

**REPLACEMENT INVESTMENT:
OPTIMAL ECONOMIC LIFE UNDER UNCERTAINTY**

by

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ABSTRACT

Replacement investment is essentially a regenerative optimal stopping problem; that is, the key decision concerns when to terminate the life of existing plant – and hence when to start over again. This paper examines this optimisation problem within a continuous time framework and studies the qualitative and quantitative impact of uncertainty on the timing of new investment (and the criteria that should be used for terminating the life of existing plant).

1. INTRODUCTION

A large part of all investment activity is in fact replacement investment; indeed in a non-growing, stationary state, economy, effectively all investment is replacement investment. The key decision concerning such replacement investment is that of timing – of deciding how long to keep existing plant and machinery. As an optimisation problem, the decision of when to renew capacity naturally depends on expectations regarding how existing and new plant operating costs change over time, and of course, on expectations regarding future capital costs and resale/salvage values. A standard approach is to envisage a replacement chain for investment in which the economic life of the first plant is determined by a trade off between expected benefits and costs associated with extending the economic life of existing plant. Keeping plant for an additional period involves incurring additional operating costs and suffering possible loss of second hand or salvage value, but reduces the present value costs associated with the chain of investments that subsequently follow. The analysis of replacement investment in this framework has been around for a considerable time and is well understood (for early examples, see e.g. Hotelling [1925], Terborgh [1949], McDowell [1960], Smith [1961], Merrett and Sykes [1965]).

Replacement investment under uncertainty has also been examined, notably in the Operational Research literature (for reviews, see Pierskalla and Voelker [1976], Sherif and Smith [1981]). This literature primarily focuses on uncertainty over the physical life of individual components, with the problem being viewed as one of determining a replacement policy, where such a policy will typically involve

replacing at least some units before the end of their physical life.¹ Optimal economic life may also depend on second hand (or salvage) values, and so a natural extension to the study of investment policy is to look at replacement investment within the context of equilibrium in the associated second hand markets (Rust [1985]). More recent work has tended to focus on other aspects of the replacement problem, notably how asymmetric information affects equilibrium values in second hand markets (following the seminal work of Akerlof [1960], see Kim [1985], Genesove [1995]).

The above literature has generally modelled the replacement investment decision within a discrete time framework; by contrast, the present paper begins by modelling the optimisation problem in a continuous time framework (and focusing on the ‘option value’ characteristics of the solution). Following this, the paper examines in some detail the qualitative and quantitative impact of uncertainty (and other parameters) on the replacement decision and on the average economic life of plant. Previous work does not seem to have examined this type of comparative statics analysis of the investment decision; broadly speaking, the OR literature has tended to feature rather case specific models, with the focus on establishing the existence of a ‘solution’, whilst the economics literature has tended to focus on the issue of characterising ‘equilibria’ in second hand markets without pursuing the further question of how such equilibria are perturbed by variation in the underlying parameter values. The equations that characterise the solution to the replacement investment decision are used in this paper as the basis for undertaking sensitivity analysis and the study of alternative ‘scenarios’.

¹ Rust [1987] is perhaps one of the best examples of this type of analysis – that paper examined the optimal replacement policy for replacing/rebuilding Bus engines, given the probability that such engines break down is a function of mileage.

In the context of the replacement chain investment, uncertainty creates option value. If the objective is to minimise expected present value costs, the decision to terminate the life of existing plant should take account of the possibility that operating and maintenance costs can go down as well as up. Dixit [1989] has shown that competitive firms, when faced with uncertainty over market price, do not exit the industry immediately price falls below average variable cost; price has to fall somewhat further; the same type of effect can be expected in the present context, namely that uncertainty can be expected to extend the economic life of plant (and within the context of the present model, it is possible to prove this result).

The present paper assumes there is a fixed and time invariant salvage value. The assumption regarding second-hand markets is either that (a) they do not exist, such that the decision is always one of replacing the old plant with new, or (b) that there is no information asymmetry and no transactions costs, such that the market price of second hand equipment represents a fair and competitive value. This latter assumption is often reasonable in quite a range of applications; asymmetric information may be of importance in used car markets, but such severe forms of asymmetric information are far from endemic.

If there is no second-hand market, clearly the firm must take the decision on when to replace existing with new plant. By contrast, with competitive second-hand markets, whilst the economic life of the plant remains the same as in the case where there is no second hand market, the decision of the individual firm (as to what age of plant to buy, and of when to sell) becomes a matter of indifference (it makes no difference

how old the plant bought is, or indeed when it is sold, so long as it is not kept beyond the time at which it would have been better to scrap it). In this latter case, it is also of interest to study the time profile for second-hand values.

The structure of the paper is as follows: Section 2 first establishes the basic model. Following this, section 3 focuses on economic life and second hand values whilst section 4 discusses alternative measures of economic life. Section 5 establishes some comparative statics results and conducts a (numerical) sensitivity analysis; section 6 then offers some concluding comments on the relevance of the work and direction for further research.

2. REPLACEMENT CHAINS UNDER UNCERTAINTY

In what follows, the solution for the deterministic case is outlined as a preliminary benchmark, prior to extending the analysis to incorporate uncertainty regarding operating costs and resale values. As explained in section 1, whether or not there are competitive second-hand markets, it suffices to analyse the economic life of plant *per se*. In the case where there are (competitive) second hand markets, equilibrium second hand values for plant can then be computed.

The Deterministic Case

The initial capital outlay is denoted K , operating/maintenance costs at time t are c_t and are assumed to increase at a constant growth rate θ whilst salvage value S is assumed constant. Let V denote the present value of all costs associated with a replacement chain, and assume that the first plant is terminated at some time T . The big assumption in replacement chain analysis is that the initial outlay and operating

cost profiles for the second and subsequent plant do not change. Under this ‘stationarity’ assumption, it follows that T is the optimal economic life for each and every plant in the replacement chain. Thus, V also represents the present value at time T of all future costs from T onward. Hence

$$V = K + \left(\int_0^T c_t e^{-rt} dt \right) + (V - S) e^{-rT}, \quad (1)$$

where r denotes the risk free discount rate (assumed constant over time). The first term on the right hand side is the initial capital outlay, the second term represents the present value of operating costs for the first plant whilst the third term represents the present value of selling the old plant for its salvage value S and then starting the chain anew (with present value cost V at that time). Rearranging (1), the present value of costs can be represented as the function

$$V(T, r, \theta, c_0, K, S) = \frac{1}{1 - e^{-rT}} \left\{ \frac{c_0}{\theta - r} (e^{(\theta-r)T} - 1) + K - S e^{-rT} \right\}. \quad (2)$$

Optimal economic life, T_{cert} is thus given as

$$T_{cert} = \text{Argmin}_{T, T > 0} V(T, r, \theta, c_0, K, S), \quad (3)$$

whilst the associated level of operating cost, denoted \bar{c}_{cert} , at which replacement investment is triggered is

$$\bar{c}_{cert} = c_0 e^{\theta T_{cert}} \quad (4)$$

(this proves of interest when making comparisons with the uncertainty case). The function (2) is non-linear but the optimisation can easily be conducted using numerical methods.² The *quantitative* sensitivity of the above certainty solution to

² It is possible to compute a first order condition associated with (2), but this in turn does not admit an explicit solution for T_{cert} ; the only gain from computing such a first order condition is that it is then possible to conduct a comparative statics exercise. This is in fact a special case of the more general comparative statics exercise presented for the uncertainty case in section 4 below.

changes in parameter values is examined at the same time as that for the uncertainty case in section 4 below.

