

HOMOLOGICAL PROPERTIES OF COCHAIN DIFFERENTIAL GRADED ALGEBRAS

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ABSTRACT. Consider a local chain Differential Graded algebra, such as the singular chain complex of a pathwise connected topological group.

In two previous papers, a number of homological results were proved for such an algebra: An Amplitude Inequality, an Auslander-Buchsbaum Equality, and a Gap Theorem. These were inspired by homological ring theory.

By the so-called looking glass principle, one would expect that analogous results exist for simply connected cochain Differential Graded algebras, such as the singular cochain complex of a simply connected topological space.

Indeed, this paper establishes such analogous results.

0. INTRODUCTION

This paper is a sequel of [5] and [6], or, more accurately, their mirror image. The papers [5] and [6] investigated the homological properties of local chain Differential Graded algebras, such as the singular chain complex of a pathwise connected topological group. Several results modelled on ring theory were proved: An Amplitude Inequality, an Auslander-Buchsbaum Equality, and a Gap Theorem for Bass numbers.

In this paper, we shall do the same thing for simply connected cochain Differential Graded algebras, such as the singular cochain complex of a simply connected topological space. The resulting cochain Auslander-Buchsbaum Equality and Gap Theorem, established in Theorems 4.7 and 4.11 below, are new. A cochain Amplitude Inequality can be found already in [10, prop. 3.11]; the proof of our Theorem 4.4 works by different methods.

2000 *Mathematics Subject Classification.* Primary 16E45; Secondary 55P62.

Key words and phrases. Amplitude Inequality, Auslander-Buchsbaum Equality, Differential Graded modules, Gap Theorem, homological dimensions, homological identities, singular cochain complexes, topological spaces.

We are grateful to the referee for pointing out that the recent paper [9] contains similar results for the (not necessarily simply) connected case. For introductions to the theory of Differential Graded (DG) algebras, we refer the reader to [1], [4], or [8].

The move from local chain to simply connected cochain DG algebras is a general phenomenon: The so-called “looking glass principle” of [2] states that each result on local chain DG algebras should have a “mirror image” for simply connected cochain DG algebras. However, proofs cannot be translated in a mechanical way.

Since a local chain DG algebra sits in non-positive cohomological degrees and a simply connected cochain DG algebra sits in non-negative cohomological degrees, the simplest statement of the looking glass principle is that it interchanges positive and negative degrees. For instance, over local chain DG algebras, in [5] we used the notion of k -projective dimension of a DG module given by

$$\mathrm{pd} M = \sup\{j \mid H_j(k \overset{L}{\otimes}_R M) \neq 0\}.$$

The looking glass principle tells us that over simply connected cochain DG algebras, we must replace this by

$$\mathrm{pcd} M = \sup\{j \mid H_{-j}(k \overset{L}{\otimes}_R M) \neq 0\} = \sup\{j \mid H^j(k \overset{L}{\otimes}_R M) \neq 0\}$$

which will be called the *projective codimension* of M . Indeed, for local chain DG algebras, the Auslander-Buchsbaum Equality is the formula

$$\mathrm{pd} M = \mathrm{depth} R - \mathrm{depth} M$$

for k -projective dimension, see [5, thm. 2.3], and for simply connected cochain DG algebras, one of its incarnations is the formula

$$\mathrm{pcd} M = \sup M - \sup R$$

for projective codimension, see Theorem 4.7.

The methods of this paper are different from the ones of [5] and [6]. They build on DG module adaptations of the ideas used by Serre to prove [11, thm. 10, p. 217] on the connection between homology and homotopy groups of topological spaces. The main tool is a list of termwise inequalities of power series given in Lemmas 3.5 and 3.6. An advantage of the present approach is that, whereas the methods of [5] and [6] fail for unbounded DG algebras, as demonstrated for example in [5, sec. 4], this paper is able to treat bounded and unbounded DG algebras on the same footing.

The paper is organized as follows: Section 1 gives notation and elementary properties for DG modules over simply connected cochain DG algebras. Sections 2 and 3 set up the DG module adaptation of Serre’s ideas from the proof of [11, thm. 10, p. 217]. Section 4 proves the cochain Amplitude Inequality, the Auslander-Buchsbaum Equality, and the Gap Theorem for Bass numbers in Theorems 4.4, 4.7, and 4.11.

Finally, Section 5 evaluates the Auslander-Buchsbaum Equality and the Gap Theorem for the singular cochain DG algebra of a topological space. The context will be a fibration of topological spaces, and we recover in Theorem 5.5 the classical fact that homological dimension is additive on fibrations. Theorem 5.6 shows that a gap of length g in the Betti numbers of the fibre space implies that the total space has cohomology in a dimension bigger than or equal to $g + 1$.

