

COTORSION PAIRS INDUCED BY DUALITY PAIRS

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ABSTRACT. We introduce the notion of a duality pair and demonstrate how the left half of such a pair is “often” covering and preenveloping. As an application, we generalize a result by Enochs et al. on Auslander and Bass classes, and we prove that the class of Gorenstein injective modules—introduced by Enochs and Jenda—is covering when the ground ring has a dualizing complex.

INTRODUCTION

What is now known as semidualizing modules were studied more than 25 years ago under other names by e.g. Foxby [15] (PG-modules of rank one), Golod [17] (suitable modules) and Vasconcelos [33] (spherical modules). As a common generalization of the notion of a semidualizing module and that of a dualizing complex—in the sense of Hartshorne [18]—Christensen [7] introduced in 2001 the notion of a semidualizing complex, cf. (1.5).

Avramov and Foxby [1] and Christensen [7] demonstrated how a semidualizing complex C over a commutative noetherian ring R gives rise to two important classes of R -modules, namely the so-called Auslander class A_0^C and Bass class B_0^C , cf. (1.6). Semidualizing complexes and their Auslander and Bass classes have caught the attention of several authors, but this paper is motivated by a result of Enochs et al. [10], for which we prove the following generalization in Theorem (3.2).

Theorem A. *Let R be a commutative noetherian ring, and let C be a semidualizing complex of R -modules. Then the following conclusions hold:*

- (a) $(A_0^C, (A_0^C)^\perp)$ is a perfect cotorsion pair, in particular, the class A_0^C is covering. Furthermore, A_0^C is preenveloping.
- (b) The class B_0^C is covering and preenveloping.

Cotorsion pairs—introduced by Salce [28]—covering classes and preenveloping classes are central notions in relative homological algebra. We refer the reader to (1.2) and (1.3), and further to the monograph [12] by Enochs and Jenda, for relevant details about these notions.

Theorem A extends the main result of [10] in two directions: In [10], C is assumed to be a semidualizing *module* (as opposed to a semidualizing *complex*), and furthermore the covering property of B_0^C is new.

To prove Theorem A, we first establish Theorem (3.1) and then combine it with the fact that A_0^C and B_0^C are parts of appropriate *duality pairs*. The latter notion is introduced in Definition (2.1). The technique used to prove Theorem A applies to

2000 *Mathematics Subject Classification.* 13D05, 13D07, 18G25.

Key words and phrases. Auslander class, Bass class, cotorsion pair, cover, duality pair, Gorenstein flat dimension, Gorenstein injective dimension, preenvelope, semidualizing complex.

show that several other classes of modules are covering and/or preenveloping; see for example Theorems B below.

Now, assume that R has a dualizing complex D in the sense of Hartshorne [18], and consider a semidualizing R -module C . Then $C^\dagger = \mathbf{R}\mathrm{Hom}_R(C, D)$ is a semidualizing complex for which the associated Auslander and Bass classes can be characterized in terms of two homological dimensions introduced in [22], and studied further by Sather-Wagstaff, Sharif, and White [29, 30, 31, 32]. More precisely, for any R -module M one has the following equivalences:

$$M \in \mathbf{A}_0^{C^\dagger}(R) \iff C\text{-Gfd}_R M \leq \dim R,$$

$$M \in \mathbf{B}_0^{C^\dagger}(R) \iff C\text{-Gid}_R M \leq \dim R.$$

Here $C\text{-Gfd}_R M$ and $C\text{-Gid}_R M$ are the so-called C -Gorenstein flat and C -Gorenstein injective dimensions of M . Naturally, Theorem A applies to the semidualizing complex C^\dagger , but in view of the equivalences above, Theorem B below—which is a special case of Theorem (3.3)—gives more information.

Theorem B. *Let R be a commutative noetherian ring with a dualizing complex, and let $n \geq 0$ be an integer. Consider the following classes of R -modules:*

$$\mathrm{GF}_n^C = \{M \mid C\text{-Gfd}_R M \leq n\},$$

$$\mathrm{GI}_n^C = \{M \mid C\text{-Gid}_R M \leq n\}.$$

Then the next conclusions hold:

- (a) $(\mathrm{GF}_n^C, (\mathrm{GF}_n^C)^\perp)$ is a perfect cotorsion pair, in particular, GF_n^C is covering. Furthermore, GF_n^C is preenveloping.
- (b) The class GI_n^C is covering and preenveloping.