Optimal Economic Life under Uncertainty

Terborgh [1949] was one of the first to model operating costs (under certainty) as increasing by a constant percentage increment; this type of assumption is both empirically plausible and also analytically fairly tractable. The natural extension is to assume that uncertainty affects the level of operating costs, c_t through a geometric Brownian motion (GBM) of the form³

$$dc_t / c_t = \theta dt + \sigma d\bar{\omega}_t . \quad (5)$$

Here θ (>0) is the trend rate of growth in operating cost and σ (>0) denotes its associated volatility. When $\sigma=0$, of course, this is the original ‘Terborghian’ operating cost process. The expected present value of operating costs at time τ , if the operating cost at this time is c_τ , for the case where the plant is never replaced is given as

$$PV_\tau(c_\tau) = E_\tau \left\{ \int_\tau^\infty c_t e^{-r(t-\tau)} dt \right\}, \quad (6)$$

where $E_\tau(\cdot)$ denotes the expectations operator for expectation formed at time τ .

When operating costs rise sufficiently, it becomes economic for plant to be replaced.

Let \bar{c} denote the level of cost at which replacement is triggered. Given the structure of the problem and the assumptions regarding the cost process, \bar{c} is a fixed deterministic value. The optimisation problem is simply that of finding the value \bar{c}

³ GBM is a common assumption in the literature. For a discussion of its pros and cons, see McDonald and Siegel [1986] and Dixit and Pindyck [1994].

which minimises expected present value costs for the replacement chain (clearly, \bar{c} will be a deterministic function of the parameters $K, S, c_0, \theta, r, \sigma^2$). By contrast, since the evolution of cost over time is governed by an Itô process, the time at which replacement is triggered is a random variable, $\tilde{t}(\bar{c})$ (to denote its dependence on \bar{c}). The present value of future costs, for the existing plant, thus depends on the current operating cost level, such that, at time τ during the life of the first plant, the expected present value of future operating costs for the chain can be written as

$$V(c_\tau) = E_\tau \left(\int_\tau^{\tilde{t}(\bar{c})} c_t e^{-r(t-\tau)} dt + e^{-r(\tilde{t}(\bar{c})-\tau)} W(\bar{c}) \right), \quad (7)$$

where $W(\bar{c})$ represents the present value at time $\tilde{t}(\bar{c})$ of all costs from that time onward; that is

$$W(\bar{c}) = V(c_0) + K - S. \quad (8)$$

In what follows, for notational compactness, subscripts and arguments are dropped wherever this does not affect intelligibility. The arbitrage condition for this problem is that⁴

$$rVdt = cdt + E(dV). \quad (9)$$

The term $E(dV)$ is evaluated in the appendix. It can be written as⁵

$$E(dV) = V'\theta cdt + \frac{1}{2}V''\sigma^2 c^2 dt, \quad (10)$$

and so (9) simplifies to give (cancelling through by dt)

$$\frac{1}{2}\sigma^2 c^2 V'' + \theta c V' - rV + c = 0, \quad (11)$$

a second order differential equation which governs the evolution of value. The general solution to this equation involves finding a particular solution to it along with

⁴ See e.g. Dixit and Pindyck [1994] for a clear exposition of stochastic dynamic programming optimality conditions.

⁵ Using the notation $V' \equiv dV/dc, V'' \equiv d^2V/dc^2$.

a general solution to the associated homogenous equation. A particular solution, (assuming $r \neq \theta$) is $V = c/(r - \theta)$ as can be easily verified.⁶ The general solution to the homogenous equation can be written as $V(c) = A_1 c^{\lambda_1} + A_2 c^{\lambda_2}$ (see appendix) and so the general solution to (11) can be written as

$$V(c) = \left(c / (r - \theta) \right) + A_1 c^{\lambda_1} + A_2 c^{\lambda_2}, \quad (12)$$

where the roots are defined as

$$\lambda_1 = (-R_1 + R_2) / \sigma^2, \quad (13)$$

$$\lambda_2 = (-R_1 - R_2) / \sigma^2, \quad (14)$$

and where

$$R_1 \equiv \left(\theta - \frac{1}{2} \sigma^2 \right), \quad (15)$$

$$R_2 \equiv \left(R_1^2 + 2\sigma^2 r \right)^{1/2}. \quad (16)$$

Notice that $2\sigma^2 r > 0$ if $\sigma^2 > 0$, so $\lambda_2 < 0 < \lambda_1$; the roots are real and of opposite sign when uncertainty is present. The two arbitrary constants A_1, A_2 are determined by boundary conditions. As $c \rightarrow 0$, since $\lambda_2 < 0$ and since value must be finite, this implies $A_2 = 0$ (Dixit [1993] discusses this sort of boundary condition in more detail). Thus (12) simplifies to

$$V(c) = \left(c / (r - \theta) \right) + A_1 c^{\lambda_1}, \quad (17)$$

where A_1 is determined by an analysis of smooth pasting conditions at the boundary where replacement investment takes place; these smooth pasting conditions require

⁶ To see this, note that if $V = c/(r - \theta)$, then $V' = 1/(r - \theta)$ and $V'' = 0$; substitute these into (11).

equality of value and equality of the first derivatives (with respect to c_t) of the value functions from regimes 1 and 2 (Dumas [1991]). Thus, at a hitting time \tilde{t} , this entails

$$V(\bar{c}) = W(\bar{c}), \quad (18)$$

and

$$\partial V(\bar{c}) / \partial c = \partial W(\bar{c}) / \partial c. \quad (19)$$

From (17), condition (18) becomes

$$V(\bar{c}) = (\bar{c} / (r - \theta)) + A_1 \bar{c}^{\lambda_1} = V(c_0) + K = (c_0 / (r - \theta)) + A_1 c_0^{\lambda_1} + K - S, \quad (20)$$

whilst $\partial V(c) / \partial c = (1 / (r - \theta)) + A_1 \lambda_1 c^{\lambda_1 - 1}$, $\partial W(\bar{c}) / \partial c = \partial [V(c_0) + K - S] / \partial c = 0$, so

(19) gives

$$(1 / (r - \theta)) + A_1 \lambda_1 \bar{c}^{\lambda_1 - 1} = 0. \quad (21)$$

After some rearrangement, (20) and (21) give the solution

$$\bar{c} \left((1 - \lambda_1) - (c_0 / \bar{c})^{\lambda_1} \right) + [c_0 + (K - S)(r - \theta)] \lambda_1 = 0. \quad (22)$$

This non-linear equation defines the level of operating cost \bar{c} at which replacement investment is triggered. In view of (13), (15), (16), the value of \bar{c} is a function of the parameters $r, \theta, c_0, K, S, \sigma$. As in the deterministic case, an explicit analytic expression for \bar{c} cannot be obtained, although it is possible to obtain some qualitative comparative statics result for the effects of parameters on the value of \bar{c} and on expected economic life. It is also straightforward to solve (22) numerically, and the quantitative impact of parameter variations can be studied numerically. Section 4 presents these results. However, prior to this, section 3 examines economic life under uncertainty.

