

# CLUSTER TILTING VS. WEAK CLUSTER TILTING IN DYNKIN TYPE A INFINITY

THORSTEN HOLM AND PETER JØRGENSEN

ABSTRACT. This paper shows a new phenomenon in higher cluster tilting theory. For each positive integer  $d$ , we exhibit a triangulated category  $\mathcal{C}$  with the following properties.

On one hand, the  $d$ -cluster tilting subcategories of  $\mathcal{C}$  have very simple mutation behaviour: Each indecomposable object has exactly  $d$  mutations. On the other hand, the weakly  $d$ -cluster tilting subcategories of  $\mathcal{C}$  which lack functorial finiteness can have much more complicated mutation behaviour: For each  $0 \leq \ell \leq d - 1$ , we show a weakly  $d$ -cluster tilting subcategory  $\mathcal{T}_\ell$  which has an indecomposable object with precisely  $\ell$  mutations.

The category  $\mathcal{C}$  is the algebraic triangulated category generated by a  $(d + 1)$ -spherical object and can be thought of as a higher cluster category of Dynkin type  $A_\infty$ .

## 0. INTRODUCTION

This paper shows a new phenomenon in higher cluster tilting theory. For each integer  $d \geq 1$ , we exhibit a triangulated category  $\mathcal{C}$  whose  $d$ -cluster tilting subcategories have *very simple* mutation behaviour, but whose weakly  $d$ -cluster tilting subcategories can have *much more complicated* mutation behaviour which we can control precisely.

To make sense of this, recall that if  $\mathcal{T}$  is a full subcategory of a triangulated category, then  $\mathcal{T}$  is called *weakly  $d$ -cluster tilting* if it satisfies the following conditions where  $\Sigma$  is the suspension functor.

$$\begin{aligned} t \in \mathcal{T} &\Leftrightarrow \operatorname{Hom}(\mathcal{T}, \Sigma t) = \cdots = \operatorname{Hom}(\mathcal{T}, \Sigma^d t) = 0, \\ t \in \mathcal{T} &\Leftrightarrow \operatorname{Hom}(t, \Sigma \mathcal{T}) = \cdots = \operatorname{Hom}(t, \Sigma^d \mathcal{T}) = 0. \end{aligned}$$

If  $\mathcal{T}$  is also left- and right-approximating in the ambient category in the sense of Remark 2.3, then it is called  *$d$ -cluster tilting*. These definitions are due to Iyama [6] and have given rise to an extensive homological theory, see for instance [1] and [7]. Note that if  $\mathcal{T} = \operatorname{add} t$  for an object  $t$ , then  $\mathcal{T}$  is automatically left- and right-approximating, but we will study subcategories which are not of this form since they have infinitely many isomorphism classes of indecomposable objects.

One remarkable property of  $d$ -cluster tilting theory is *mutation*. If  $t \in \mathcal{T}$  is an indecomposable object, then it is sometimes possible to remove  $t$  from  $\mathcal{T}$  and insert an indecomposable object  $t^* \not\cong t$  in such a way that the subcategory remains (weakly)  $d$ -cluster tilting. This is called *mutation of  $\mathcal{T}$  at  $t$* , see [7, sec. 5].

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2010 *Mathematics Subject Classification.* 13F60, 16G20, 16G70, 18E30.

*Key words and phrases.* Auslander-Reiten quiver,  $d$ -Calabi-Yau category,  $d$ -cluster tilting subcategory, Fomin-Zelevinsky mutation, functorial finiteness, left-approximating subcategory, right-approximating subcategory, spherical object, weakly  $d$ -cluster tilting subcategory.

In good cases, there are exactly  $d$  different choices of  $t^*$  up to isomorphism. That is, there are  $d$  ways of mutating  $\mathbb{T}$  at  $t$ , see [7, sec. 5].

To be more precise, one hopes(!) that this happens for  $d$ -cluster tilting subcategories. Indeed, it does happen for  $d = 1$  by [7, thm. 5.3], but can fail for  $d \geq 2$ , see [7, thms. 9.3 and 10.2]. The situation for weakly  $d$ -cluster tilting subcategories is less clear.

We can now explain the opening paragraph of the paper. Let us first define  $\mathbb{C}$  which, as we will explain below, can be thought of as a  $d$ -cluster category of type  $A_\infty$ .

**Definition 0.1.** For the rest of the paper,  $k$  is an algebraically closed field,  $d \geq 1$  is an integer, and  $\mathbb{C}$  is a  $k$ -linear algebraic triangulated category which is idempotent complete and classically generated by a  $(d + 1)$ -spherical object  $s$ ; that is,

$$\dim_k \mathbb{C}(s, \Sigma^\ell s) = \begin{cases} 1 & \text{for } \ell = 0, d + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $\mathbb{C}(-, -)$  is short for the Hom functor in  $\mathbb{C}$ .

We prove the following three theorems about  $\mathbb{C}$ , where Theorems A and B show *very simple*, respectively *much more complicated* mutation behaviour.

**Theorem A.** *Let  $\mathbb{T}$  be a  $d$ -cluster tilting subcategory of  $\mathbb{C}$  and let  $t \in \mathbb{T}$  be indecomposable. Then  $\mathbb{T}$  can be mutated at  $t$  in precisely  $d$  ways.*

**Theorem B.** *Let  $0 \leq \ell \leq d - 1$  be given. Then there exists a weakly  $d$ -cluster tilting subcategory  $\mathbb{T}_\ell$  of  $\mathbb{C}$  with an indecomposable object  $t$  such that  $\mathbb{T}_\ell$  can be mutated at  $t$  in precisely  $\ell$  ways.*

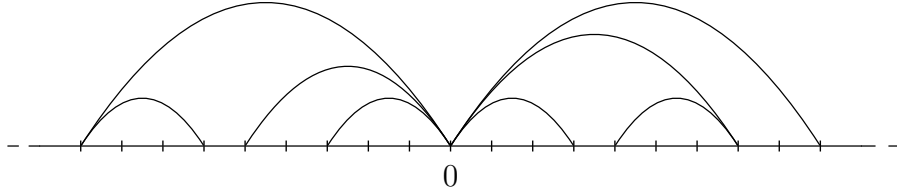
**Theorem C.** *Let  $\mathbb{T}$  be a weakly  $d$ -cluster tilting subcategory of  $\mathbb{C}$  and let  $t \in \mathbb{T}$  be indecomposable. Then  $\mathbb{T}$  can be mutated at  $t$  in at most  $d$  ways.*

The interest of Theorems A and C depends on a rich supply of (weakly)  $d$ -cluster tilting subcategories in  $\mathbb{C}$ . Indeed, such a supply exists by the following two theorems. As a prelude, note that there is a bijection between subcategories  $\mathbb{T} \subseteq \mathbb{C}$  closed under direct sums and summands, and sets of  $d$ -admissible arcs  $\mathfrak{T}$ ; see Section 1, in particular Proposition 1.4. A  $d$ -admissible arc is an arc connecting two integers  $t, u$  with  $u - t \geq 2$  and  $u - t \equiv 1 \pmod{d}$ .

**Theorem D.** *The subcategory  $\mathbb{T}$  is weakly  $d$ -cluster tilting if and only if the corresponding set of  $d$ -admissible arcs  $\mathfrak{T}$  is a  $(d + 2)$ -angulation of the  $\infty$ -gon.*

**Theorem E.** *The subcategory  $\mathbb{T}$  is  $d$ -cluster tilting if and only if the corresponding set of  $d$ -admissible arcs  $\mathfrak{T}$  is a  $(d + 2)$ -angulation of the  $\infty$ -gon which is either locally finite or has a fountain.*

We defer the definition of “ $(d + 2)$ -angulation of the  $\infty$ -gon” and other unexplained notions to Definition 1.3 and merely bring Figure 1 which shows part of a 4-angulation of the  $\infty$ -gon with a fountain at 0. Note how the arcs divide the upper half plane into a collection of ‘quadrangular’ regions, each with four integers as ‘vertices’. Some of the vertices sit at cusps.

FIGURE 1. Part of a 4-angulation of the  $\infty$ -gon.