For $C = R$ and $n = 0$, Theorem B(a) asserts that the class of Gorenstein flat modules is the left half of a perfect cotorsion pair, and that it is preenveloping. The first of these results is proved by Enochs and López-Ramos [14, cor. 2.11] and the other follows immediately by combining [14, prop. 2.10, thm. 2.5, and rmk. 3] with [8, thm. 5.7]. For $C = R$ and $n = 0$, the second part of Theorem B(b) asserts that the class of Gorenstein injective modules is preenveloping. This is proved in [14, cor. 2.7]. Actually, Krause [24, thm. 7.12] proves the existence of special Gorenstein injective preenvelopes.

1. PRELIMINARIES

In this section we introduce our terminology and recall a few notions relevant for this paper.

(1.1) **Setup.** Throughout, R denotes a ring with identity, and R° its opposite ring. Unless otherwise specified, all modules under consideration are unitary left modules. Recall that a right R -module can be identified with a left R° -module. We write $\mathrm{Mod}(R)$ for the category of all (left) R -modules.

(1.2) **Covers and envelopes.** The following notions were coined by Enochs [9].

Let \mathbf{M} be any class of R -modules. An \mathbf{M} -precover of an R -module N is a homomorphism $\varphi: M \rightarrow N$, where M is in \mathbf{M} , with the property that for every homomorphism $\varphi': M' \rightarrow N$, where M' is in \mathbf{M} , there exists a (not necessarily unique)

homomorphism $\psi: M' \rightarrow M$ with $\varphi' = \varphi\psi$. An \mathbf{M} -precover $\varphi: M \rightarrow N$ is an \mathbf{M} -cover if every homomorphism $\psi: M \rightarrow M$ satisfying $\varphi = \varphi\psi$ is an automorphism. The class \mathbf{M} is called *(pre)covering* if every R -module has an \mathbf{M} -(pre)cover.

The notion of an \mathbf{M} -*(pre)envelope* is categorically dual to that of an \mathbf{M} -(pre)cover, and thus we will omit the definition here.

(1.3) **Cotorsion pairs.** For a class \mathbf{M} of R -modules one defines:

$$\begin{aligned} {}^\perp\mathbf{M} &= \{X \in \text{Mod}(R) \mid \text{Ext}_R^1(X, M) = 0 \text{ for all } M \in \mathbf{M}\}, \\ \mathbf{M}^\perp &= \{Y \in \text{Mod}(R) \mid \text{Ext}_R^1(M, Y) = 0 \text{ for all } M \in \mathbf{M}\}. \end{aligned}$$

A *cotorsion pair* is a pair (\mathbf{M}, \mathbf{N}) of classes of R -modules with $\mathbf{M} = {}^\perp\mathbf{N}$ and $\mathbf{M}^\perp = \mathbf{N}$. A cotorsion pair (\mathbf{M}, \mathbf{N}) is called *perfect* if \mathbf{M} is covering and \mathbf{N} is enveloping. These notions go back to Salce [28].

(1.4) **The derived category.** We denote by $\mathbf{D}(R)$ the *derived category* of the abelian category $\text{Mod}(R)$. We write $\mathbf{D}_b(R)$ for the full subcategory of $\mathbf{D}(R)$ whose objects have bounded homology. The right derived Hom functor and the left derived tensor product functor are denoted by $\mathbf{R}\text{Hom}_R(-, -)$ and $-\otimes_R^{\mathbf{L}}-$, respectively. The reader is referred to Weibel [34, chap. 10] for further details.

(1.5) **Semidualizing complexes.** The following is from Christensen [7, def. (2.1)].

Assume that R is commutative and noetherian. A complex $C \in \mathbf{D}_b(R)$ with degreewise finitely generated homology is *semidualizing* if the natural homothety morphism $R \rightarrow \mathbf{R}\text{Hom}_R(C, C)$ is an isomorphism in the derived category $\mathbf{D}(R)$.

(1.6) **Auslander and Bass classes.** Assume that R is commutative and noetherian, and let C be a semidualizing R -complex. The following definitions are due to Avramov and Foxby [1, (3.1)] and Christensen [7, def. (4.1)].

