5 Decision Theory

In this chapter we will introduce the concept of **Decision Theory** briefly.

5.1 Expectation

If X is a discrete random variable with the probability function

$$P(X = x_i) = p_i, \text{ for } i = 1, 2, \dots,$$

the **expected value** (or **expectation**, or **mean**) of X is

$$E(X) = \sum_i x_i p_i.$$

Correspondingly, if X is a continuous random variable with density function $f(x)$, the **expected value** of X is

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx.$$

- If a and b are constant, then $E(aX + b) = aE(X) + b$;
- $E(X + Y) = E(X) + E(Y)$.

Example 1. If X has a Binomial distribution $B(n, p)$, then

$$p_i = P(X = i) = \left(\begin{array}{c} n \\ i \end{array} \right) p^i (1-p)^{n-i},$$

then

$$E(X) = \sum_{i=1}^{n} i p_i = \sum_{i=1}^{n} \left[i \frac{n!}{(n-i)!i!} p^i (1-p)^{n-i} \right] = \dots = np.$$

Example 2. A company of leather goods must decide whether to expand the capacity. The advisors provided the following information: if the company expands now and the recession remains, there will be a loss of 40,000 during the next fiscal year; but if the economic conditions will be improved, there will be a profit of 164,000. If we know that the probability of 'remain recession' is 2/3, what is the expected value of the profit in the next fiscal year?

Solutions. Let X be the profit in the next fiscal year, then

X	164,000	-40,000
p_i	1/3	2/3

Thus,

$$E(X) = 164,000 \times \frac{1}{3} + (-40,000) \times \frac{2}{3} = 28,000.$$

Example 3. If X has a uniform distribution $U(a, b)$, then it has the following density function

$$f(x) = \frac{1}{b-a}, \quad \text{for} \quad a < x < b.$$

The expectation of X is

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_a^b \frac{x}{b-a} dx \\ &= \frac{1}{b-a}\left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{a+b}{2}. \end{aligned}$$

5.2 The decision model

Our concern is decision.

- What course of action should be taken when uncertainty exists regarding the **true state of nature**? i.e., true state θ may be one of the outcome in the sample space $\Theta = (\theta_1, \theta_2, \ldots, \theta_m)$. In Example 2, θ could be either

  $$\theta_1 = \{\textbf{the economic conditions will be improved}\}$$

  or

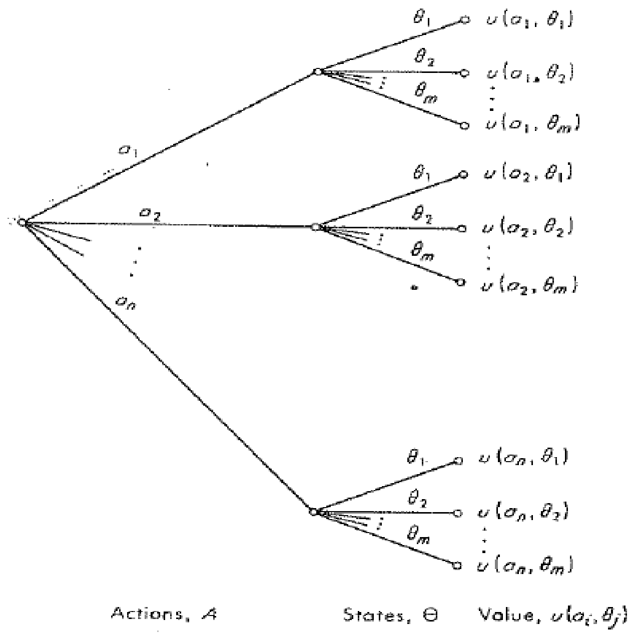
  $$\theta_2 = \{\textbf{remain recession}\}.$$

- Choose an action from an **action space**: $A = \{a_1, \ldots, a_n\}$. In Example 2, the action could be either

  $$a_1 = \{\textbf{expand now}\} \text{ or } a_2 = \{\textbf{delay expansion}\}.$$

- Once the decision has been made, the decision maker (engineer) can only wait to see which state of nature θ is the true one.

The figure below displays a decision tree for discrete-state space; where $u(a, \theta)$ is called **utility function**, a numerical measure (say pounds) of the consequences of this action-state pair.