We end the introduction with a few remarks about the category  $\mathbf{C}$  which has been studied intensively in a number of recent papers [2], [4], [5], [8], [9], [11], [13]. It is determined up to triangulated equivalence by [9, thm. 2.1]. It is a Krull-Schmidt and  $(d + 1)$ -Calabi-Yau category by [5, rmk. 1 and prop. 1.8], and a number of other properties can be found in [5, secs. 1 and 2]. Theorems A and E are two reasons for viewing  $\mathbf{C}$  as a cluster category of type  $A_\infty$ , since they are infinite versions of the corresponding theorems in type  $A_n$ ; see [12, thm. 3] for Theorem A and [10, prop. 2.13] and [12, thm. 1] for Theorem E. See also [4] for the case  $d = 1$ .

The paper is organised as follows: Section 1 introduces  $d$ -admissible arcs into the study of the triangulated category  $\mathbf{C}$  and proves Theorem D. Section 2 proves Theorem E. Section 3 shows some technical results on  $(d + 2)$ -angulations of the  $\infty$ -gon. Section 4 proves Theorems A, B, and C.

**Notation 0.2.** We write  $\text{ind}(\mathbf{C})$  for the set of isomorphism classes of indecomposable objects in  $\mathbf{C}$ . We will follow the custom of being lax about the distinction between *indecomposable objects* and *isomorphism classes of indecomposable objects*. This makes the language a bit less precise, but avoids excessive elaborations.

The word *subcategory* will always mean *full subcategory closed under isomorphisms, direct sums, and direct summands*. In particular, a subcategory is determined by the indecomposable objects it contains.

## 1. THE ARC PICTURE OF $\mathbf{C}$

**Remark 1.1.** By [5, prop. 1.10], the Auslander-Reiten (AR) quiver of  $\mathbf{C}$  consists of  $d$  components, each of which is a copy of  $\mathbb{Z}A_\infty$ , and  $\Sigma$  acts cyclically on the set of components.

**Construction 1.2.** We pick a component of the AR quiver of  $\mathbf{C}$  and impose the coordinate system in Figure 2. We think of coordinate pairs as indecomposable objects of  $\mathbf{C}$ , and extend the coordinate system to the other components of the quiver by setting

$$\Sigma(t, u) = (t - 1, u - 1). \quad (1)$$

By [5, prop. 1.8], the Serre functor of  $\mathbf{C}$  is  $S = \Sigma^{d+1}$ . The actions of  $S$  and the AR translation  $\tau = S\Sigma^{-1}$  are given on objects by

$$S(t, u) = (t - d - 1, u - d - 1), \quad \tau(t, u) = (t - d, u - d). \quad (2)$$

Like  $\Sigma$ , the Serre functor  $S$  acts cyclically on the set of components of the AR quiver. Indeed, since there are  $d$  components, the two functors have the same action on the set

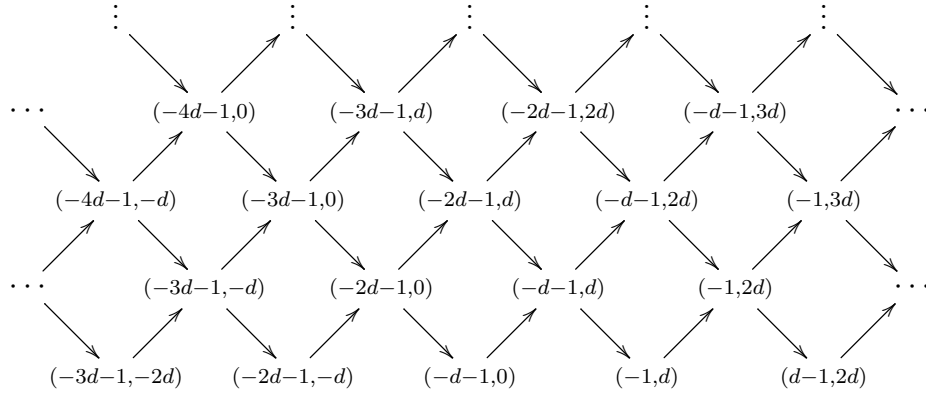


FIGURE 2. The coordinate system on one of the components of the AR quiver of  $\mathbf{C}$ .

of components. The AR translation  $\tau$  is given on each component of the AR quiver by moving one vertex to the left.

We also think of the coordinate pair  $(t, u)$  as an arc connecting the integers  $t$  and  $u$ . The ensuing geometrical picture is illustrated by Figure 1. However, not all values of  $(t, u)$  are possible. Indeed, it is easy to check that the coordinate pairs which occur in Construction 1.2 are precisely the  $d$ -admissible arcs in the following definition.

**Definition 1.3.** A pair of integers  $(t, u)$  with  $u - t \geq 2$  and  $u - t \equiv 1 \pmod{d}$  is called a  $d$ -admissible arc.

The *length* of the arc  $(t, u)$  is  $u - t$ .

The arcs  $(r, s)$  and  $(t, u)$  *cross* if  $r < t < s < u$  or  $t < r < u < s$ . Moreover,  $(r, s)$  is an *overarc* of  $(t, u)$  if  $(r, s) \neq (t, u)$  and  $r \leq t < u \leq s$ .

Let  $\mathfrak{A}$  be a set of  $d$ -admissible arcs.

We say that  $\mathfrak{A}$  is a  $(d + 2)$ -angulation of the  $\infty$ -gon if it is a maximal set of pairwise non-crossing  $d$ -admissible arcs.

We say that  $\mathfrak{A}$  is *locally finite* if, for each integer  $t$ , there are only finitely many arcs of the form  $(s, t)$  and  $(t, u)$  in  $\mathfrak{A}$ .

An integer  $t$  is a *left-fountain* of  $\mathfrak{A}$  if  $\mathfrak{A}$  contains infinitely many arcs of the form  $(s, t)$ , and  $t$  is a *right-fountain* of  $\mathfrak{A}$  if  $\mathfrak{A}$  contains infinitely many arcs of the form  $(t, u)$ . We say that  $t$  is a *fountain* of  $\mathfrak{A}$  if it is both a left- and a right-fountain of  $\mathfrak{A}$ .

The first part of the following proposition is a consequence of what we did above. The second part follows from the first because our subcategories are determined by the indecomposable objects they contain, see Notation 0.2.

**Proposition 1.4.** *Construction 1.2 gives a bijective correspondence between  $\text{ind}(\mathbf{C})$  and the set of  $d$ -admissible arcs.*

*This extends to a bijective correspondence between (i) subcategories of  $\mathbf{C}$  and (ii) subsets of the set of  $d$ -admissible arcs.*

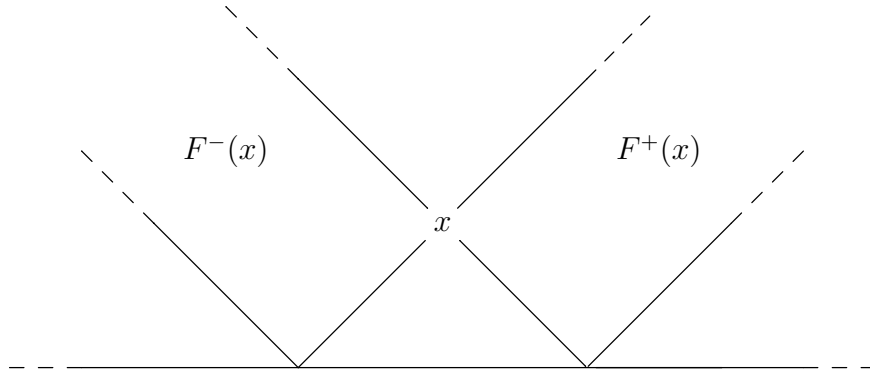


FIGURE 3. The sets  $F^\pm(x)$ .

**Definition 1.5.** Let  $x \in \text{ind}(\mathbf{C})$  be given. Figure 3 defines two infinite sets  $F^\pm(x)$  consisting of vertices in the same component of the AR quiver as  $x$ . Each set contains  $x$  and all other vertices inside the indicated boundaries; the boundaries are included in the sets.

Recall that  $S = \Sigma^{d+1}$  is the Serre functor of  $\mathbf{C}$ .