This paper supersedes the manuscript “Homological identities for Differential Graded Algebras, II” from the spring of 2002. That manuscript suffers from technical problems which remain unsolved, and it was never submitted.

Anders J. Frankild, my coauthor and friend of many years, died in June 2007, before the present, more successful approach took the form of this paper.

He will be bitterly missed, but his memory will live on.

Since Anders has not been able to check the final version of the paper, the responsibility for any mistakes rests with me.

1. BACKGROUND

This section gives notation and elementary properties for DG modules over simply connected cochain DG algebras. For introductions to the theory of DG algebras, see [1], [4], or [8]. The notation will stay close to [5], [6], and [7].

Setup 1.1. By k is denoted a field and by R a cochain DG algebra over k which has the form

$$\cdots \rightarrow 0 \rightarrow k \rightarrow 0 \rightarrow R^2 \rightarrow R^3 \rightarrow \cdots$$

and satisfies $\dim_k H^i(R) < \infty$ for each i .

Remark 1.2. In particular, $R^0 = k$, $R^1 = 0$, $H^0(R) = k$, and $H^1(R) = 0$.

Notation 1.3. There are derived categories $\mathbf{D}(R)$ of DG left- R -modules and $\mathbf{D}(R^\circ)$ of DG right- R -modules which support the derived functors $\overset{\mathbf{L}}{\otimes}_R$ and \mathbf{RHom}_R . We define full subcategories of $\mathbf{D}(R)$,

$$\begin{aligned} \mathbf{D}^+(R) &= \{ M \in \mathbf{D}(R) \mid H^j(M) = 0 \text{ for } j \ll 0 \}, \\ \mathbf{D}^-(R) &= \{ M \in \mathbf{D}(R) \mid H^j(M) = 0 \text{ for } j \gg 0 \}, \end{aligned}$$

and similarly for $\mathbf{D}(R^\circ)$.

A DG left- R -module M is *compact* precisely if it is finitely built from R in $\mathbf{D}(R)$ using distinguished triangles, (de)suspensions, finite direct sums, and direct summands, cf. [8, thm. 5.3]. The full subcategory of $\mathbf{D}(R)$ consisting of compact DG modules is denoted $\mathbf{D}^c(R)$, and similarly for $\mathbf{D}(R^\circ)$.

The suspension functor on DG modules is denoted by Σ .

The operation $(-)^{\natural}$ forgets the differential of a complex; it sends DG algebras and DG modules to graded algebras and graded modules.

A DG R -module M is called *locally finite* if it satisfies $\dim_k H^j(M) < \infty$ for each j .

The *infimum* and the *supremum* of a DG module are defined by

$$\inf M = \inf\{ j \mid H^j(M) \neq 0 \}, \quad \sup M = \sup\{ j \mid H^j(M) \neq 0 \},$$

and the *amplitude* is

$$\text{amp } M = \sup M - \inf M.$$

Note that we use the convention $\inf(\emptyset) = \infty$ and $\sup(\emptyset) = -\infty$, so $\inf(0) = \infty$, $\sup(0) = -\infty$, and $\text{amp}(0) = -\infty$. In fact, these special values occur precisely when a DG module has zero cohomology, and this property characterizes DG modules which are zero in the derived category, so

$$\begin{aligned} M \cong 0 \text{ in the derived category} \\ \Leftrightarrow \inf M = \infty \Leftrightarrow \sup M = -\infty \Leftrightarrow \text{amp } M = -\infty. \end{aligned} \quad (1.a)$$

Moreover,

$$M \not\cong 0 \text{ in the derived category} \Rightarrow \inf M \leq \sup M. \quad (1.b)$$

We can view k as a DG bi- R -module concentrated in cohomological degree 0. The *Betti numbers* of a DG left- R -module M are

$$\beta_R^j(M) = \dim_k H^j(k \overset{\mathbf{L}}{\otimes}_R M). \quad (1.c)$$

The *projective codimension* of M is

$$\mathrm{pcd}_R M = \sup(k \overset{\mathrm{L}}{\otimes}_R M) = \sup\{j \mid \beta_R^j(M) \neq 0\}; \quad (1.d)$$

it is an integer or ∞ or $-\infty$.

If β is a cardinal number, then a direct sum of β copies of M will be denoted by $M^{(\beta)}$.

Various incarnations of the following lemma are well known; see [3, appendix] or [7, lem. 3.3].

Lemma 1.4. *Let M be in $\mathrm{D}^+(R)$. There is a semi-free resolution*

$$\varphi : F \rightarrow M$$

with semi-free filtration

$$0 = F(-1) \subseteq F(0) \subseteq F(1) \subseteq \dots \subseteq F$$

where the free quotients $F(j)/F(j-1)$ are direct sums of (de)suspensions $\Sigma^\ell R$ with $\ell \leq -\inf M$.

