Rightarrow$

$$\xi^{\lambda_2} = \frac{-\gamma\left(1-C_0\bar{\xi}\right)\left(\lambda_2-\lambda_1\right)\lambda_1(\lambda_1+\gamma)}{\bar{\xi}^{1-\lambda_2}\left(\lambda_2-\lambda_1\right)\left[C_0\lambda_1+B_0\left(1-\lambda_1\right)\right]\left\{\left(\gamma^2+\lambda_1\lambda_2+\gamma\left(\lambda_2+\lambda_1\right)\right)\right\}}$$

$\Rightarrow$

$$\xi^{\lambda_2} = \frac{-\gamma\lambda_1(\lambda_1+\gamma)\left(1-C_0\bar{\xi}\right)}{\bar{\xi}^{1-\lambda_2}\left[C_0\lambda_1+B_0\left(1-\lambda_1\right)\right]\left\{\left(\gamma^2+\lambda_1\lambda_2+\gamma\left(\lambda_2+\lambda_1\right)\right)\right\}} \tag{A.146}$$

The aim now is to simplify this to obtain the formula in result 3.
Denote

$$\xi_c = \left(\delta+\theta+r\right) \tag{A.147}$$

as the certainty competitive entry trigger relative price,

$$\xi_u = \left(\tfrac{\lambda_2-1}{\lambda_2}\right)\left(\delta+\theta+r\right) = \left(\tfrac{\lambda_1}{\lambda_1-1}\right)\left(\theta+r-\mu_p\right) = \left(\tfrac{\lambda_2-1}{\lambda_2}\right)\xi_c \tag{A.148}$$

as the uncertainty competitive entry trigger relative price and

$$\xi_M = \left(\tfrac{\gamma}{1+\gamma}\right)\xi_u \tag{A.149}$$

as the uncertainty unconstrained monopoly entry trigger relative price, then the aim is to show that the price cap monopoly relative entry market clearing price can be written as in result 3; that is, as

$$\xi = \left(\left(\frac{\bar{\xi}-\xi_c}{\xi_M-\xi_c}\right)\xi_M\bar{\xi}^{\lambda_2-1}\right)^{1/\lambda_2} . \tag{A.150}$$

So, returning to (A.146), and dividing by $C_0$ gives

$$\xi^{\lambda_2} = \frac{-\gamma\lambda_1(\lambda_1+\gamma)\left((1/C_0)-\bar{\xi}\right)}{\bar{\xi}^{1-\lambda_2}\left[\lambda_1+\left(B_0/C_0\right)\left(1-\lambda_1\right)\right]\left\{\left(\gamma^2+\lambda_1\lambda_2+\gamma\left(\lambda_2+\lambda_1\right)\right)\right\}} . \tag{A.151}$$

Dividing by $(\lambda_1-1)$ and substituting for $B_0, C_0$ then gives

$$\xi^{\lambda_2} = \frac{\gamma \frac{\lambda_1}{(1-\lambda_1)}(\lambda_1+\gamma)\left(\xi_u - \overline{\xi}\right)\overline{\xi}^{\lambda_2-1}}{\left[\frac{\lambda_1}{(\lambda_1-1)} - \left(\frac{(\theta+\delta+r)}{\left(r+\theta-\mu_p\right)}\right)\right]\left\{\left(\gamma^2+\lambda_1\lambda_2+\gamma\left(\lambda_2+\lambda_1\right)\right)\right\}} \tag{A.152}$$

$\Rightarrow$

$$\xi^{\lambda_2} = \frac{\gamma \frac{\lambda_1}{(1-\lambda_1)}\left(r+\theta-\mu_p\right)(\lambda_1+\gamma)\left(\xi_c - \overline{\xi}\right)\overline{\xi}^{\lambda_2-1}}{\left[\frac{\lambda_1}{(\lambda_1-1)}\left(r+\theta-\mu_p\right) - (\theta+\delta+r)\right]\left\{\left(\gamma^2+\lambda_1\lambda_2+\gamma\left(\lambda_2+\lambda_1\right)\right)\right\}} \tag{A.153}$$