The *Auslander class* $A^C(R)$ consists of all $M \in \mathbf{D}_b(R)$ such that $C \otimes_R^{\mathbf{L}} M \in \mathbf{D}_b(R)$ and the canonical morphism $M \rightarrow \mathbf{R}\text{Hom}_R(C, C \otimes_R^{\mathbf{L}} M)$ in $\mathbf{D}(R)$ is an isomorphism.

The *Bass class* $B^C(R)$ consists of all $N \in \mathbf{D}_b(R)$ such that $\mathbf{R}\text{Hom}_R(C, N) \in \mathbf{D}_b(R)$ and the canonical morphism $C \otimes_R^{\mathbf{L}} \mathbf{R}\text{Hom}_R(C, N) \rightarrow N$ in $\mathbf{D}(R)$ is an isomorphism.

We write $A_0^C(R)$ and $B_0^C(R)$ —or simply A_0^C and B_0^C if the ground ring is understood—for the class of R -modules which, when considered as objects in $\mathbf{D}(R)$, belong to $A^C(R)$ and $B^C(R)$, respectively.

(1.7) **Remark.** If C is a semidualizing R -module then it is possible to define $A_0^C(R)$ and $B_0^C(R)$ directly in $\text{Mod}(R)$ without using $\mathbf{D}(R)$, cf. [7, obs. (4.10)].

(1.8) **Homological dimensions.** Let M be an arbitrary R -module.

We write $\text{fd}_R M$ and $\text{id}_R M$ for the *flat* and *injective* dimension of M . These classical notions go back to Cartan and Eilenberg [4].

We write $\text{Gfd}_R M$ and $\text{Gid}_R M$ for the *Gorenstein flat* and *Gorenstein injective* dimension of M . These notions were introduced by Enochs, Jenda et al. [11, 13] and have subsequently been studied by several authors.

When R is commutative and noetherian, the definitions of Enochs, Jenda et al. mentioned above have been extended in [22]: For a semidualizing R -module C , cf. (1.5), [22] introduces a *C-Gorenstein flat* dimension $C\text{-Gfd}_R M$ and a *C-Gorenstein injective* dimension $C\text{-Gid}_R M$. For $C = R$ these invariants agree with $\text{Gfd}_R M$ and $\text{Gid}_R M$, respectively. The C -Gorenstein dimensions have been studied by e.g. Sather-Wagstaff, Sharif, and White [29, 30, 31, 32].

(1.9) **Depth and width.** Assume that (R, \mathfrak{m}, k) is commutative noetherian local. The *depth* of a finitely generated R -module $M \neq 0$, that is, the length of a maximal M -regular sequence, can be computed as

$$\text{depth}_R M = \inf\{m \in \mathbb{Z} \mid \text{Ext}_R^m(k, M) \neq 0\}.$$

Foxby [16] defines the depth of an arbitrary R -module M by the equality above, and Yassemi [36] studies the dual notion of *width*, which is defined by

$$\text{width}_R M = \inf\{m \in \mathbb{Z} \mid \text{Tor}_m^R(k, M) \neq 0\}.$$

Note that $\text{depth}_R 0 = \text{width}_R 0 = \infty$.

2. DUALITY PAIRS

In this section we define duality pairs and give several examples. In the next section we will prove how suitable duality pairs induce cotorsion pairs. For unexplained notions and notation, the reader is referred to Section 1.

(2.1) **Definition.** A *duality pair* over R is a pair (\mathbf{M}, \mathbf{C}) , where \mathbf{M} is a class of R -modules and \mathbf{C} is a class of R° -modules, subject to the following conditions:

- (1) For an R -module M , one has $M \in \mathbf{M}$ if and only if $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \in \mathbf{C}$.
- (2) \mathbf{C} is closed under direct summands and finite direct sums.

A duality pair (\mathbf{M}, \mathbf{C}) is called *(co)product-closed* if the class \mathbf{M} is closed under (co)products in the category of all R -modules.

A duality pair (\mathbf{M}, \mathbf{C}) is called *perfect* if it is coproduct-closed, if \mathbf{M} is closed under extensions, and if R belongs to \mathbf{M} .

(2.2) **Example.** Consider for each integer $n \geq 0$ the following module classes:

$$\begin{aligned} \mathbf{F}_n &= \{M \in \text{Mod}(R) \mid \text{fd}_R M \leq n\}, \\ \mathbf{l}_n &= \{M \in \text{Mod}(R^\circ) \mid \text{id}_{R^\circ} M \leq n\}. \end{aligned}$$

The following examples of duality pairs are well-known.