**Proposition 1.6.** Let  $x, y \in \text{ind}(\mathbf{C})$ . Then

$$\dim_k \mathbf{C}(x, y) = \begin{cases} 1 & \text{for } y \in F^+(x) \cup F^-(Sx), \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* See [5, prop. 2.2]. □

In other words,  $x$  has non-zero maps to a region  $F^+(x)$  in the same component of the AR quiver as itself, and to a region  $F^-(Sx)$  in the “next” component of the AR quiver. Note that if  $d = 1$  then the quiver has only one component so  $F^-(Sx)$  is in the same component as  $x$ .

**Remark 1.7.** It is not hard to check that  $y \in F^+(x) \Leftrightarrow x \in F^-(y)$ . So the proposition is equivalent to

$$\dim_k \mathbf{C}(x, y) = \begin{cases} 1 & \text{for } x \in F^+(S^{-1}y) \cup F^-(y), \\ 0 & \text{otherwise.} \end{cases}$$

The following proposition is simple but crucial since it leads straight to Theorem D.

**Proposition 1.8.** Let  $\mathfrak{x}, \mathfrak{y}$  be  $d$ -admissible arcs corresponding to  $x, y \in \text{ind}(\mathbf{C})$ . Then  $\mathfrak{x}$  and  $\mathfrak{y}$  cross if and only if at least one of the Hom-spaces

$$\mathbf{C}(x, \Sigma^1 y) , \dots , \mathbf{C}(x, \Sigma^d y)$$

is non-zero.

*Proof.* For  $1 \leq \ell \leq d$ , the condition that  $\mathbf{C}(x, \Sigma^\ell y) \neq 0$  is equivalent to  $\Sigma^\ell y \in F^+(x)$  or  $\Sigma^\ell y \in F^-(Sx)$  by Proposition 1.6. If we write  $\mathfrak{x} = (r, s)$ ,  $\mathfrak{y} = (t, u)$ , then, using

equations (1) and (2) and the coordinate system on the AR quiver of  $\mathbf{C}$ , it is elementary to check that

$$\Sigma^\ell y \in F^+(x) \Leftrightarrow \begin{cases} u \equiv s + \ell \pmod{d}, \\ r + \ell \leq t \leq s + \ell - d - 1, \\ s + \ell \leq u, \end{cases} \quad (3)$$

$$\Sigma^\ell y \in F^-(Sx) \Leftrightarrow \begin{cases} u \equiv s + \ell - 1 \pmod{d}, \\ t \leq r + \ell - d - 1, \\ r + \ell \leq u \leq s + \ell - d - 1. \end{cases} \quad (4)$$

The condition that at least one of the Hom spaces  $\mathbf{C}(x, \Sigma^1 y), \dots, \mathbf{C}(x, \Sigma^d y)$  is non-zero is hence equivalent to the existence of at least one  $\ell$  with  $1 \leq \ell \leq d$  such that the right hand side of (3) or (4) is true. It is again elementary to check that this is equivalent to the condition that  $\mathfrak{x} = (r, s)$  and  $\mathfrak{y} = (t, u)$  cross.  $\square$

**Proof of Theorem D.** Combine the definition of weakly  $d$ -cluster tilting subcategories with Propositions 1.4 and 1.8.  $\square$

## 2. LEFT- AND RIGHT-APPROXIMATING SUBCATEGORIES

**Proposition 2.1.** *Let  $x, y \in \text{ind}(\mathbf{C})$  be such that  $y \in F^+(x)$ .*

- (i) *Each morphism  $x \rightarrow y$  is a scalar multiple of a composition of irreducible morphisms.*
- (ii) *A morphism  $x \rightarrow y$  which is a composition of irreducible morphisms is non-zero.*

*Keeping  $x, y$  as above, let  $z \in \text{ind}(\mathbf{C})$  be such that  $z \in F^+(x) \cap F^+(y)$ .*

- (iii) *Non-zero morphisms  $x \rightarrow y, y \rightarrow z$  compose to a non-zero morphism  $x \rightarrow z$ .*
- (iv) *If  $y \xrightarrow{\psi} z$  is a non-zero morphism, then each morphism  $x \rightarrow z$  factors as  $x \rightarrow y \xrightarrow{\psi} z$ .*

*Proof.* (i) Let  $x \xrightarrow{\varphi} y$  be a morphism. If  $y \cong x$  then it follows from Proposition 1.6 that  $\varphi$  is a scalar multiple of the identity, and then we can take the claimed composition of irreducible morphisms to be empty.

If  $y \not\cong x$ , then let  $\tau y \rightarrow y_1 \xrightarrow{\theta} y$  be the AR triangle ending in  $y$ . Since  $x, y$  are indecomposable, the morphism  $\varphi$  is not a split epimorphism so it factors as  $x \rightarrow y_1 \xrightarrow{\theta} y$ .

We can repeat this factorization process for the direct summands of  $y_1$  to which  $x$  has non-zero morphisms, that is, the direct summands of  $y_1$  which are in the rectangle  $R$  shown in Figure 4; cf. Proposition 1.6. Successive repetitions show that the morphism  $\varphi$  is a linear combination of compositions of irreducible morphisms within  $R$ . However, the mesh relations imply that any two such compositions are scalar multiples of each other, so  $\varphi$  is a composition of irreducible morphisms.

(ii) By Proposition 1.6 there is a non-zero morphism  $x \rightarrow y$ . By part (i), it is a scalar multiple of a composition of irreducible morphisms. But as remarked in the proof of

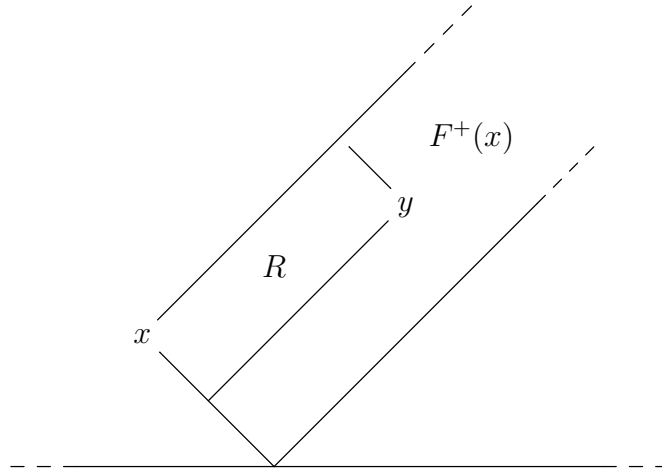


FIGURE 4. The rectangle  $R$  spanned by  $x$  and  $y$ .

part (i), two morphisms  $x \rightarrow y$  which are both compositions of irreducible morphisms are scalar multiples of each other, so it follows that any such composition is non-zero.

(iii) By part (i), each of the morphisms  $x \rightarrow y$  and  $y \rightarrow z$  is a scalar multiple of a composition of irreducible morphisms, so the same is true for the composition  $x \rightarrow z$ . But  $z$  is in  $F^+(x)$ , so  $x \rightarrow z$  is non-zero by part (ii).

(iv) By Proposition 1.6 there is a non-zero morphism  $x \xrightarrow{\varphi} y$ . The composition  $x \xrightarrow{\psi\varphi} z$  is non-zero by part (iii). But the space  $\mathbf{C}(x, z)$  is 1-dimensional by Proposition 1.6, so any morphism  $x \rightarrow z$  can be factored as  $\psi \circ \alpha\varphi$  with  $\alpha$  a scalar.  $\square$

**Proposition 2.2.** *Let  $x, y, z \in \text{ind}(\mathbf{C})$  be such that  $x \in F^+(S^{-1}y) \cap F^+(S^{-1}z)$  and  $z \in F^+(y)$ .*

*If  $y \xrightarrow{\psi} z$  is a non-zero morphism, then each morphism  $x \rightarrow z$  factors as  $x \rightarrow y \xrightarrow{\psi} z$ .*

*Proof.* We must show that  $\mathbf{C}(x, \psi) : \mathbf{C}(x, y) \rightarrow \mathbf{C}(x, z)$  is surjective. By Serre duality, it is equivalent to show that  $\mathbf{C}(\psi, Sx) : \mathbf{C}(z, Sx) \rightarrow \mathbf{C}(y, Sx)$  is injective. For this it is enough to show that  $\mathbf{C}(\psi, Sx)$  is non-zero, since the Hom spaces  $\mathbf{C}(z, Sx)$  and  $\mathbf{C}(y, Sx)$  have dimension 0 or 1 over the ground field  $k$  by Proposition 1.6.