Using (A.147) and (A.148), this gives

$$\xi^{\lambda_2} = \frac{\left(\overline{\xi}-\xi_c\right)\xi_u\overline{\xi}^{\lambda_2-1}}{\left(\xi_u-\xi_c\right)\left(\frac{\gamma^2+\lambda_1\lambda_2+\gamma\left(\lambda_2+\lambda_1\right)}{\gamma(\lambda_1+\gamma)}\right)} \tag{A.154}$$

$\Rightarrow$

$$\xi^{\lambda_2} = \frac{\left(\overline{\xi}-\xi_c\right)\xi_u\overline{\xi}^{\lambda_2-1}}{\left(\xi_u-\xi_c\right)\left(\frac{\gamma\left(\lambda_1+\gamma\right)+\lambda_2(\lambda_1+\gamma)}{\gamma(\lambda_1+\gamma)}\right)} \tag{A.155}$$

so

$$\xi^{\lambda_2} = \frac{\left(\overline{\xi}-\xi_c\right)\xi_u\overline{\xi}^{\lambda_2-1}}{\left(\xi_u-\xi_c\right)\left(1+\eta\lambda_2\right)}$$

(where $\eta = 1/\gamma$). Writing $(1+\eta)\xi_M = \xi_u$, then

$$\xi^{\lambda_2} = \frac{\left(\overline{\xi}-\xi_c\right)(1+\eta)\xi_M\overline{\xi}^{\lambda_2-1}}{\left((1+\eta)\xi_M-\xi_c\right)\left(1+\eta\lambda_2\right)} = \frac{\left(\overline{\xi}-\xi_c\right)\xi_M\overline{\xi}^{\lambda_2-1}}{\left(\xi_M-\frac{\xi_c}{1+\eta}\right)\left(1+\eta\lambda_2\right)}$$

$$= \frac{\left(\overline{\xi}-\xi_c\right)\xi_M\overline{\xi}^{\lambda_2-1}}{\left(\xi_M-\frac{\xi_c}{1+\eta}+\frac{\eta\lambda_2\xi_u}{1+\eta}-\frac{\xi_c\eta\lambda_2}{1+\eta}\right)} = \frac{\left(\overline{\xi}-\xi_c\right)\xi_M\overline{\xi}^{\lambda_2-1}}{\left(\xi_M-\frac{\xi_c}{1+\eta}+\frac{\eta\lambda_2\left(\frac{\lambda_2-1}{\lambda_2}\right)\xi_c}{1+\eta}-\frac{\xi_c\eta\lambda_2}{1+\eta}\right)}$$

$$= \frac{\left(\overline{\xi}-\xi_c\right)\xi_M\overline{\xi}^{\lambda_2-1}}{\left(\xi_M-\frac{\xi_c}{1+\eta}+\frac{(\lambda_2-1)\eta\xi_c}{1+\eta}-\frac{\xi_c\eta\lambda_2}{1+\eta}\right)} = \frac{\left(\overline{\xi}-\xi_c\right)\xi_M\overline{\xi}^{\lambda_2-1}}{\left(\xi_M-\xi_c\right)}$$

Hence

$$\xi = \left\{\frac{\left(\overline{\xi}-\xi_c\right)\xi_M\overline{\xi}^{\lambda_2-1}}{\left(\xi_M-\xi_c\right)}\right\}^{1/\lambda_2} \tag{A.156}$$

which is Result 3.

**A7 Proof for Result 4.**

From the formula for $\xi(\bar{\xi})$ in Result 3 was

$$\xi = \left\{ \left(\bar{\xi} - \xi_c\right) \xi_M \bar{\xi}^{\lambda_2 - 1} \Big/ \left(\xi_M - \xi_c\right) \right\}^{1/\lambda_2} \tag{A.157}$$

Differentiating with respect to $\bar{\xi}$ gives

$$\frac{d\xi}{d\bar{\xi}} = (1/\lambda_2)\left\{ \left( \frac{\bar{\xi} - \xi_c}{\xi_M - \xi_c} \right) \xi_M \bar{\xi}^{\lambda_2 - 1} \right\}^{(1/\lambda_2)-1} \left( \frac{\xi_M}{\xi_M - \xi_c} \right) \frac{d}{d\bar{\xi}} \left( \left(\bar{\xi} - \xi_c\right) \bar{\xi}^{\lambda_2 - 1} \right) \tag{A.158}$$