- (a) $(\mathbf{F}_n, \mathbf{l}_n)$ is a perfect duality pair. If R is right coherent then this pair is product-closed by a classical result of Chase [5, thm. 2.1].
- (b) If R is right noetherian then $(\mathbf{l}_n, \mathbf{F}_n)$ is a product- and coproduct-closed duality pair (over R°), cf. Xu [35, lem. 3.1.4] and Bass [2, thm. 1.1].

(2.3) **Example.** Let \mathbf{B} be a class of finitely presented R -modules. Following Lenzing [26, §2] and [19, def. 2.3], we consider the class \mathbf{M} of modules with *support* in \mathbf{B} , and the class \mathbf{C} of modules with *cosupport* in \mathbf{B} defined by:

$$\begin{aligned} \mathbf{M} &= \varinjlim \mathbf{B}, \\ \mathbf{C} &= \text{Prod}\{\text{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \mid B \in \mathbf{B}\}. \end{aligned}$$

Then (\mathbf{M}, \mathbf{C}) is a coproduct-closed duality pair by [26, prop. 2.1] and [19, thm. 1.4]. For example, if \mathbf{B} is the class of all finitely generated projective R -modules, then $(\mathbf{M}, \mathbf{C}) = (\mathbf{F}_0, \mathbf{l}_0)$ by a classical result of Lazard [25].

(2.4) **Proposition.** Assume that R is commutative and noetherian, and let C be a semidualizing R -complex. Then one has:

- (a) $(\mathbf{A}_0^C, \mathbf{B}_0^C)$ is a perfect and product-closed duality pair.
- (b) $(\mathbf{B}_0^C, \mathbf{A}_0^C)$ is a product- and coproduct-closed duality pair.

Proof. That (A_0^C, B_0^C) and (B_0^C, A_0^C) are duality pairs follow from (the proof of) [6, lem. (3.2.9)]. That A_0^C and B_0^C are closed under products and coproducts follow from (the proof of) [8, lem. 5.6]. The class A_0^C clearly contains R , and it is closed under extensions by (the proof of) [6, lem. (3.1.13)]. \square

(2.5) **Lemma.** *Consider for each integer $n \geq 0$ the following module classes:*

$$\begin{aligned} \mathbf{GF}_n &= \{M \in \mathbf{Mod}(R) \mid \mathbf{Gfd}_R M \leq n\}, \\ \mathbf{GI}_n &= \{M \in \mathbf{Mod}(R^\circ) \mid \mathbf{Gid}_{R^\circ} M \leq n\}. \end{aligned}$$

Then the following conclusions hold:

- (a) *If R is right coherent then $(\mathbf{GF}_n, \mathbf{GI}_n)$ is a perfect duality pair. If R is commutative noetherian with a dualizing complex then this duality pair is product-closed.*
- (b) *If R is commutative noetherian with a dualizing complex then $(\mathbf{GI}_n, \mathbf{GF}_n)$ is a product- and coproduct-closed duality pair (over R°).*

Proof. (a): Since R is right coherent, it follows by [20, prop. 3.11] that the given pair is a duality pair. The class \mathbf{GF}_n is closed under coproducts by [20, prop. 3.13], and to see that \mathbf{GF}_n is closed under extensions one applies [20, thm. 3.14 and 3.15]. It is clear that R belongs to \mathbf{GF}_n .

If R is commutative and noetherian with a dualizing complex then \mathbf{GF}_n is closed under products by [8, thm. 5.7].

(b): Since R is commutative with a dualizing complex, it follows by (the proof of) [8, prop. 5.1] that the given pair is a duality pair. The class \mathbf{GI}_n is closed under products by [20, thm. 2.6], and it is closed under coproducts by [8, thm. 6.9]. \square

(2.6) **Proposition.** *Assume that R is commutative and noetherian, and let C be a semidualizing R -module. Consider for each $n \geq 0$ the following module classes:*

$$\begin{aligned} \mathbf{GF}_n^C &= \{M \in \mathbf{Mod}(R) \mid C\text{-Gfd}_R M \leq n\}, \\ \mathbf{GI}_n^C &= \{M \in \mathbf{Mod}(R) \mid C\text{-Gid}_R M \leq n\}. \end{aligned}$$