We must hence show that if  $z \rightarrow Sx$  is non-zero, then so is the composition  $y \xrightarrow{\psi} z \rightarrow Sx$ . And this holds by Proposition 2.1(iii) since we have  $z \in F^+(y)$  and  $Sx \in F^+(y) \cap F^+(z)$ ; the latter condition holds because it is equivalent to the assumption  $x \in F^+(S^{-1}y) \cap F^+(S^{-1}z)$ .  $\square$

**Remark 2.3.** Recall that if  $\mathbf{S}$  is a subcategory of  $\mathbf{C}$  and  $x \in \mathbf{C}$  is an object, then a right- $\mathbf{S}$ -approximation of  $x$  is a morphism  $s \xrightarrow{\sigma} x$  with  $s \in \mathbf{S}$  such that each morphism  $s' \rightarrow x$  with  $s' \in \mathbf{S}$  factors through  $\sigma$ .

If each  $x \in \mathbf{C}$  has a right- $\mathbf{S}$ -approximation, then  $\mathbf{S}$  is called right-approximating. There are dual notions with “left” instead of “right”.

The following is a generalization of [4, thm. 4.4] and [11, thm. 2.2], and we follow the proofs of those results.

**Proposition 2.4.** *Let  $\mathbf{S}$  be a subcategory of  $\mathbf{C}$  and let  $\mathfrak{S}$  be the corresponding set of  $d$ -admissible arcs. The following conditions are equivalent.*

- (i) *The subcategory  $\mathbf{S}$  is right-approximating.*
- (ii) *Each right-fountain of  $\mathfrak{S}$  is a left-fountain of  $\mathfrak{S}$ .*

*Proof.* For  $d = 1$  this is [11, thm. 2.2] so assume  $d \geq 2$ .

Recall the notion of a slice: If  $(t, u)$  is a vertex on the base line of the AR quiver of  $\mathbf{C}$ , then the slice starting at  $(t, u)$  is  $(t, *)$ ; that is, it consists of the vertices with coordinates of the form  $(t, u')$ . The slice ending at  $(t, u)$  is  $(*, u)$ .

This means that  $t \in \mathbb{Z}$  is a right-fountain of  $\mathfrak{S}$  if and only if  $\mathbf{S}$  has infinitely many indecomposable objects on the slice  $(t, *)$  starting at  $(t, t + d + 1)$ . Likewise,  $t$  is a left-fountain of  $\mathfrak{S}$  if and only if  $\mathbf{S}$  has infinitely many indecomposable objects on the slice  $(*, t)$  ending at  $(t - d - 1, t) = S(t, t + d + 1)$ . Hence (ii) is equivalent to the following condition on  $\mathbf{S}$ .

- (ii') Let  $v \in \text{ind}(\mathbf{C})$  be on the base line of the AR quiver of  $\mathbf{C}$ . If  $\mathbf{S}$  has infinitely many indecomposable objects on the slice starting at  $v$ , then it has infinitely many indecomposable objects on the slice ending at  $Sv$ .

Strictly speaking, we should say “infinitely many isomorphism classes of indecomposable objects” but as mentioned in Notation 0.2 we are lax about this.

(i)  $\Rightarrow$  (ii'). Let  $v \in \text{ind}(\mathbf{C})$  be on the base line of the AR quiver. Note that  $v$  and  $Sv$  are in different components of the AR quiver since there are  $d \geq 2$  components and  $S$  moves vertices to the “next” component; cf. Construction 1.2. Figure 5 shows the components of the quiver containing  $v$  and  $Sv$ . As indicated,  $b$  is the slice starting at  $v$  and  $a$  the slice ending at  $Sv$ . Assume that (i) holds and that  $\text{ind}(\mathbf{S}) \cap b$  is infinite. To show (ii'), we must show that  $\text{ind}(\mathbf{S}) \cap a$  is infinite.

Let  $z$  be an indecomposable object on  $a$  with right- $\mathbf{S}$ -approximation  $s \xrightarrow{\sigma} z$ . If  $s_1$  is an object on  $b$  then as shown by outlines in the figure we have  $z \in F^-(Ss_1)$ . Hence there is a non-zero morphism  $s_1 \rightarrow z$  by Proposition 1.6. So each of the infinitely many objects in  $\text{ind}(\mathbf{S}) \cap b$  has a non-zero morphism to  $z$ , and each such morphism factors through  $\sigma$  because  $\sigma$  is a right- $\mathbf{S}$ -approximation. Since  $\mathbf{C}$  is a Krull-Schmidt category, this implies that there is an indecomposable direct summand  $s'$  of  $s$  such that the component  $s' \xrightarrow{\sigma'} z$  of  $\sigma$  is non-zero and such that there are infinitely many objects  $s_1, s_2, s_3, \dots$  in  $\text{ind}(\mathbf{S}) \cap b$  which have non-zero morphisms to  $s'$ . Note that  $s' \in \mathbf{S}$  since  $\mathbf{S}$  is closed under direct summands by assumption.

We claim that this forces  $s'$  to be on  $a$ , higher up than  $z$ . Hence, by moving  $z$  upwards we obtain infinitely many objects in  $\text{ind}(\mathbf{S}) \cap a$ .

To prove the claim, note that since  $s' \xrightarrow{\sigma'} z$  is non-zero, Remark 1.7 gives  $s' \in F^+(S^{-1}z) \cup F^-(z)$ . The sets  $F^+(S^{-1}z)$  and  $F^-(z)$  are outlined in Figure 5. And  $s' \in F^+(S^{-1}z)$  is impossible because there would not be infinitely many objects in

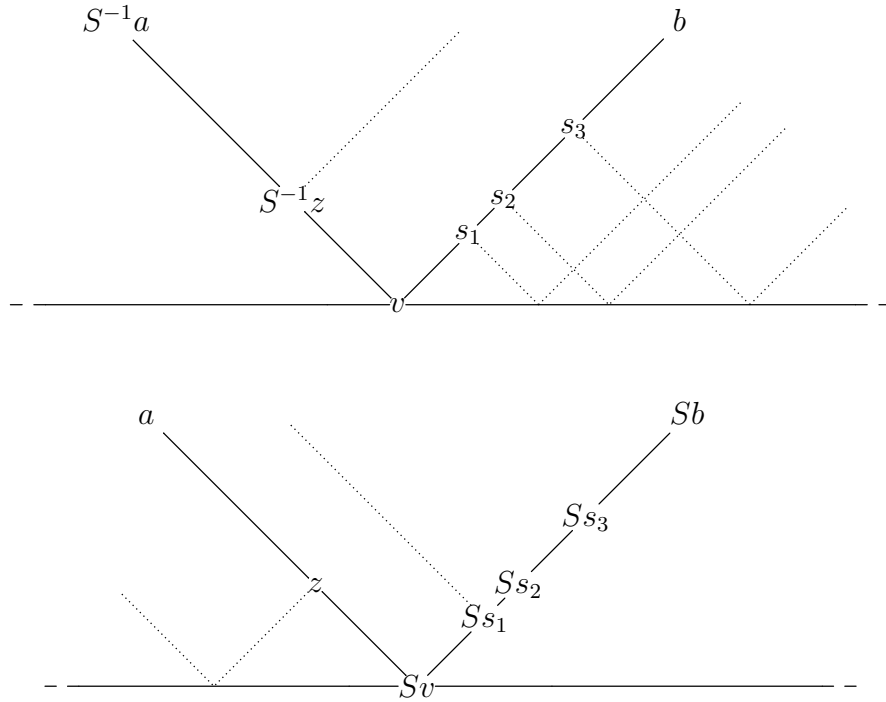


FIGURE 5. Objects in two components of the AR quiver.

$\text{ind}(\mathbf{S}) \cap b$  with a non-zero morphism to  $s'$ , as one sees by considering the sets  $F^+(s_i)$  which are also outlined in the figure.