3. ECONOMIC LIFE AND SECOND HAND VALUES

Under uncertainty, the economic life of a plant $\tilde{t}(\bar{c})$ is a random variable, and as a consequence, economic life may turn out to be longer or shorter than in the deterministic case. In what follows, we focus on the average or expected life of a plant, $E_0(\tilde{t}(\bar{c}))$. An analytic expression for this expected life is extremely difficult to obtain,⁷ but it is straightforward to derive a numerical approximation for it by running a simple simulation model. The simulation model repeatedly generates paths for c_t ; for the i^{th} run, it is possible to compute the time T_i taken for c_t to reach the trigger level \bar{c} . An estimator for $E_0(\tilde{t}(\bar{c}))$ is thus given as

$$\bar{T} = \sum_{i=1}^n T_i / n \quad (23)$$

if the simulation is run n times. In conducting this simulation, the operating cost process can be approximated as a discrete process using the stochastic difference equation⁸

$$\ln(c_t) = \ln(c_{t-1}) + \left(\theta_m - \frac{1}{2}\sigma_m^2\right) + \sigma_m \varepsilon_t \quad (24)$$

where $\varepsilon_t \sim N(0,1)$. Here, if σ represents an annualised value for volatility of the cost process, then setting periods to months, such that $\sigma_m^2 = \sigma^2 / 12$, and $\theta_m = e^{\theta/12} - 1$, then σ_m, θ_m represent the equivalent monthly rates for the variables σ, θ .

⁷ Cox and Miller [1965, pp. 220-222] discuss the form of calculation required for a special case involving simple Brownian motion. In the more complex case involved in this paper, a closed form explicit solution cannot be obtained – getting a solution thus requires numerical methods at some point.

⁸ A Fortran program which runs this simulation is available from the author on request. The average economic life was computed for each case reported in the sensitivity analysis in section 4 below using $N = 10,000$ runs.

An alternative and much simpler to calculate proxy measure for economic life, denoted \hat{T} , involves calculating the time it takes for *expected operating cost* to reach the level \bar{c} . Thus, given that $E_0(c_t) = c_0 e^{\theta t}$, this proxy measure is computed by setting

$$c_0 e^{\theta \hat{T}} = \bar{c} \quad \Rightarrow \quad \hat{T} = (1/\theta) \ln(\bar{c}/c_0) \quad (25)$$

This proxy under-estimates average economic life because the distribution for $\tilde{t}(\bar{c})$ is skewed to the left. For low levels of volatility, the proxy is quite close to the estimate of $E_0(\tilde{t}(\bar{c}))$ established using simulation but the quality of the approximation deteriorates at higher volatilities (see next section).

In the case where there is secondary trading, it is possible to relate the selling price p_t of used equipment to the current level of operating cost it manifests, c_t using the value function (17). As previously remarked, given the competitive price, there is no reason *per se* for a firm to wish to sell such equipment on such a market (since it is a matter of indifference as to whether to sell or not, and for ongoing business, if plant is sold, another plant of some age must also be bought). In a market with zero sunk costs and no information asymmetries, the prime source of such equipment would presumably be ‘distress’ stock; that is, equipment coming to market because firms have ceased trading, or where there has been a fall in demand for their products. So long as there is always some positive demand for replacement investment, second hand prices will not be affected by product market fluctuations, and will be determined solely by cost characteristics. From (17), using (21) to replace the constant A_1 , value can be written as

$$V(c) = \left(\frac{c}{\theta - r} \right) \left((1/\lambda_1)(c/\bar{c})^{\lambda_1 - 1} - 1 \right) \quad (26)$$

and hence second hand market price $p(c)$ for used equipment is given, using this, as

$$p(c) = K - [V(c) - V(c_0)]. \quad (27)$$

Thus second hand prices are related to their current level of operating cost rather than age *per se*. From (27), second hand price is clearly K for new equipment, and used equipment then declines in value as operating costs increase, to the point when, with operating cost \bar{c} , in view of (18) which implies $V(\bar{c}) = V(c_0) + K - S$, clearly

$$p(\bar{c}) = K - [V(\bar{c}) - V(c_0)] = K - [V(c_0) + K - S - V(c_0)] = S. \quad (28)$$

That is, when operating cost reaches \bar{c} , second hand value has fallen to salvage value (and the machine is scrapped). Although second hand price is determined by ‘quality’ rather than age *per se*, it is possible to relate the second hand price in (28) to the estimated average age of plant with any given level of current operating cost, and results for this are reported in the next section).

4. COMPARATIVE STATICS AND SENSITIVITY ANALYSIS

A comparative statics analysis can be conducted on equation (22), bearing in mind the fact that λ_1 is itself a function of the various parameters, in view of equations (13), (15) and (16). The results for the level of operating cost at which plant is scrapped are as follows (derivations are given in the appendix).

Table 1: Comparative Statics Results

Parameter	ψ	$d\bar{c} / d\psi$
Capital Cost	K	+
Scrap Value	S	-
Initial operating cost	c_0	+
Volatility	σ	+
Discount rate	r	?
Growth rate in operating cost	θ	?

Increases in capital costs or decreases in scrap value tend to increase \bar{c} , the level of operating cost at which replacement is triggered as one would expect. Increases in the initial operating cost of new plant also tend to increase it. Volatility has the expected effect - increases in volatility tend to increase \bar{c} because of the option value effect. That is, relative to the certainty case, it pays to hang on a little longer, and wait for a higher level of operating cost before scrapping plant - simply because there is the possibility that costs may also fall. This is the same type of argument as, when considering the competitive firm under uncertainty, it is not optimal to shut down when price falls to average variable cost; price must fall a bit more before it is optimal to shut down (as in Dixit [1989]). It is not possible to sign the impact of the discount rate, nor, more curiously, of the growth rate in operating cost – at least not without making further assumptions regarding the relative magnitude of various parameters (the relative values of θ and r are naturally of some importance here). However, it is fair to say that, for a plausible range of parameter values, an increase in θ tends to increase the operating cost \bar{c} at which replacement is triggered – and this is also true for an increase in the discount rate r . These observations are illustrated in the numerical results reported in the sensitivity analysis below (Table 2).