Then one has the following conclusions:

- (a) *$(\mathbf{GF}_n^C, \mathbf{GI}_n^C)$ is a perfect duality pair. If R has a dualizing complex then this duality pair is product-closed.*
- (b) *If R has a dualizing complex then $(\mathbf{GI}_n^C, \mathbf{GF}_n^C)$ is a product- and coproduct-closed duality pair.*

Proof. (a): We denote by $R \times C$ the trivial extension of R by C , cf. [3, §3.3]. For any R -module M , it follows by [22, thm. 2.16] that

$$(\dagger) \quad C\text{-Gfd}_R M = \mathbf{Gfd}_{R \times C} M \quad \text{and} \quad C\text{-Gid}_R M = \mathbf{Gid}_{R \times C} M.$$

Combining this with [20, prop. 3.11] one gets that

$$\begin{aligned} C\text{-Gid}_R \mathbf{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) &= \mathbf{Gid}_{R \times C} \mathbf{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \\ &= \mathbf{Gfd}_{R \times C} M \\ &= C\text{-Gfd}_R M, \end{aligned}$$

from which we conclude that $(\mathbf{GF}_n^C, \mathbf{GI}_n^C)$ is a duality pair. By Lemma (2.5)(a), the class $\mathbf{GF}_n(R \times C)$ is closed under coproducts and extensions; and combining this

with the first equality in (†), it follows that \mathbf{GF}_n^C is closed under coproducts and extensions as well. Also note that R belongs to \mathbf{GF}_n^C by [22, exa. 2.8(c)].

If R has a dualizing complex then so has $R \times C$, since it is a module finite extension of R . Hence $\mathbf{GF}_n(R \times C)$ is closed under products by Lemma (2.5)(a), and by the first equality in (†) we then conclude that \mathbf{GF}_n^C is closed under products.

(b): Similar to the proof of part (a), but using the second equality in (†) instead of the first, and using that for any R -module M one has:

$$\begin{aligned} C\text{-Gfd}_R \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) &= \operatorname{Gfd}_{R \times C} \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \\ &= \operatorname{Gid}_{R \times C} M \\ &= C\text{-Gid}_R M. \end{aligned}$$

The equalities above follow from (†) and (the proof of) [8, prop. 5.1]. \square

(2.7) **Proposition.** *Let (R, \mathfrak{m}, k) be commutative noetherian local. Consider for each integer $n \geq 0$ the following module classes:*

$$\begin{aligned} \mathbf{D}_n &= \{M \in \operatorname{Mod}(R) \mid \operatorname{depth}_R M \geq n\}, \\ \mathbf{W}_n &= \{M \in \operatorname{Mod}(R) \mid \operatorname{width}_R M \geq n\}. \end{aligned}$$

Then the following conclusions hold:

- (a) $(\mathbf{D}_n, \mathbf{W}_n)$ is a product- and coproduct-closed duality pair. If $n \leq \operatorname{depth} R$ then this duality pair is perfect.
- (b) $(\mathbf{W}_n, \mathbf{D}_n)$ is a product- and coproduct-closed duality pair.

Proof. For every R -module M one has:

$$\begin{aligned} \operatorname{depth}_R \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) &= \operatorname{width}_R M, \\ \operatorname{width}_R \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) &= \operatorname{depth}_R M, \end{aligned}$$

from which it follows that the pairs in (a) and (b) are duality pairs. It is trivial from the definitions of depth and width, cf. (1.9), that \mathbf{D}_n and \mathbf{W}_n are closed under products, coproducts, direct summands and extensions. \square

We end this section by noting that the next easily proved result can be applied to construct new duality pairs from existing ones.

(2.8) **Proposition.** *Let (M_μ, C_μ) be a family of duality pairs over R . Then their intersection $(\bigcap M_\mu, \bigcap C_\mu)$ is also a duality pair. Furthermore, the following hold:*

- (a) If each (M_μ, C_μ) is (co)product-closed then so is $(\bigcap M_\mu, \bigcap C_\mu)$.
- (b) If each (M_μ, C_μ) is perfect then so is $(\bigcap M_\mu, \bigcap C_\mu)$. \square

3. EXISTENCE OF PREENVELOPES, COVERS, AND COTORSION PAIRS

The main result of this section, Theorem (3.1), shows that the left half of a duality pair is “often” preenveloping and covering. We apply this result to a few of the duality pairs found in Section 2.