So we have  $s' \in F^-(z)$ . We already know  $s' \in F^-(Ss_i)$  for each  $i$ . Hence, as one sees in Figure 5, we have  $s'$  on  $a$ . Finally, since there is a non-zero morphism  $s' \xrightarrow{\sigma'} z$ , it follows that  $s'$  is higher up on  $a$  than  $z$ .

(ii')  $\Rightarrow$  (i). Assume that (ii') holds and that  $z \in \text{ind}(\mathbf{C})$  is given. We will show (i) by constructing a right- $\mathbf{S}$ -approximation  $s \xrightarrow{\sigma} z$ . We must ensure that each morphism  $s' \rightarrow z$  with  $s' \in \mathbf{S}$  factors through  $\sigma$ , and we will do so by considering the possibilities for  $s'$  and building up  $\sigma$  accordingly.

We only need to consider those  $s' \in \text{ind}(\mathbf{S})$  which have non-zero morphisms to  $z$ . By Remark 1.7 there are the cases  $s' \in F^-(z)$  and  $s' \in F^+(S^{-1}z)$ , see Figure 6. Note that  $z$  and  $S^{-1}z$  are in different components of the AR quiver; cf. the previous part of the proof.

First, assume  $s' \in \text{ind}(\mathbf{S}) \cap F^-(z)$ . The slice  $a$  in Figure 6 determines a half line  $F^-(z) \cap a$ . If there are objects of  $\mathbf{S}$  on this half line, then let  $s_a$  be the one which is closest to the base line of the quiver and let  $s_a \xrightarrow{\sigma_a} z$  be a non-zero morphism. If  $s'$  is on  $a$  then it is above  $s_a$  and Proposition 2.1(iv) implies that each morphism  $s' \rightarrow z$  factors through  $\sigma_a$ . There are only finitely many slices  $a$  intersecting  $F^-(z)$ . Including the corresponding morphisms  $\sigma_a$  as components of  $\sigma$  ensures that each morphism  $s' \rightarrow z$  with  $s' \in \text{ind}(\mathbf{S}) \cap F^-(z)$  factors through  $\sigma$ .

Secondly, assume  $s' \in \text{ind}(\mathbf{S}) \cap F^+(S^{-1}z)$ . The slice  $b$  in Figure 6 determines a half line  $F^+(S^{-1}z) \cap b$ , and we split into two cases.

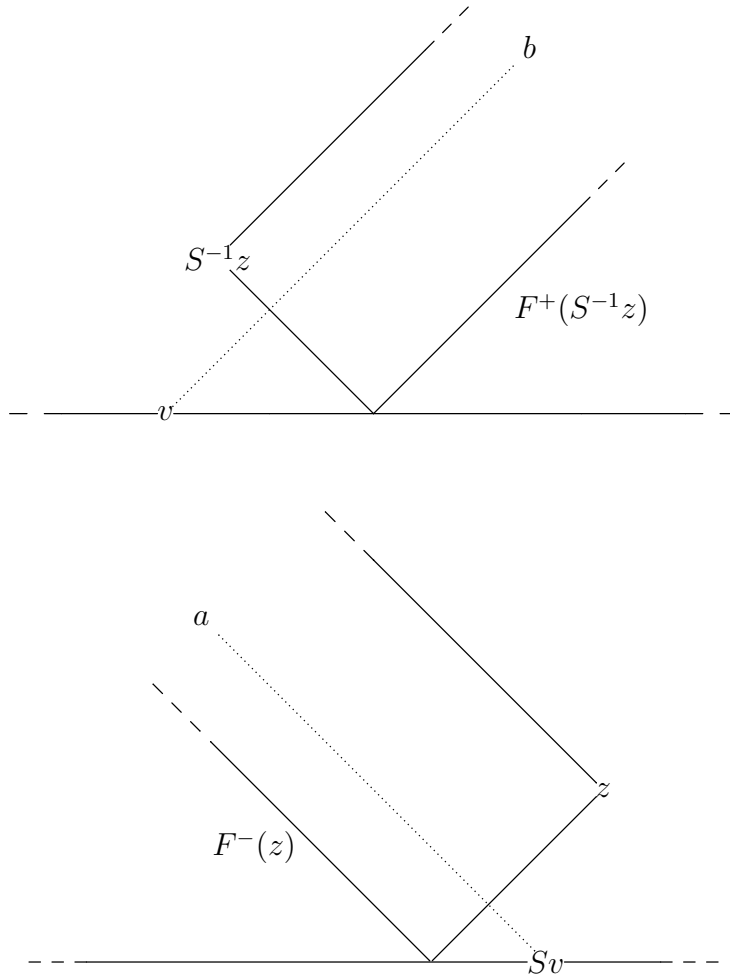


FIGURE 6. Another view of objects in two components of the AR quiver.

The case where  $\mathbf{S}$  has finitely many objects on  $F^+(S^{-1}z) \cap b$ : Let  $s_b$  be the direct sum of these objects and let each component of the morphism  $s_b \xrightarrow{\sigma_b} z$  be non-zero. If  $s'$  is on  $b$ , then  $s'$  is one of the direct summands of  $s_b$  and each morphism  $s' \rightarrow z$  factors through  $\sigma_b$  since each non-zero Hom space in  $\mathbf{C}$  is 1-dimensional.

The case where  $\mathbf{S}$  has infinitely many objects on  $F^+(S^{-1}z) \cap b$ : Then  $\text{ind}(\mathbf{S}) \cap b$  is infinite. If  $b$  is the slice starting at  $v$  and  $a$  the slice ending at  $Sv$ , then condition (ii') says that  $\text{ind}(\mathbf{S}) \cap a$  is also infinite. In particular it is non-empty so we have already included the non-zero morphism  $s_a \xrightarrow{\sigma_a} z$  as a component of  $\sigma$  in the previous part of the proof. If  $s'$  is on  $b$  then it is straightforward to use Proposition 2.2 to check that each morphism  $s' \rightarrow z$  factors through  $\sigma_a$ .

As above, there are only finitely many slices  $b$  intersecting  $F^+(S^{-1}z)$ . Including the relevant morphisms  $\sigma_b$  as components of  $\sigma$  ensures that each morphism  $s' \rightarrow z$  with  $s' \in \text{ind}(\mathbf{S}) \cap F^+(S^{-1}z)$  factors through  $\sigma$ .  $\square$

A similar proof establishes the following dual result.

**Proposition 2.5.** *Let  $\mathbf{S}$  be a subcategory of  $\mathbf{C}$  and let  $\mathfrak{S}$  be the corresponding set of  $d$ -admissible arcs. The following conditions are equivalent.*

- (i) *The subcategory  $\mathcal{S}$  is left-approximating.*
- (ii) *Each left-fountain of  $\mathcal{S}$  is a right-fountain of  $\mathcal{S}$ .*

**Proof of Theorem E.** Given a subcategory  $\mathcal{T}$  of  $\mathcal{C}$  and the corresponding set of  $d$ -admissible arcs  $\mathfrak{T}$ , Theorem D says that  $\mathcal{T}$  is weakly  $d$ -cluster tilting if and only if  $\mathfrak{T}$  is a  $(d + 2)$ -angulation of the  $\infty$ -gon. It is not hard to see that since  $\mathfrak{T}$  is a set of pairwise non-crossing  $d$ -admissible arcs, it is locally finite or has a fountain if and only if it satisfies conditions (ii) in Propositions 2.4 and 2.5. By the propositions, this happens if and only if  $\mathcal{T}$  is left- and right-approximating.  $\square$

### 3. ARC COMBINATORICS

**Construction 3.1.** Let  $\mathfrak{T}$  be a  $(d+2)$ -angulation of the  $\infty$ -gon and let  $p_0 \in \mathbb{Z}$  be given. We define integers  $p_1, p_2, \dots$  inductively as follows: If  $p_\ell$  has already been defined, then

- if  $\mathfrak{T}$  contains no arcs of the form  $(p_\ell, q)$ , then let  $p_{\ell+1} = p_\ell + 1$ ;
- if  $\mathfrak{T}$  contains a non-zero, finite number of arcs of the form  $(p_\ell, q)$ , then let  $(p_\ell, p_{\ell+1})$  be the one with maximal length;
- if  $\mathfrak{T}$  contains infinitely many arcs of the form  $(p_\ell, q)$ , that is, if  $p_\ell$  is a right-fountain of  $\mathfrak{T}$ , then stop the algorithm and do not define  $p_{\ell+1}$ .