where, using the definition for $\xi_u$,

$$\frac{d}{d\bar{\xi}} \left( \left(\bar{\xi} - \xi_c\right) \bar{\xi}^{\lambda_2 - 1} \right) = \frac{d}{d\bar{\xi}} \left( \bar{\xi}^{\lambda_2} - \xi_c \bar{\xi}^{\lambda_2 - 1} \right) = \left( \lambda_2 \bar{\xi}^{\lambda_2 - 1} - (\lambda_2 - 1) \xi_c \bar{\xi}^{\lambda_2 - 2} \right)$$

$$= \lambda_2 \bar{\xi}^{\lambda_2 - 2} \left( \bar{\xi} - \xi_u \right) \tag{A.159}$$

so

$$\frac{d\xi}{d\bar{\xi}} = \underbrace{\left\{ \left( \frac{\bar{\xi} - \xi_c}{\xi_M - \xi_c} \right) \xi_M \bar{\xi}^{\lambda_2 - 1} \right\}^{(1/\lambda_2)-1}}_{(+)} \underbrace{\left( \frac{\xi_M}{\xi_M - \xi_c} \right)}_{(+)} \underbrace{\bar{\xi}^{\lambda_2 - 2}}_{(+)} \left( \bar{\xi} - \xi_u \right) \tag{A.160}$$

hence

$$\frac{d\xi}{d\bar{\xi}} \gtreqless 0 \quad \text{as } \bar{\xi} \gtreqless \xi_u . \tag{A.161}$$

This completes the proof for result 4(i). As $\bar{\xi} \downarrow \xi_c$ in equation (A.157), the term in brackets $\{ \ \} \to 0$; since $1/\lambda_2 < 0$, it follows that $\xi \to +\infty$, which is result 4(ii). Letting $\bar{\xi} \to \xi_M$ in (A.157), clearly $\xi(\bar{\xi}) \to \xi_M$, which is result 4(iii). Setting $\bar{\xi} = \xi_u$, from (A.157),

$$\xi^{\lambda_2} = \left\{ \left(\xi_u - \xi_c\right) \xi_M \xi_u^{\lambda_2 - 1} \Big/ \left(\xi_M - \xi_c\right) \right\} \tag{A.162}$$

Now, $\xi(\xi_u) \gtreqless \xi_u$ as $\xi(\xi_u)^{\lambda_2} \lesseqgtr \xi_u^{\lambda_2}$ (since $\lambda_2 < 0$). Using (A.162) this implies

$\xi(\xi_u) \gtreqless \xi_u$ as $\left(\xi_u - \xi_c\right) \xi_M \xi_u^{\lambda_2 - 1} \Big/ \left(\xi_M - \xi_c\right) \lesseqgtr \xi_u^{\lambda_2}$

$$\Rightarrow \left(\xi_u - \xi_c\right) \xi_M \lesseqgtr \xi_u \left(\xi_M - \xi_c\right) \Rightarrow \xi_M - \xi_u \gtreqless 0$$

In fact $\xi_M - \xi_u > 0$ and hence $\xi(\xi_u) > \xi_u$, which is Result 4 (iv).

To establish Result 4 (v), first substitute in (A.162) using result 2, for $\xi_u = (\lambda_2 - 1)\xi_u / \lambda_2$ and $\xi_M = (\lambda_2 - 1)\xi_u / \lambda_2 (1+\eta)$ to get $\xi(\xi_u) = \left(1 + \eta\lambda_2\right)^{-1/\lambda_2} \xi_u$. From the definition for $\lambda_2$, note that $Lim_{\sigma\to 0}\lambda_2 = -\infty$, and so $Lim_{\sigma \to 0}\left(\frac{\lambda_2 - 1}{\lambda_2}\right) = 1$. Hence from result 3, $Lim_{\sigma\to 0}\ \xi_u = \xi_c$. Also $Lim_{\sigma\to 0}\left(1+\eta\lambda_2\right)^{-1/\lambda_2} = Lim_{\lambda_2 \to -\infty}\left(1+\eta\lambda_2\right)^{-1/\lambda_2} = 1$.

Hence $$Lim_{\sigma\to 0}\,\xi\left(\xi_u\right) = Lim_{\sigma\to 0}\left(1+\eta\lambda_2\right)^{-1/\lambda_2} Lim_{\sigma\to 0}\ \xi_u = \xi_c \tag{A.163}$$

which is Result 4 (v).