Decision tree

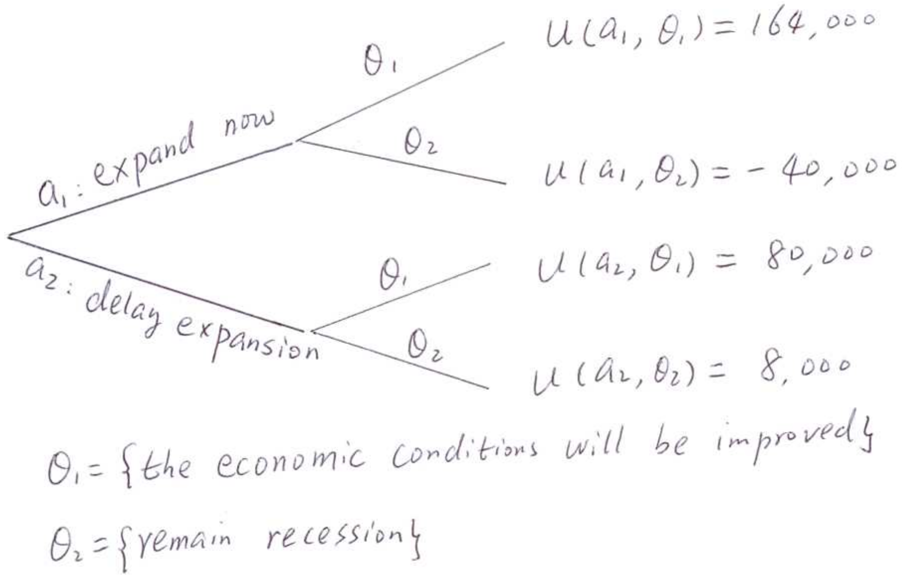
Example 2 (continued) Denote actions as

$$a_1 = \{\text{expand now}\} \quad \text{and} \quad a_2 = \{\text{delay expansion}\},$$

and the state of nature as

$$\begin{aligned} \theta_1 &= \{\text{the economic conditions will be improved}\} \text{ and} \\ \theta_2 &= \{\text{remain recession}\}. \end{aligned}$$

We further know that if the company waits at least another year to expand and the recession remains, there will be a profit of 8,000; and there will be a profit of 80,000 if the economic conditions will be improved. What decision should be made (expand now or delay expansion)? How to draw a decision tree?



Decision tree for Example 2

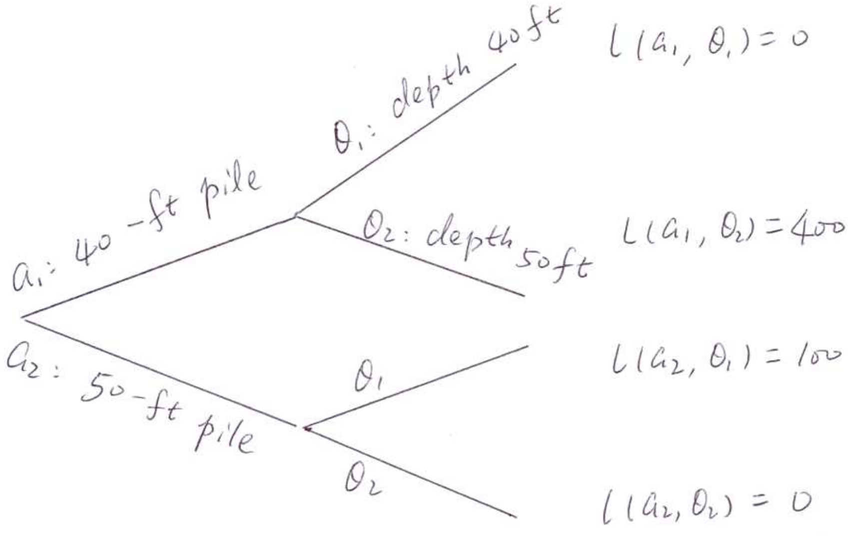
Example 4. A construction engineer needs to select a steel pile length when the depth to rock is uncertain. The available actions are driving a 40- or 50-ft pile, and the possible states of nature are a 40- or 50-ft depth to bedrock. The consequences of any action-state pair can be given in a *payoff* table:

	a_1 (drive 40-ft pile)	a_2 (drive 50-ft pile)
θ_1 depth 40 ft	correct decision no loss	cut off 10-ft piece of pile 100-unit loss
θ_2 depth 50 ft	splice and weld 10-ft pile 400-unit loss	correct decision no loss

In this example, we can define either a **loss function** $l(a_i, \theta_j)$ or a **utility function** $u(a_i, \theta_j)$:

loss function	utility function
$l(a_1, \theta_1) = 0$	$u(a_1, \theta_1) = 0$
$l(a_1, \theta_2) = 400$	$u(a_1, \theta_2) = -400$
$l(a_2, \theta_1) = 100$	$u(a_2, \theta_1) = -100$
$l(a_2, \theta_2) = 0$	$u(a_2, \theta_2) = 0$

How to draw a decision tree?