The effects on expected economic life are less easy to establish. Notice that changes in capital cost, or salvage value, have no impact on the operating cost process. An increase in the value for \bar{c} thus necessarily means that on any realisation of the cost trajectory, it will take longer to reach this value. Hence an increase in \bar{c} is associated with an increase in $E_0(\tilde{t}(\bar{c}))$. Hence from table 1, clearly $dE_0(\tilde{t}(\bar{c}))/dK > 0$ and $dE_0(\tilde{t}(\bar{c}))/dS < 0$. Changing the discount rate r also involves no impact on the cost process, but the effect on \bar{c} was ambiguous in Table 1, and hence so too is the sign for $dE_0(\tilde{t}(\bar{c}))/dr$. Likewise from table 1, $dE_0(\tilde{t}(\bar{c}))/d\theta$ is ambiguous. An increase in initial operating cost, c_0 increases \bar{c} which *ceteris paribus* would increase expected economic life. However, the change in c_0 also affects the operating cost process, and by raising operating cost, tends to lead to earlier hitting times. Around the benchmark values used in Table 2, the tendency is for this latter effect to more than offset the raising of the threshold, such that the average economic life declines when c_0 is increased. Raising volatility also raises \bar{c} , so *ceteris paribus* tending to increase economic life; however, again, volatility also affects the operating cost process, although in this case the effect tends to be in the same direction. That is, if \bar{c} was kept fixed whilst volatility was increased, this would also tend to increase expected economic life. Overall then, the increase in volatility tends to increase average economic life.

Table 2 illustrates the quantitative impact of varying parameter values on the level for \bar{c} , the level of operating cost at which replacement investment is triggered, the economic life under certainty T_{cert} , the proxy for economic life under uncertainty \hat{T} ,

and the estimate of average economic life \bar{T} (based on the simulation). The final column reports the estimated standard deviation for \bar{T} (given that the number of simulations on which \bar{T} is calculated is 10,000, the confidence interval for \bar{T} is $\pm 1.96s_{\bar{T}}$). The quantitative results confirm the comparative statics analysis reported in table 1, of course. It is worth noting the significant impact of volatility on average economic life, particularly for volatility in excess of 20%. The first two panels in Table 2 illustrate the impact of varying the average rate of growth in cost, θ and the level of volatility, σ . Panel (c) then examines the impact of unilaterally varying each parameter from its benchmark value by 10%. These results are used in the computation of elasticities for $\bar{c}, T_{cert}, \hat{T}, \bar{T}$ which are then reported in Table 3. Thus for example, $(d\hat{T}/dS)(S/K) = -0.04$. In general, all these elasticities are fairly inelastic. Notice also that uncertainty has relatively little impact on these elasticities; that is, the elasticities reported for the certainty case (column 3) are fairly close to those estimated under uncertainty (columns 4, 5 when σ is 20%). Finally, to illustrate how uncertainty impacts on depreciation, table 4 shows how value, expected economic life and second hand price vary with the current level of operating cost. Clearly, as current operating cost rises toward the level at which replacement is activated, average life falls to zero, as does the second hand value. Although the determinant of second value is the current level of operating cost, it is of some interest to consider the implied relationship between the second hand value and the average life expectancy. Interestingly, the rate of depreciation under uncertainty, for the benchmark values, shows a more linear rate of depreciation than would occur if depreciation was exponential (a constant rate depreciation curve is included - in column 4 of table 4 - for comparison purposes).

5. CONCLUDING COMMENTS

It was probably George Terborgh [1949] who first advanced the assumption, when examining the replacement investment decision, that operating expenses tend to increase at a constant rate over time. This paper extends this type of analysis to the case where the evolution of operating cost is uncertain. The impact of uncertainty has been treated before in the literature, but generally in different ways (usually in a discrete time framework, and usually featuring components which have a risk of catastrophic failure) and there has been relatively little (if any) study of comparative statics properties of such models. This paper extends Terborgh's 'exponential model' to the uncertainty case, and then conducts a systematic study of both qualitative and quantitative properties of the model.

The aim in undertaking this analysis was to provide a model which is relatively easy to work with numerically (the basic equation which determines the replacement decision can be readily solved using standard spreadsheet functions (such as SOLVER in EXCEL). Furthermore, it was shown that the proxy for average economic life (equation (25)) was usually a fairly good approximation,⁹ and again, this was readily computable using a simple spreadsheet. For applications where the underlying assumptions are reasonably plausible, it could be used as a basis for assessing the timing of replacement investment. Naturally, there are many other considerations outwith those considered here which might affect such decisions; in such circumstances it may be possible to use the model to put an opportunity cost on

⁹ Table 2 indicates that the proxy only starts to seriously underestimate average economic life when there is high volatility and/or very low rates of increase in operating cost.

those considerations. For example, consider a fleet of company cars (or a bus company); it is possible to assess not only the trigger points at which replacement should be undertaken, but it is also possible to estimate the costs of varying from these choices. For example, if the company does not wish to see its sales force in cars older than a certain age, it is possible, using this model, to quantify the cost of shortening the age at which such company cars are replaced.

Apart from the intrinsic importance of the decision itself, replacement investment is of course important from a macro-economic perspective, in that it might cast some light on the relationship between interest rates and levels of investment (how steep the marginal efficiency of capital schedule is). In thinking about such issues, it is useful to recognise the importance of uncertainty. In the above model, an increase in the steady state rate of interest not only affects costs directly; it also has an effect in that firms will choose to extend the economic life of the plant and equipment they operate; at the benchmark values for example, the elasticity of (average) economic life to changes in the rate of interest was in the region of 0.1 under certainty, and in the same ballpark under uncertainty. This is not only fairly inelastic, but is also a fairly typical value, given plausible values for volatility and the other parameters involved. Thus, the model is also suggestive that changes in the rate of interest do not have great impacts on replacement timing. Such observations are supportive of the case that, if there is a relationship at all, the marginal efficiency of capital schedule may be fairly steep (and hence so too the IS curve, in an ISLM framework).¹⁰

¹⁰ An increase in interest rates naturally raises financing costs and so there could be longer run output consequences which would also need to be considered; the above discussion presumes the 'quantity' of plant in service at a point in time is unaffected by the change in the level of interest. In such

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circumstances, a rise in interest rates tends to raise economic life (albeit only slightly); in a steady state, that would mean less per period investment in new equipment.