If the algorithm stops, then it defines a sequence with finitely many elements,

$$p_0 < \dots < p_m.$$

If it does not stop, then it defines a sequence with infinitely many elements,

$$p_0 < p_1 < \dots,$$

and we set  $m = \infty$ . Let us sum up the properties of the sequence.

- (i) If  $m < \infty$ , then  $p_m$  is a right-fountain of  $\mathfrak{T}$ .
- (ii)  $(p_\ell, p_{\ell+1})$  is either a pair of consecutive integers or an arc in  $\mathfrak{T}$ .
- (iii)  $p_\ell - p_0 \equiv \ell \pmod{d}$ .

To see (iii), note that the length of a  $d$ -admissible arc is  $\equiv 1 \pmod{d}$ .

Collin Bleak proved that a triangulation of the  $\infty$ -gon has a left-fountain if and only if it has a right-fountain, and his method also works for  $(d + 2)$ -angulations. We thank him for permitting us to bring a proof of the following lemma which establishes the “only if” direction. “If” follows by symmetry. See also [3, lem. 4.11].

**Lemma 3.2.** *Let  $\mathfrak{T}$  be a  $(d + 2)$ -angulation of the  $\infty$ -gon. Suppose that  $p_0$  is a left-fountain of  $\mathfrak{T}$  and perform Construction 3.1.*

- (i) *The construction gives a finite sequence  $p_0 < \dots < p_m$  with  $0 \leq m \leq d$ .*
- (ii)  *$p_m$  is a right-fountain of  $\mathfrak{T}$ .*
- (iii) *Let  $\mathfrak{t} \in \mathfrak{T}$  and assume  $\mathfrak{t} \neq (p_\ell, p_{\ell+1})$  for  $\ell \in \{0, \dots, m - 1\}$ . Then  $\mathfrak{t}$  has an overarc  $\mathfrak{r} \in \mathfrak{T}$ .*

*Proof.* (i) Assume to the contrary that  $m \geq d+1$ ; this includes the possibility  $m = \infty$ . Construction 3.1(iii) shows that  $(p_0, p_{d+1})$  is a  $d$ -admissible arc. If  $(p_0, p_1)$  is a  $d$ -admissible arc, then  $(p_0, p_{d+1})$  has strictly greater length than  $(p_0, p_1)$  and by Construction 3.1 we have  $(p_0, p_{d+1}) \notin \mathfrak{T}$ . If  $(p_0, p_1)$  is not a  $d$ -admissible arc, then by Construction 3.1 there are no arcs in  $\mathfrak{T}$  of the form  $(p_0, q)$  so we have  $(p_0, p_{d+1}) \notin \mathfrak{T}$  again. In either case there must be an arc  $(r, s) \in \mathfrak{T}$  which crosses  $(p_0, p_{d+1})$ , that is,  $r < p_0 < s < p_{d+1}$  or  $p_0 < r < p_{d+1} < s$ . But  $p_0$  is a left-fountain of  $\mathfrak{T}$  so  $r < p_0 < s < p_{d+1}$  is impossible since it would imply that  $(r, s)$  crossed an arc in  $\mathfrak{T}$ .

We must therefore have  $p_0 < r < p_{d+1} < s$ . However, this also leads to a contradiction: We cannot have  $p_\ell < r < p_{\ell+1}$  for any  $\ell \in \{0, \dots, d\}$ , for if we did then  $(p_\ell, p_{\ell+1})$  would not be consecutive integers whence  $(p_\ell, p_{\ell+1}) \in \mathfrak{T}$  by Construction 3.1(ii), but this arc would cross  $(r, s) \in \mathfrak{T}$ . So we must have  $r = p_\ell$  for an  $\ell \in \{1, \dots, d\}$ . Hence  $\mathfrak{T}$  contains arcs of the form  $(p_\ell, q)$ , and by Construction 3.1 the one with maximal length is  $(p_\ell, p_{\ell+1})$ . But this contradicts  $(r, s) \in \mathfrak{T}$  because we know  $r = p_\ell$  and  $p_{\ell+1} \leq p_{d+1} < s$ .

(ii) See Construction 3.1(i).

(iii) Let us write  $\mathbf{t} = (t, u)$  and search for  $\mathbf{r}$ .

Since  $p_0$  and  $p_m$  are a left-fountain and a right-fountain of  $\mathfrak{T}$ , we must have  $t < u \leq p_0$  or  $p_0 \leq t < u \leq p_m$  or  $p_m \leq t < u$ .

If  $t < u \leq p_0$ , then we can choose  $\mathbf{r} = (r, p_0) \in \mathfrak{T}$  with  $r < t$ . If  $p_m \leq t < u$ , then we can choose  $\mathbf{r} = (p_m, s) \in \mathfrak{T}$  with  $u < s$ .

Now assume  $p_0 \leq t < u \leq p_m$ .

If  $t = p_\ell$  for an  $\ell \in \{0, \dots, m-1\}$ , then  $\mathbf{t} \in \mathfrak{T}$  is an arc of the form  $(p_\ell, q)$ . Among the arcs in  $\mathfrak{T}$  of this form, by Construction 3.1 the one with maximal length is  $(p_\ell, p_{\ell+1})$ . Since  $\mathbf{t} \neq (p_\ell, p_{\ell+1})$  by assumption, we get that  $\mathbf{r} = (p_\ell, p_{\ell+1}) \in \mathfrak{T}$  is an overarc of  $\mathbf{t}$ .

If  $p_\ell < t < p_{\ell+1}$  for an  $\ell \in \{0, \dots, m-1\}$ , then we must have  $u \leq p_{\ell+1}$ , since otherwise  $(t, u) \in \mathfrak{T}$  and  $(p_\ell, p_{\ell+1}) \in \mathfrak{T}$  would cross. But then  $\mathbf{r} = (p_\ell, p_{\ell+1}) \in \mathfrak{T}$  is again an overarc of  $\mathbf{t} = (t, u)$ .  $\square$

**Lemma 3.3.** *Let  $\mathfrak{T}$  be a  $(d+2)$ -angulation of the  $\infty$ -gon. Then  $\mathfrak{T}$  is either locally finite or has precisely one left-fountain and one right-fountain.*

*Proof.* If  $\mathfrak{T}$  is not locally finite, then it has a left- or a right-fountain. By Lemma 3.2(ii) and its mirror image, it has both a left- and a right-fountain. And it is easy to see that in any event, it has at most one left- and at most one right-fountain.  $\square$

**Lemma 3.4.** *Let  $\mathfrak{T}$  be a  $(d+2)$ -angulation of the  $\infty$ -gon which is locally finite or has a fountain. Then each arc  $\mathbf{t} \in \mathfrak{T}$  has an overarc  $\mathbf{r} \in \mathfrak{T}$ .*

*Proof.* The case where  $\mathfrak{T}$  has a fountain at  $p_0$ : Then we must have  $m = 0$  in Lemma 3.2, and Lemma 3.2(iii) implies the present result.

The case where  $\mathfrak{T}$  is locally finite: Let us write  $\mathbf{t} = (t, u)$  and search for  $\mathbf{r}$ . We can assume that, among the arcs in  $\mathfrak{T}$  of the form  $(t, v)$ , the one of maximal length is  $(t, u)$ , since otherwise there is obviously an overarc. Let  $p_0 = t$  and perform Construction 3.1; then  $(p_0, p_1) = (t, u)$ . Since  $\mathfrak{T}$  is locally finite, it has no right-fountain, so Construction

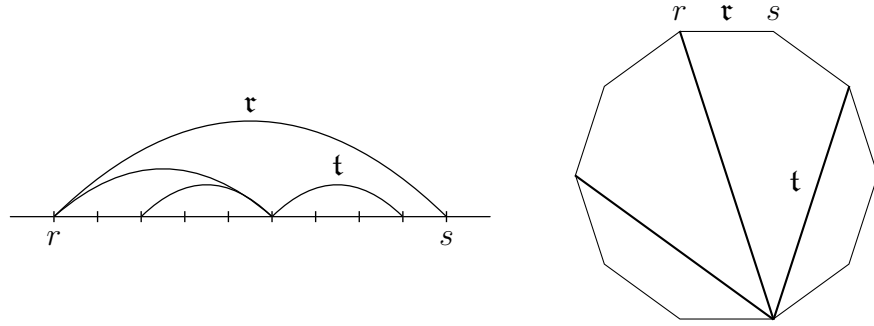


FIGURE 7. If  $\tau = (r, s)$  is a  $d$ -admissible arc then  $r, r + 1, \dots, s$  can be viewed as the vertices of a polygon  $R$ . If  $t = (t, u)$  has  $\tau$  as an overarc, then  $t$  can be viewed as a  $d$ -admissible diagonal of  $R$ .