Decision tree for Example 4

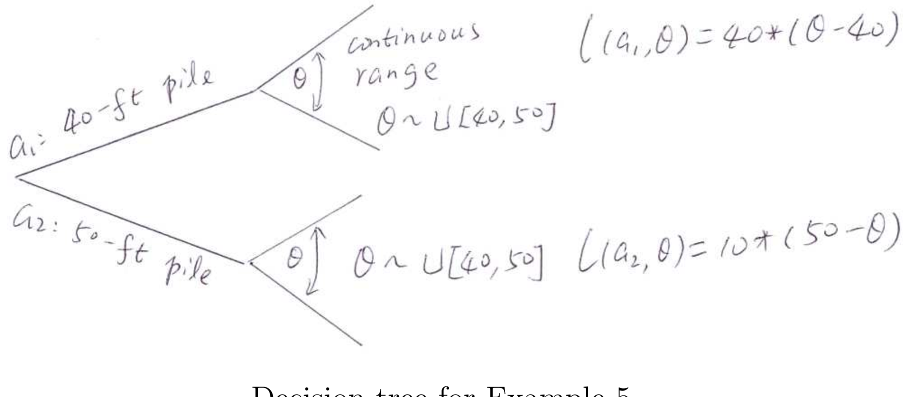
Example 5. Same as Example 4, the engineer can take action to drive either a 40- or 50-ft pile, but the possible states of nature θ has a uniform distribution $U(40, 50)$. We may consider the following loss function:

$$l(a_1, \theta) = 40 * (\theta - 40), \quad \theta \sim U(40, 50);$$

and

$$l(a_2, \theta) = 10 * (50 - \theta), \quad \theta \sim U(40, 50).$$

How to draw a decision tree?



Decision tree for Example 5

5.3 Decision Theory

1. Minimax criterion

We take an action to minimize loss function (or maximize utility function) in the worst-case scenario.

Example 2 (continued) If we don't know the probability of θ_1 = {the economic conditions will be improved} and θ_2 = {remain recession}, what decision should be made? We can define the utility function (or loss function) as the profit achieved in the worst-case scenario. Thus

$$u(a_1) = \min\{u(a_1, \theta_1), u(a_1, \theta_2)\} = -40,000,$$

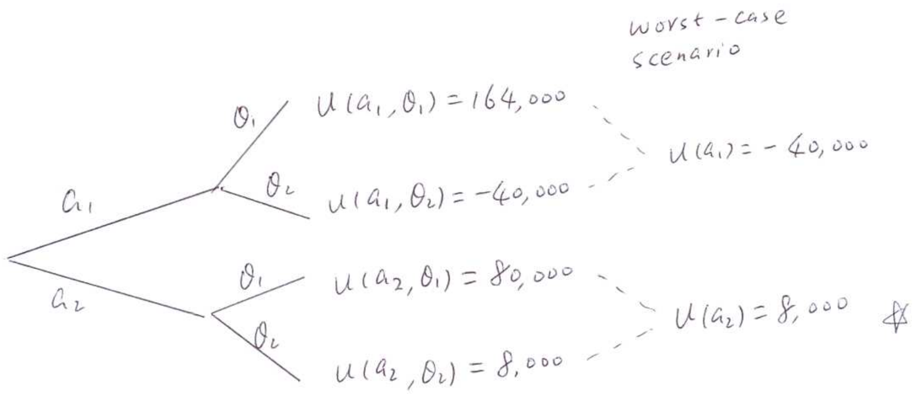
or

$$\text{Utility(Expand now)} = -40,000.$$

Similarly,

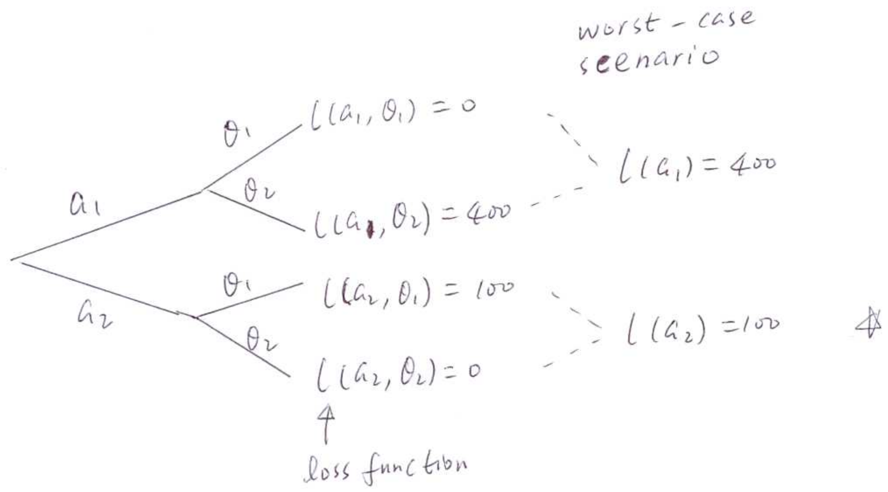
$$\begin{aligned}\text{Utility(Delay expansion)} &= u(a_2) \\ &= \min\{u(a_2, \theta_1), u(a_2, \theta_2)\} = 8,000.\end{aligned}$$