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Table 2: Economic Life as a function of various parameters

Parameter	Value	\bar{c}	T_{cert}	\hat{T}	\bar{T}	$s_{\bar{T}}$
	Bench Mark	35.870	8.197	8.515	10.09	0.048
Panel (a)						
θ	0.05	29.888	18.326	21.897	37.445	0.411
	0.1	31.382	10.979	11.437	14.723	0.096
	0.15	35.870	8.197	8.515	10.09	0.048
	0.25	42.344	5.688	5.773	6.383	0.022
	0.5	56.713	3.457	3.471	3.665	0.008
	1	80.639	2.085	2.087	2.149	0.003
Panel (b)						
σ	0.000001	34.198	8.197	8.197	8.197	0.000
	0.01	34.202	8.197	8.198	8.199	0.002
	0.1	34.606	8.197	8.276	8.649	0.020
	0.2	35.870	8.197	8.515	10.100	0.049
	0.5	45.781	8.197	10.142	55.753	1.000
Panel (c)						
θ	0.165	36.860	7.656	7.906	9.168	0.041
r	0.11	36.454	8.280	8.623	10.112	0.049
K	110	37.782	8.533	8.862	10.417	0.050
S	11	35.676	8.162	8.479	10.054	0.049
c_0	11	37.695	7.902	8.211	9.685	0.047
σ	0.22	36.234	8.197	8.583	10.479	0.057

Benchmark Parameter values: $r = 0.1, \theta = 0.15, K = 100, S = 10, c_0 = 10, \sigma = 0.2$

Table 3: Elasticities at benchmark values

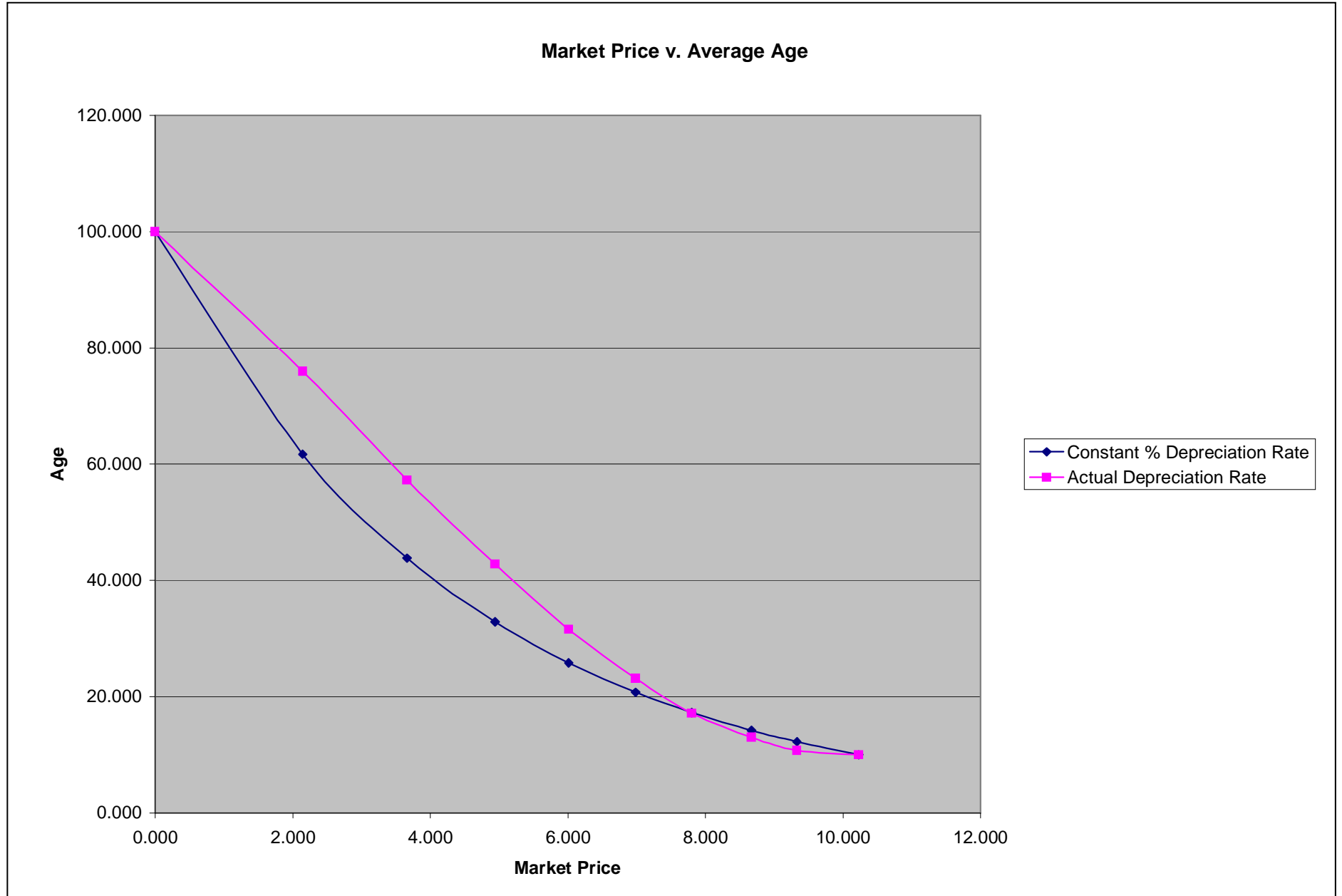
Parameter	\bar{c}	T_{cert}	\hat{T}	\bar{T} #
θ	0.28	-0.66	-0.72	-0.91
r	0.16	0.10	0.13	0.02
K	0.53	0.41	0.41	0.32
S	-0.05	-0.04	-0.04	-0.04
c_0	0.51	-0.36	-0.36	-0.40
σ	0.10	0.00	0.08	0.39

The elasticities calculated here are subject only to machine error in numerical computations of solutions – except for those for \bar{T} . Given that \bar{T} is estimated by simulation and manifests a standard error of around 0.05, the estimates of elasticity in the final column are significantly less robust.

Table 4: Example of Depreciation under Uncertainty

c_0	\bar{T}	Average age	Second hand Price if constant (exponential) depreciation	Theoretical second hand price $p(c)$
10.000	10.230	0.000	100.000	100.000
12.875	8.084	2.146	61.687	75.928
15.749	6.567	3.664	43.842	57.278
18.623	5.289	4.941	32.886	42.776
21.498	4.215	6.015	25.827	31.602
24.372	3.242	6.988	20.745	23.180
27.247	2.428	7.802	17.271	17.095
30.121	1.557	8.673	14.198	13.027
32.996	0.896	9.334	12.235	10.728
35.870	0.000	10.230	10.000	10.000

Figure 1



APPENDIX:

1. Derivation of the fundamental equation

The arbitrage condition, repeated here for convenience, is

$$rV(c)dt = cdt + E(dV(c)). \quad (\text{A1.1})$$

The next step is to evaluate $E(dV(c))$, using Itô's lemma. Thus

$$dV(c) = V'(c)dc + \frac{1}{2}V''(c)dc^2, \quad (\text{A1.2})$$

whilst

$$dc = \theta cdt - \sigma cd\bar{\omega} \Rightarrow dc^2 = (\theta cdt - \sigma cd\bar{\omega})^2 = c^2\sigma^2 dt, \quad (\text{A1.3})$$