3.1(i) implies  $m = \infty$ . Construction 3.1(iii) implies that  $(p_0, p_{d+1})$  is a  $d$ -admissible arc. It has strictly greater length than  $(p_0, p_1)$ , so  $(p_0, p_{d+1}) \notin \mathfrak{T}$  follows.

There must hence be an arc  $(r, s) \in \mathfrak{T}$  which crosses  $(p_0, p_{d+1})$ , that is,  $r < p_0 < s < p_{d+1}$  or  $p_0 < r < p_{d+1} < s$ .

First, assume  $r < p_0 < s < p_{d+1}$ . Note that we cannot have  $p_0 < s < p_1$  since then  $(r, s) \in \mathfrak{T}$  and  $(p_0, p_1) \in \mathfrak{T}$  would cross. So  $p_1 \leq s$  whence  $\tau = (r, s) \in \mathfrak{T}$  is an overarc of  $(p_0, p_1) = (t, u)$ .

Secondly, assume  $p_0 < r < p_{d+1} < s$ . This leads to a contradiction in the same way as in the second paragraph of the proof of Lemma 3.2.  $\square$

**Construction 3.5.** Let  $\mathfrak{T}$  be a  $(d + 2)$ -angulation of the  $\infty$ -gon and let  $\tau = (r, s) \in \mathfrak{T}$ . We can view  $\{r, \dots, s\}$  as the vertices of an  $(s - r + 1)$ -gon  $R$ . Each pair  $(r, r + 1), (r + 1, r + 2), \dots, (s - 1, s)$  is viewed as an edge of  $R$ , and so is the arc  $(r, s)$ ; that is,  $r$  and  $s$  are viewed as consecutive vertices of  $R$ . Each  $d$ -admissible arc  $t$  of which  $\tau = (r, s)$  is an overarc is viewed as a  $d$ -admissible diagonal of  $R$ . See Figure 7. In particular, the set

$$\mathfrak{R} = \{t \in \mathfrak{T} \mid \tau \text{ is an overarc of } t\}$$

is a  $(d + 2)$ -angulation of  $R$ .

Observe that  $\mathfrak{R}$  divides  $R$  into  $(d + 2)$ -gons, and that one of these  $(d + 2)$ -gons, say  $T$ , has  $r$  and  $s$  among its vertices. We can write the whole set of vertices of  $T$  as

$$r < t_1 < \dots < t_d < s,$$

and hence each of

$$(r, t_1), (t_1, t_2), \dots, (t_{d-1}, t_d), (t_d, s)$$

is either a pair of consecutive integers or a diagonal in  $\mathfrak{R}$ , that is, an arc in  $\mathfrak{T}$ .

**Lemma 3.6.** *Let  $\mathfrak{T}$  be a  $(d + 2)$ -angulation of the  $\infty$ -gon and let  $t \in \mathfrak{T}$ .*

- (i) *If  $\mathfrak{U}$  is a set of  $d$ -admissible arcs not in  $\mathfrak{T} \setminus t$  such that  $(\mathfrak{T} \setminus t) \cup \mathfrak{U}$  is a  $(d + 2)$ -angulation of the  $\infty$ -gon, then  $\mathfrak{U} = \{t^*\}$  for a single  $d$ -admissible arc  $t^*$ .*
- (ii) *If  $t$  has an overarc in  $\mathfrak{T}$  then there are  $d + 1$  choices of  $t^*$ .*
- (iii) *If  $t$  has no overarc in  $\mathfrak{T}$  then there are  $\leq d$  choices of  $t^*$ .*

*Proof.* Suppose that  $\mathfrak{t}$  has the overarc  $\mathfrak{r} \in \mathfrak{T}$ . We will establish (i) and (ii) for  $\mathfrak{t}$ .

The set of all arcs in  $\mathfrak{T}$  of which  $\mathfrak{r} \in \mathfrak{T}$  is an overarc can be viewed as a  $(d+2)$ -angulation  $\mathfrak{R}$  of a polygon  $R$  by Construction 3.5. When  $(\mathfrak{T} \setminus \mathfrak{t}) \cup \mathfrak{U}$  is a  $(d+2)$ -angulation of the  $\infty$ -gon,  $\mathfrak{r}$  is an overarc of each arc in  $\mathfrak{U}$  since removing  $\mathfrak{t}$  does not create any room above its overarc  $\mathfrak{r}$ . It follows that  $\mathfrak{U}$  can be viewed as a set of  $d$ -admissible diagonals of  $R$  such that  $(\mathfrak{R} \setminus \mathfrak{t}) \cup \mathfrak{U}$  is a  $(d+2)$ -angulation of  $R$ . Then it is well known that, as desired,  $\mathfrak{U}$  has one element which can be chosen in  $d+1$  different ways.

Now suppose that  $\mathfrak{t}$  has no overarc in  $\mathfrak{T}$ . We will establish (i) and (iii) for  $\mathfrak{t}$ .

Lemma 3.4 shows that  $\mathfrak{T}$  is not locally finite and does not have a fountain. Lemma 3.3 shows that  $\mathfrak{T}$  has a left-fountain  $p_0$  which is not a right-fountain. We can perform Construction 3.1. By Lemma 3.2(i+ii) this gives a sequence  $p_0 < \dots < p_m$  with  $m \leq d$  where  $p_m$  is a right-fountain of  $\mathfrak{T}$ . Note that  $1 \leq m$  since  $p_0$  is not a right-fountain. By Construction 3.1(ii), each  $(p_\ell, p_{\ell+1})$  is either a pair of consecutive integers or an arc in  $\mathfrak{T}$ , and it follows from Lemma 3.2(iii) that  $\mathfrak{t} = (p_j, p_{j+1})$  for a  $j \in \{0, \dots, m-1\}$ .

By Construction 3.5 applied to  $\mathfrak{t} = (p_j, p_{j+1})$ , there is a sequence of integers  $p_j < q_1 < \dots < q_d < p_{j+1}$  such that each of  $(p_j, q_1)$ ,  $(q_1, q_2)$ ,  $\dots$ ,  $(q_{d-1}, q_d)$ ,  $(q_d, p_{j+1})$  is either a pair of consecutive integers or an arc in  $\mathfrak{T}$ . We hence have a sequence of integers

$$p_0 < p_1 < \dots < p_j < q_1 < \dots < q_d < p_{j+1} < \dots < p_m \quad (5)$$

where each pair of neighbouring elements is either a pair of consecutive integers or an arc in  $\mathfrak{T} \setminus \mathfrak{t}$ . In particular, each pair of neighbouring elements has a difference which is  $\equiv 1 \pmod{d}$ .

Now consider a  $d$ -admissible arc  $\mathfrak{t}^* = (v, w) \notin \mathfrak{T} \setminus \mathfrak{t}$  which crosses no arc in  $\mathfrak{T} \setminus \mathfrak{t}$ .

We cannot have  $w \leq p_0$ . For if we did, then  $\mathfrak{t}^* = (v, w)$  would not cross  $\mathfrak{t} = (p_j, p_{j+1})$ , and hence  $\mathfrak{t}^*$  would cross no arc in  $\mathfrak{T}$  whence  $\mathfrak{t}^* \in \mathfrak{T}$ . Since  $\mathfrak{t}^* \notin \mathfrak{T} \setminus \mathfrak{t}$ , this would force  $\mathfrak{t}^* = \mathfrak{t}$ , but this contradicts  $v < w \leq p_0 \leq p_j$ . We also cannot have  $v < p_0 < w$  because  $p_0$  is a left-fountain of  $\mathfrak{T}$ . Similarly, we cannot have  $p_m \leq v$  or  $v < p_m < w$ .