Using the minimax criterion, we should choose the action with the maximum value of $u(a_i), i = 1, 2$. We should therefore choose a_2, i.e. delay expansion. The decision is made based on maximising the utility function for the worst case.



Decision tree for Example 2 (continued).

Example 4 (continued). It is quite obvious that we should take action a_2 based on Minimax criterion (minimize loss function). Why?



Decision tree for Example 4 (continued).

2. Expected-value decisions

How to make decision when a decision situation involves uncertainty? We can calculate the expected-value of utility function or loss function.

Let $u(a_i, \theta)$ be a utility function for action a_i. It depends on a random variable θ (state of nature).

- If θ is a discrete random variable with probability function $P(\theta = \theta_j) = p_j, j = 1, \ldots, m$, then the expected utility is calculated by

$$u(a_i) = E(u(a_i, \theta)) = \sum_{j=1}^{m} u(a_i, \theta_j)p_j;$$

- If θ is a continues random variable with density function $f(\theta)$, then the expected utility is calculated by

$$u(a_i) = E(u(a_i, \theta)) = \int u(a_i, \theta)f(\theta)d\theta.$$

Optimal decision is to choose the action with the maximum utility.

Example 2 (continued) From the discuss in Section 5.1, we know that

$$\begin{aligned} P(\theta = \theta_1) &= P(\text{the economic conditions will be improved}) = 1/3 \\ P(\theta = \theta_2) &= P(\text{remain recession}) = 2/3 \end{aligned}$$

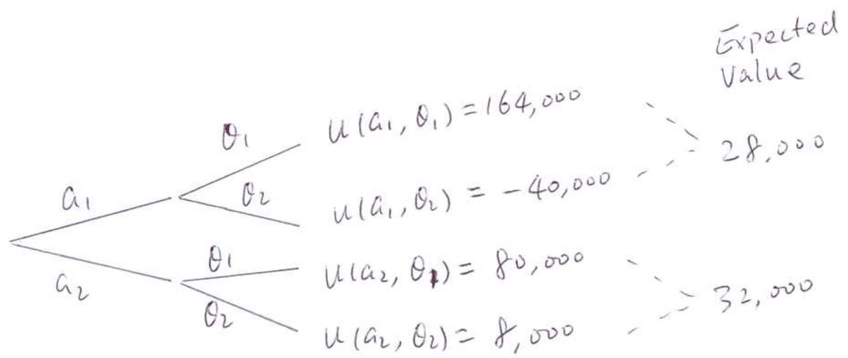
The expected value of the profit if the company **expand now**:

$$u(a_1) = E(u(a_1, \theta)) = 164,000 \times \frac{1}{3} + (-40,000) \times \frac{2}{3} = 28,000.$$

The expected value of the profit if the company **waits at least another year to expand**

$$u(a_2) = E(u(a_2, \theta)) = 80,000 \times \frac{1}{3} + 8,000 \times \frac{2}{3} = 32,000.$$

Based on the expected values of the **utility** function (profit), we should take action a_2, i.e., delay expansion. The decision is made based on maximising the expected-value of utility function.



Decision tree for Example 2 (continued).

Example 5 (continued). Since $\theta \sim U(40, 50)$, we know that (from Example 3)

$$E(\theta) = (40 + 50)/2 = 45.$$

Thus, the expected values of the loss functions are:

$$l(a_1) = E(l(a_1, \theta)) = 40 * (E(\theta) - 40) = 40 * (45 - 40) = 200.$$

and

$$l(a_2) = E(l(a_2, \theta)) = 10 * (50 - E(\theta)) = 10 * (50 - 45) = 50.$$

We then should take action a_2, i.e., driving a 50-ft pile. The decision is made based on minimising the expected-value of loss function.

3. *Bayesian decision

The probability function $P(\theta = \theta_j)$ or density function $f(\theta)$ are not always easy to obtain. Using Bayesian approach, if we have observed a set of data, we can calculate the posterior distribution $f(\theta|\text{data})$. Bayesian decision is based on the expected-values of the utilities calculated by using the posterior distribution.