(setting terms $\theta^2 c^2 dt^2, \theta c\sigma cdt d\bar{\omega} = 0$). Thus (abbreviating the notation a little)

$$dV = V'[\theta cdt - \sigma cd\bar{\omega}] + \frac{1}{2}V''[c^2\sigma^2 dt]. \quad (\text{A1.4})$$

Taking expectations,

$$E(dV) = \theta cV'dt + \frac{1}{2}c^2\sigma^2V''dt. \quad (\text{A1.5})$$

Hence the arbitrage equation becomes

$$rVdt = cdt + \theta cV'dt + \frac{1}{2}c^2\sigma^2V''dt \quad (\text{A1.6})$$

Thus

$$\frac{1}{2}\sigma^2c^2V'' + \theta cV' - rV + c = 0 \quad (\text{A1.7})$$

which is equation (11) in the paper. The general solution to the homogenous equation

$$\frac{1}{2}\sigma^2c^2V'' + V'c - rV = 0 \quad (\text{A1.8})$$

is derived here. Consider a trial solution of the form

$$V(c) = c^\lambda \quad (\text{A1.9})$$

Thus, $V'(c) = \lambda c^{\lambda-1}$ and $V''(c) = \lambda(\lambda-1)c^{\lambda-2}$. Substituting into (A1.8) gives

$$\frac{1}{2}\sigma^2\lambda(\lambda-1)c^\lambda + \lambda\theta c^\lambda - rc^\lambda = 0 \quad (\text{A1.10})$$

which would hold if

$$\frac{1}{2}\sigma^2\lambda^2 + \left(\theta - \frac{1}{2}\sigma^2\right)\lambda - r = 0 \quad (\text{A1.11})$$

It is convenient to define

$$R_1 \equiv \left(\theta - \frac{1}{2}\sigma^2\right) \quad (\text{A1.12})$$

$$R_2 \equiv \left(R_1^2 + 2r\sigma^2\right)^{1/2} \quad (\text{A1.13})$$

so the roots to the quadratic equation are

$$\lambda_1 = (-R_1 + R_2) / \sigma^2 \quad (\text{A1.14})$$

and

$$\lambda_2 = (-R_1 - R_2) / \sigma^2. \quad (\text{A1.15})$$

The general solution to (A1.7) is formed as the sum of the solution to the homogenous equation and the particular solution given in the paper. It thus takes the form

$$V(c) = c / (r - \theta) + A_1x^{\lambda_1} + A_2x^{\lambda_2} \quad (\text{A1.16})$$

where λ_1, λ_2 are as defined above. The arbitrary constants are determined by boundary conditions.

2. Comparative Statics Analysis

There are two equations determining \bar{c} and λ_1 ; the latter is the positive root of the fundamental equation (A11). Defining $v \equiv \sigma^2$ and the function

$$f(\lambda) \equiv \frac{1}{2}v\lambda^2 + (\theta - \frac{1}{2}v)\lambda - r \quad (\text{A2.1})$$

then λ_1 satisfies

$$f(\lambda_1) = 0 \quad (\text{A2.2})$$

Note that λ_1 is defined as the positive root of the equation (A2.2), such that

$$\lambda_1 = -\frac{(\theta - \frac{1}{2}v)}{v} + \frac{1}{v} \left[(\theta - \frac{1}{2}v)^2 + 2vr \right]^{1/2}. \quad (\text{A2.3})$$

Note also that λ_1 is strictly increasing in r . If we set $r = \theta$, then (A2.3) simplifies to give

$$\begin{aligned} \lambda_1 &= -\frac{(\theta - \frac{1}{2}v)}{v} + \frac{1}{v} \left[\theta^2 - \theta v + \frac{1}{4}v^2 + 2v\theta \right]^{1/2} \\ &= -\frac{(\theta - \frac{1}{2}v)}{v} + \frac{1}{v} \left[(\theta + \frac{1}{2}v)^2 \right]^{1/2} = -\frac{(\theta - \frac{1}{2}v)}{v} + \frac{(\theta + \frac{1}{2}v)}{v} = 1 \end{aligned} \quad (\text{A2.4})$$

It thus follows that, since $\partial\lambda_1/\partial r > 0$, that

$$\theta \geq r \Rightarrow \lambda_1 \leq 1 \quad (\text{A2.5})$$

The other condition established in the paper is that, defining the function

$$g(c, \lambda) = (1 - \lambda)c - c_0^\lambda c^{1-\lambda} + [c_0 + (K - S)(r - \theta)]\lambda \quad (\text{A2.6})$$

that

$$g(\bar{c}, \lambda_1) = 0 \quad (\text{A2.7})$$

Denote a generic parameter as ψ (i.e. $\psi = r, \theta, S, K, c_0, v$). Then

$$\frac{\partial f(\lambda_1)}{\partial \lambda} \frac{d\lambda_1}{d\psi} + \frac{\partial f(\lambda_1)}{\partial \psi} = 0 \Rightarrow \frac{d\lambda_1}{d\psi} = -\frac{\partial f(\lambda_1)}{\partial \psi} \Big/ \frac{\partial f(\lambda_1)}{\partial \lambda} \quad (\text{A2.8})$$

and

$$\frac{\partial g(\bar{c}, \lambda_1)}{\partial \lambda} \frac{d\lambda_1}{d\psi} + \frac{\partial g(\bar{c}, \lambda_1)}{\partial c} \frac{d\bar{c}}{d\psi} + \frac{\partial g(\lambda_1)}{\partial \psi} = 0 \quad (\text{A2.9})$$

so the comparative statics derivative is given as

$$\frac{d\bar{c}}{d\psi} = -\left(\frac{\partial g(\bar{c}, \lambda_1)}{\partial \lambda} \frac{d\lambda_1}{d\psi} + \frac{\partial g(\lambda_1)}{\partial \psi} \right) \Big/ \frac{\partial g(\bar{c}, \lambda_1)}{\partial c} \quad (\text{A2.10})$$

which, using (A2.8), means that

$$\frac{d\bar{c}}{d\psi} = -\left(\frac{\partial g(\lambda_1)}{\partial \psi} - \left(\frac{\partial g(\bar{c}, \lambda_1)}{\partial \lambda} \times \frac{\partial f(\lambda_1)}{\partial \psi} \Big/ \frac{\partial f(\lambda_1)}{\partial \lambda} \right) \right) \Big/ \frac{\partial g(\bar{c}, \lambda_1)}{\partial c} \quad (\text{A2.11})$$