(3.1) **Theorem.** *Let (M, C) be a duality pair. Then M is closed under pure submodules, pure quotients, and pure extensions. Furthermore, the following hold:*

- (a) If (M, C) is product-closed then M is preenveloping.
- (b) If (M, C) is coproduct-closed then M is covering.
- (c) If (M, C) is perfect then (M, M^\perp) is a perfect cotorsion pair.

Proof. First we prove that \mathbf{M} is closed under pure submodules, pure quotients, and pure extensions, that is, given a pure exact sequence of R -modules,

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

then M is in \mathbf{M} if and only if M', M'' are in \mathbf{M} . Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ to the sequence above, we get by Jensen and Lenzing [23, thm. 6.4] a split exact sequence,

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M'', \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M', \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

By (2.1)(2) it follows that $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is in \mathbf{C} if and only if $\mathrm{Hom}_{\mathbb{Z}}(M', \mathbb{Q}/\mathbb{Z})$ and $\mathrm{Hom}_{\mathbb{Z}}(M'', \mathbb{Q}/\mathbb{Z})$ both are in \mathbf{C} . The desired conclusion now follows by (2.1)(1).

(a): We have proved that \mathbf{M} is closed under pure submodules. Since \mathbf{M} is also closed under products by assumption, it follows by Rada and Saorín [27, cor. 3.5(c)] that \mathbf{M} is preenveloping.

(b): We have proved that \mathbf{M} is closed under pure quotients. By assumption, \mathbf{M} is also closed under coproducts, and therefore it follows by [21, thm. 2.5] that \mathbf{M} is covering.

(c): We have proved that \mathbf{M} is closed under pure submodules and pure quotients. By assumption, \mathbf{M} is also closed under coproducts and extensions, and R belongs to \mathbf{M} . Thus [21, thm. 3.4] implies that $(\mathbf{M}, \mathbf{M}^{\perp})$ is a perfect cotorsion pair. \square

As mentioned in the introduction, in the case where C is a semidualizing *module* (as opposed to as semidualizing *complex*), the following result—except the first assertion in part (b)—is proved by Enochs et al. [10].

(3.2) Theorem. *Assume that R is commutative and noetherian, and let C be a semidualizing R -complex. Then the following conclusions hold:*

- (a) $(\mathbf{A}_0^C, (\mathbf{A}_0^C)^{\perp})$ is a perfect cotorsion pair, in particular, the class \mathbf{A}_0^C is covering. Furthermore, \mathbf{A}_0^C is preenveloping.
- (b) The class \mathbf{B}_0^C is covering and preenveloping.

Proof. (a): By Proposition (2.4)(a), the class \mathbf{A}_0^C is the left half of a perfect and product-closed duality pair. Thus the conclusions follow from Theorem (3.1)(c,a).

(b): By Proposition (2.4)(b), the class \mathbf{B}_0^C is the left half of a product- and coproduct-closed duality pair. The conclusions follow from Theorem (3.1)(a,b). \square

(3.3) Theorem. *Assume that R is commutative and noetherian, let C be a semidualizing R -module, and let $n \geq 0$ be an integer. Then one has:*

- (a) $(\mathbf{GF}_n^C, (\mathbf{GF}_n^C)^{\perp})$ is a perfect cotorsion pair, in particular, \mathbf{GF}_n^C is covering. If, in addition, R has a dualizing complex then \mathbf{GF}_n^C is preenveloping.
- (b) If R has a dualizing complex then \mathbf{GI}_n^C is covering and preenveloping.

Proof. (a): By Proposition (2.6)(a), the class \mathbf{GF}_n^C is the left half of a perfect duality pair. Thus the claimed perfect cotorsion pair exists by Theorem (3.1)(c). Under the assumption of the existence of a dualizing R -complex, \mathbf{GF}_n^C is also product-closed by Proposition (2.6)(a), and therefore \mathbf{GF}_n^C is preenveloping by Theorem (3.1)(a).

(b): If R has a dualizing complex, Proposition (2.6)(b) gives that \mathbf{GI}_n^C is the left half of a product- and coproduct-closed duality pair. Thus the assertions follow from Theorem (3.1)(b,a). \square

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