We conclude that  $p_0 \leq v < w \leq p_m$ . We claim that, in fact,  $v$  and  $w$  must be among the elements of the sequence (5).

Namely, assume that at least one of  $v$  and  $w$  is not an element of the sequence. Then it is strictly between two such elements. For the sake of argument, say  $q_\ell < v < q_{\ell+1}$ . Then we cannot have  $q_{\ell+1} < w$ , for then  $\mathfrak{t}^* = (v, w)$  and  $(q_\ell, q_{\ell+1}) \in \mathfrak{T} \setminus \mathfrak{t}$  would cross. So we must have  $v < w \leq q_{\ell+1}$ . Hence  $(q_\ell, q_{\ell+1})$  is an overarc of  $\mathfrak{t}^* = (v, w)$ .

However, the set of all arcs in  $\mathfrak{T}$  of which  $(q_\ell, q_{\ell+1}) \in \mathfrak{T} \setminus \mathfrak{t}$  is an overarc can be viewed as a  $(d+2)$ -angulation  $\mathfrak{R}'$  of a polygon  $R'$  by Construction 3.5, and  $\mathfrak{t}^*$  can be viewed as a  $d$ -admissible diagonal of this polygon. Note that  $(q_\ell, q_{\ell+1})$  is not an overarc of  $\mathfrak{t} = (p_j, p_{j+1})$  and so  $\mathfrak{R}' \subseteq \mathfrak{T} \setminus \mathfrak{t}$ . Hence the assumption that  $\mathfrak{t}^*$  crosses none of the arcs in  $\mathfrak{T} \setminus \mathfrak{t}$  means that it crosses none of the diagonals in  $\mathfrak{R}'$ . But then  $\mathfrak{t}^* \in \mathfrak{R}'$  whence  $\mathfrak{t}^* \in \mathfrak{T} \setminus \mathfrak{t}$  which is a contradiction.

So we have  $\mathfrak{t}^* = (v, w)$  with  $v, w$  elements in the sequence (5). However, we saw that each pair of neighbouring elements in this sequence has a difference which is  $\equiv 1 \pmod{d}$ . Hence, for  $\mathfrak{t}^*$  to be a  $d$ -admissible arc,  $v$  and  $w$  must either be neighbours in

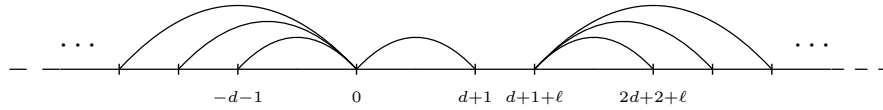


FIGURE 8. A  $(d + 2)$ -angulation  $\mathfrak{T}_\ell$  of the  $\infty$ -gon where the arc  $\mathbf{t} = (0, d + 1)$  can be replaced in  $\ell$  ways.

the sequence, or  $nd + 1$  steps apart for an integer  $n \geq 1$ . But they cannot be neighbours for then we would have  $\mathbf{t}^* \in \mathfrak{T} \setminus \mathbf{t}$ , so  $v$  and  $w$  must be  $nd + 1$  steps apart in the sequence.

Since  $m \leq d$  by Lemma 3.2(i), the sequence (5) has  $m + 1 + d \leq 2d + 1$  elements. It follows that  $n = 1$  and hence two different choices of  $\mathbf{t}^*$  must cross each other; this shows part (i) of the present lemma. It also follows that there are at most  $(2d + 1) - (d + 1) = d$  different choices for  $\mathbf{t}^*$ , showing part (iii) of the present lemma.  $\square$

#### 4. PROOFS OF THEOREMS A, B, AND C

**Remark 4.1.** Let  $\mathbb{T}$  be a weakly  $d$ -cluster tilting subcategory of the triangulated category  $\mathbb{C}$  and let  $\mathfrak{T}$  be the corresponding  $(d + 2)$ -angulation of the  $\infty$ -gon.

Lemma 3.6(i) says that if we drop one arc from  $\mathfrak{T}$ , then we must add precisely one other  $d$ -admissible arc to get a new  $(d + 2)$ -angulation.

So if we drop one indecomposable object from  $\mathbb{T}$ , then we must add precisely one other indecomposable object to get a new weakly  $d$ -cluster tilting subcategory of  $\mathbb{C}$ .

That is, “mutating  $\mathbb{T}$  at  $t$ ” has the expected effect of replacing  $t$  by a single other indecomposable object.

**Proof of Theorem A.** By Theorem E which was already proved in Section 2, a  $d$ -cluster tilting subcategory  $\mathbb{T}$  of  $\mathbb{C}$  corresponds to a  $(d + 2)$ -angulation of the  $\infty$ -gon  $\mathfrak{T}$  which is locally finite or has a fountain.

By Lemma 3.4, each  $\mathbf{t} \in \mathfrak{T}$  has an overarc  $\mathbf{r} \in \mathfrak{T}$ .

By Lemma 3.6(ii), this means that there are  $d + 1$  different choices of a  $d$ -admissible arc  $\mathbf{t}^*$  such that  $(\mathfrak{T} \setminus \mathbf{t}) \cup \mathbf{t}^*$  is a  $(d + 2)$ -angulation of the  $\infty$ -gon.

Excluding the trivial choice  $\mathbf{t}^* = \mathbf{t}$  leaves  $d$  choices for  $\mathbf{t}^*$  and translating back to  $\mathbb{T}$  shows Theorem A.  $\square$

**Proof of Theorem B.** Let  $\ell \in \{0, \dots, d - 1\}$  be given. Figure 8 shows part of a  $(d + 2)$ -angulation  $\mathfrak{T}_\ell$  of the  $\infty$ -gon. It contains the arc  $\mathbf{t} = (0, d + 1)$  and has a left-fountain at 0 and a right-fountain at  $d + 1 + \ell$ .

There are  $\ell + 1$  different choices of a  $d$ -admissible arc  $\mathbf{t}^*$  such that  $(\mathfrak{T} \setminus \mathbf{t}) \cup \mathbf{t}^*$  is a  $(d + 2)$ -angulation of the  $\infty$ -gon; namely,  $\mathbf{t}^* = (p, p + d + 1)$  for  $p \in \{0, \dots, \ell\}$ .

Excluding the trivial choice  $\mathbf{t}^* = \mathbf{t}$  leaves  $\ell$  choices. By Theorem D which was already proved in Section 2, the  $(d + 2)$ -angulation  $\mathfrak{T}_\ell$  therefore corresponds to a weakly  $d$ -cluster tilting subcategory  $\mathbb{T}_\ell$  with the property claimed in Theorem B.  $\square$

**Proof of Theorem C.** Similar to the proof of Theorem A, but  $t \in \mathfrak{T}$  may or may not have an overarc, so both parts (ii) and (iii) of Lemma 3.6 are needed.  $\square$

**Acknowledgement.** We are grateful to Collin Bleak for permitting us to bring a proof that a  $(d+2)$ -angulation of the  $\infty$ -gon has a left-fountain if and only if it has a right-fountain, see Lemmas 3.2 and 3.3. Collin Bleak originally proved this for  $d = 1$ .

This work was supported by grant number HO 1880/4-1 under the research priority programme SPP 1388 *Darstellungstheorie* of the Deutsche Forschungsgemeinschaft (DFG).

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INSTITUT FÜR ALGEBRA, ZAHLENTHEORIE UND DISKRETE MATHEMATIK, FAKULTÄT FÜR MATHEMATIK UND PHYSIK, LEIBNIZ UNIVERSITÄT HANNOVER, WELFENGARTEN 1, 30167 HANNOVER, GERMANY

*E-mail address:* [holm@math.uni-hannover.de](mailto:holm@math.uni-hannover.de)

*URL:* <http://www.iazd.uni-hannover.de/~tholm>

SCHOOL OF MATHEMATICS AND STATISTICS, NEWCASTLE UNIVERSITY, NEWCASTLE UPON TYNE NE1 7RU, UNITED KINGDOM

*E-mail address:* [peter.jorgensen@ncl.ac.uk](mailto:peter.jorgensen@ncl.ac.uk)

*URL:* <http://www.staff.ncl.ac.uk/peter.jorgensen>