To begin, we need to establish the various partial derivatives (evaluating these at \bar{c}, λ_1 , as follows: from (A2.1),

$$\partial f(\lambda_1)/\partial K = 0, \quad (\text{A2.12})$$

$$\partial f(\lambda_1)/\partial S = 0, \quad (\text{A2.13})$$

$$\partial f(\lambda_1)/\partial c_0 = 0, \quad (\text{A2.14})$$

$$\partial f(\lambda_1)/\partial r = -1 < 0 \quad (\text{A2.15})$$

$$\partial f(\lambda_1)/\partial \theta = \lambda_1 > 0 \quad (\text{A2.16})$$

$$\partial f(\lambda_1)/\partial v = \frac{1}{2} \lambda_1 (\lambda_1 - 1) \quad (\text{A2.17})$$

whilst from (A2.6)

$$\partial g(\bar{c}, \lambda_1)/\partial K = (r - \theta) \lambda_1 < 0 \quad (\text{A2.18})$$

$$\partial g(\bar{c}, \lambda_1)/\partial S = -(r - \theta) \lambda_1 > 0 \quad (\text{A2.19})$$

$$\partial g(\bar{c}, \lambda_1)/\partial c_0 = -\lambda_1 c_0^{\lambda_1 - 1} \bar{c}^{1 - \lambda_1} + \lambda_1 = \lambda_1 \left(1 - (c_0/\bar{c})^{\lambda_1 - 1} \right) \quad (\text{A2.20})$$

$$\partial g(\bar{c}, \lambda_1)/\partial r = (K - S) \lambda_1 > 0 \quad (\text{A2.21})$$

$$\partial g(\bar{c}, \lambda_1)/\partial \theta = -(K - S) \lambda_1 < 0 \quad (\text{A2.22})$$

$$\partial g(\bar{c}, \lambda_1)/\partial v = 0 \quad (\text{A2.23})$$

Finally, note that from (A2.1),

$$\partial f(\lambda_1)/\partial \lambda = v \lambda_1 + \left(\theta - \frac{1}{2} v \right). \quad (\text{A2.24})$$

From (A2.3) (in which the positive square root is taken), clearly this is positive. From (A2.6),

$$\partial g(\bar{c}, \lambda_1)/\partial c = (1 - \lambda_1) - (1 - \lambda_1) c_0^{\lambda_1} \bar{c}^{-\lambda_1} = (1 - \lambda_1) \left(1 - (c_0/\bar{c})^{\lambda_1} \right). \quad (\text{A2.25})$$

Given this is the denominator in (A2.11), its sign is crucial for comparative statics

results. Since $c_0/\bar{c} < 1$ and $\lambda_1 > 0$, clearly $1 - (c_0/\bar{c})^{\lambda_1} > 0$, and so from (A2.5),

$$\theta \geq r \Rightarrow \lambda_1 \leq 1 \Rightarrow \partial g(\bar{c}, \lambda_1)/\partial c \geq 0 \quad (\text{A2.26})$$

The term $\partial g/\partial \lambda$ is a little more tricky. From (A2.6),

$$\partial g(c, \lambda)/\partial \lambda = -c - \frac{\partial}{\partial \lambda} \left(c_0^{\lambda} c^{1 - \lambda} \right) + [c_0 + (K - S)(r - \theta)] \quad (\text{A2.27})$$

Now,

$$\frac{\partial}{\partial \lambda} \left(c_0^{\lambda} c^{1 - \lambda} \right) = c_0^{\lambda} \frac{\partial}{\partial \lambda} \left(c^{1 - \lambda} \right) + c^{1 - \lambda} \frac{\partial}{\partial \lambda} \left(c_0^{\lambda} \right) \quad (\text{A2.28})$$

and

$$\frac{\partial}{\partial \lambda} \left(c^{1 - \lambda} \right) = -c^{1 - \lambda} \ln c \quad (\text{A2.29})$$

$$\frac{\partial}{\partial \lambda} \left(c_0^{\lambda} \right) = c_0^{\lambda} \ln c_0 \quad (\text{A2.30})$$

so

$$\begin{aligned} \partial g/\partial \lambda &= -c - c_0^{\lambda} \frac{\partial}{\partial \lambda} \left(c^{1 - \lambda} \right) - c^{1 - \lambda} \frac{\partial}{\partial \lambda} \left(c_0^{\lambda} \right) + [c_0 + (K - S)(r - \theta)] \\ &= -c + c_0^{\lambda} c^{1 - \lambda} \ln c - c^{1 - \lambda} c_0^{\lambda} \ln c_0 + [c_0 + (K - S)(r - \theta)] \end{aligned} \quad (\text{A2.31})$$

so evaluating at \bar{c}, λ_1 and simplifying a little gives

$$\partial g(\bar{c}, \lambda_1)/\partial \lambda = \bar{c} \left[(c_0/\bar{c})^{\lambda_1} \ln(\bar{c}/c_0) - 1 \right] + [c_0 + (K - S)(r - \theta)]. \quad (\text{A2.32})$$

Now (A2.6) and (A2.7) imply that :

$$g(\bar{c}, \lambda_1) = (1 - \lambda_1) \bar{c} - c_0^{\lambda_1} \bar{c}^{1 - \lambda_1} + [c_0 + (K - S)(r - \theta)] \lambda_1 = 0 \quad (\text{A2.33})$$

and so

$$c_0 + (K - S)(r - \theta) = - \frac{\left[(1 - \lambda_1) \bar{c} - c_0^{\lambda_1} \bar{c}^{1 - \lambda_1} \right]}{\lambda_1}. \quad (\text{A2.34})$$

Using this to replace the term $[c_0 + (K - S)(r - \theta)]$ in (A2.32) gives

$$\partial g(\bar{c}, \lambda_1) / \partial \lambda = \bar{c} \left[(c_0 / \bar{c})^{\lambda_1} \ln(\bar{c} / c_0) - 1 \right] - \left[\frac{[(1 - \lambda_1) \bar{c} - c_0^{\lambda_1} c^{1 - \lambda_1}]}{\lambda_1} \right] \quad (\text{A2.35})$$

\Rightarrow

$$\begin{aligned} \partial g(\bar{c}, \lambda_1) / \partial \lambda &= \bar{c} \left[(c_0 / \bar{c})^{\lambda_1} \ln(\bar{c} / c_0) - 1 \right] - \bar{c} \left[\frac{[1 - (c_0 / \bar{c})^{\lambda_1}]}{\lambda_1} - 1 \right] \\ &= \frac{\bar{c}}{\lambda_1} \left(\lambda_1 \left[(c_0 / \bar{c})^{\lambda_1} \ln(\bar{c} / c_0) - 1 \right] - \left[1 - (c_0 / \bar{c})^{\lambda_1} \right] + \lambda_1 \right) \end{aligned}$$

\Rightarrow

$$\partial g(\bar{c}, \lambda_1) / \partial \lambda = \frac{\bar{c}}{\lambda_1} \left(\lambda_1 (c_0 / \bar{c})^{\lambda_1} \ln(\bar{c} / c_0) - \left[1 - (c_0 / \bar{c})^{\lambda_1} \right] \right)$$

\Rightarrow

$$\partial g(\bar{c}, \lambda_1) / \partial \lambda = \frac{\bar{c}}{\lambda_1} \left((c_0 / \bar{c})^{\lambda_1} \left[1 - \ln(c_0 / \bar{c})^{\lambda_1} \right] - 1 \right) \quad (\text{A2.36})$$

Write $z \equiv (c_0 / \bar{c})^{\lambda_1}$, so that, with $0 < c_0 / \bar{c} < 1$ and $\lambda_1 > 0$, clearly $0 < z < 1$. The sign of above expression then depends on the term

$$W(z) \equiv z[1 - \ln z] - 1 \quad (\text{A2.37})$$

First note that $dW(z) / dz = [1 - \ln z] + z[-1/z] = -\ln z > 0$ (since $z \in (0, 1)$) so $W(z)$ is strictly increasing on $(0, 1)$. Also note that $W(1) = 0$, hence it follows, for all z such that $0 < z < 1$, that $W(z) < 0$. Given $\bar{c} / \lambda_1 > 0$ this implies

$$\partial g(\bar{c}, \lambda_1) / \partial \lambda < 0 \quad (\text{A2.38})$$

This completes the preliminaries necessary for determining the comparative statics results, which can now be computed using (A2.11).

Capital cost:

Here, $\partial f / \partial K = 0$, and $\partial g / \partial K = (r - \theta)\lambda$ so (A2.11) becomes

$$\begin{aligned} d\bar{c} / dK &= -(\partial g / \partial K - (\partial g / \partial \lambda \times \partial f / \partial K / \partial f / \partial \lambda)) / \partial g / \partial c \\ &= -((r - \theta)\lambda_1 - (\partial g / \partial \lambda \times 0 / \partial f / \partial \lambda)) / \partial g / \partial c \\ &= (\theta - r)\lambda_1 / \partial g / \partial c \end{aligned}$$

Now, $\lambda_1 > 0$ and $\theta \geq r \Rightarrow \theta - r \geq 0$ and (for all $\theta, r > 0$, $\theta \neq r$) from (A2.26)

$\theta \geq r \Rightarrow \partial g(\bar{c}, \lambda_1) / \partial c \geq 0$ hence, it follows that

$$\text{(for all } \theta, r > 0, \theta \neq r), d\bar{c} / dK > 0 \quad (\text{A2.39})$$

Salvage Value:

The analysis parallels that for capital cost. Here $\partial f / \partial S = 0$, and $\partial g / \partial S = -(r - \theta)\lambda_1$ so $d\bar{c} / dS = -(\theta - r)\lambda_1 / \partial g / \partial c$. Thus

$$\text{(for all } \theta, r > 0, \theta \neq r), \quad d\bar{c} / dS < 0 \quad (\text{A2.40})$$

Initial Operating Cost:

Here $\partial f / \partial c_0 = 0$, and $\partial g / \partial c_0 = \lambda_1 \left(1 - (c_0 / \bar{c})^{\lambda_1 - 1}\right)$ so

$$\begin{aligned} d\bar{c} / dc_0 &= -\left(\partial g / \partial c_0 - (\partial g / \partial \lambda \times \partial f / \partial c_0 / \partial f / \partial \lambda)\right) / \partial g / \partial c \\ &= -\left(\lambda_1 \left(1 - (c_0 / \bar{c})^{\lambda_1 - 1}\right) - (\partial g / \partial \lambda \times 0 / \partial f / \partial \lambda)\right) / \partial g / \partial c \\ &= -\lambda_1 \left(1 - (c_0 / \bar{c})^{\lambda_1 - 1}\right) / \partial g / \partial c \end{aligned}$$

Now, $\lambda_1 > 0$ and $0 < c_0 / \bar{c} < 1$ and (for all $\theta, r > 0, \theta \neq r$) from (A2.26)

$\theta \geq r \Rightarrow \partial g(\bar{c}, \lambda_1) / \partial c \geq 0$. Also, $\theta \geq r \Rightarrow \lambda_1 \leq 1 \Rightarrow (c_0 / \bar{c})^{\lambda_1 - 1} \geq 1 \Rightarrow 1 - (c_0 / \bar{c})^{\lambda_1 - 1} \leq 0$.

It thus follows that

$$\text{(for all } \theta, r > 0, \theta \neq r), \quad d\bar{c} / dc_0 > 0 \quad (\text{A2.41})$$

Volatility:

Here

$$d\bar{c} / dv = -\left(\partial g / \partial v - (\partial g / \partial \lambda \times \partial f / \partial v / \partial f / \partial \lambda)\right) / \partial g / \partial c$$

where, $\partial g / \partial v = 0$, $\partial g / \partial \lambda < 0$, and $\partial f / \partial \lambda > 0$ so

$$\text{Sign}(d\bar{c} / dv) = -\text{Sign}(\partial f / \partial v / \partial g / \partial c)$$

Now, $\partial f / \partial v = \frac{1}{2}\lambda_1(\lambda_1 - 1)$ and from (A2.5), $\theta \geq r \Rightarrow \lambda_1 \leq 1$, so

$$\theta \geq r \Rightarrow \lambda_1 \leq 1 \Rightarrow \partial f / \partial v \leq 0.$$

However, $\theta \geq r \Rightarrow \partial g(\bar{c}, \lambda_1) / \partial c \geq 0$ from (A2.26). Hence

$$\text{(for all } \theta, r > 0, \theta \neq r), \quad d\bar{c} / d\sigma > 0 \quad (\text{A2.42})$$

Interest rate:

Here,

$$d\bar{c} / dr = -\left(\partial g / \partial r - (\partial g / \partial \lambda \times \partial f / \partial r / \partial f / \partial \lambda)\right) / \partial g / \partial c$$

where $\partial f / \partial r = -1 < 0$, $\partial g / \partial r = (K - S)\lambda_1 > 0$, $\partial g(\bar{c}, \lambda_1) / \partial \lambda < 0$ and

$$\partial f / \partial \lambda = v\lambda_1 + (\theta - \frac{1}{2}v) > 0.$$

Hence, $d\bar{c} / dr$ is of ambiguous sign.

Operating cost trend:

Here

$$d\bar{c} / d\theta = -\left(\partial g / \partial \theta - (\partial g / \partial \lambda \times \partial f / \partial \theta / \partial f / \partial \lambda)\right) / \partial g / \partial c$$

where $\partial f / \partial \theta = \lambda_1 > 0$, $\partial g / \partial \theta = -(K - S)\lambda_1 < 0$, $\partial g / \partial \lambda < 0$, and $\partial f / \partial \lambda > 0$.

Hence, $d\bar{c} / d\theta$ is of ambiguous sign.