Proposition 1.5. *Let P be in $\mathrm{D}^+(R^0)$ and let M be in $\mathrm{D}^+(R)$. Then*

$$\inf(P \overset{\mathrm{L}}{\otimes}_R M) = \inf P + \inf M.$$

Proof. If P or M is zero then the equation reads $\infty = \infty$, so suppose that P and M are non-zero in the derived categories. Then $j = \inf P$ and $i = \inf M$ are integers.

By [7, lem. 3.4(i)], we can replace P with a quasi-isomorphic DG module which is zero in cohomological degrees $< j$. Lemma 1.4 says that M has a semi-free resolution F with a semi-free filtration where the successive quotients are direct sums of DG modules $\Sigma^\ell R$ with $\ell \leq -i$. This implies that F^\natural is a direct sum of graded modules $\Sigma^\ell R^\natural$ with $\ell \leq -i$, so $(P \otimes_R F)^\natural = P^\natural \otimes_{R^\natural} F^\natural$ is a direct sum of graded modules $\Sigma^\ell P^\natural$ with $\ell \leq -i$. Since P is zero in cohomological degrees $< j$, this implies that $(P \otimes_R F)^\natural$ is zero in cohomological degrees $< j + i$. In particular we have $\inf(P \otimes_R F) \geq j + i$, that is,

$$\inf(P \overset{\mathrm{L}}{\otimes}_R M) \geq j + i = \inf P + \inf M. \quad (1.e)$$

On the other hand, a morphism of DG left- R -modules $\Sigma^{-i}R \rightarrow M$ is determined by the image z of $\Sigma^{-i}1_R$, and z is a cycle in M^i , the i th component of M . Since $H^i(\Sigma^{-i}R) \cong H^0(R) \cong k$, the induced map $H^i(\Sigma^{-i}R) \rightarrow H^i(M)$ is just the map $k \rightarrow H^i(M)$ which sends 1_k to the cohomology class of z . Hence if we pick cycles z_α such that

the corresponding cohomology classes form a k -basis of $H^i(M)$ and construct a morphism $\Sigma^{-i}R^{(\beta)} \rightarrow M$ by sending the elements $\Sigma^{-i}1_R$ to the z_α , then the induced map $H^i(\Sigma^{-i}R^{(\beta)}) \rightarrow H^i(M)$ is an isomorphism. Complete to a distinguished triangle

$$\Sigma^{-i}R^{(\beta)} \rightarrow M \rightarrow M'' \rightarrow; \quad (1.f)$$

since we have $H^{i+1}(\Sigma^{-i}R^{(\beta)}) \cong H^1(R^{(\beta)}) = 0$, the long exact cohomology sequence shows

$$\inf M'' \geq i + 1. \quad (1.g)$$

Tensoring the distinguished triangle (1.f) with P gives

$$\Sigma^{-i}P^{(\beta)} \rightarrow P \overset{\mathbb{L}}{\otimes}_R M \rightarrow P \overset{\mathbb{L}}{\otimes}_R M'' \rightarrow$$

whose long exact cohomology sequence contains

$$H^{j+i-1}(P \overset{\mathbb{L}}{\otimes}_R M'') \rightarrow H^{j+i}(\Sigma^{-i}P^{(\beta)}) \rightarrow H^{j+i}(P \overset{\mathbb{L}}{\otimes}_R M). \quad (1.h)$$

The inequality (1.e) can be applied to P and M'' ; because of (1.g), this gives $\inf(P \overset{\mathbb{L}}{\otimes}_R M'') \geq j + i + 1$ so the first term of the exact sequence (1.h) is zero. The second term is $H^{j+i}(\Sigma^{-i}P^{(\beta)}) \cong H^j(P^{(\beta)})$ which is non-zero since $j = \inf P$. So the third term is non-zero whence

$$\inf(P \overset{\mathbb{L}}{\otimes}_R M) \leq j + i = \inf P + \inf M.$$

Combining with (1.e) completes the proof. \square

Lemma 1.6. *Let P be non-zero in $D^+(R^\circ)$ and let M be in $D^+(R)$.*

- (i) $M \cong 0$ in $D^+(R) \Leftrightarrow P \overset{\mathbb{L}}{\otimes}_R M \cong 0$ in $D(k)$.
- (ii) $M \not\cong 0$ in $D^+(R) \Rightarrow \inf M \leq \text{pcd } M$.

Proof. (i) Follows from Proposition 1.5 and equation (1.a).

(ii) When M is non-zero in $D^+(R)$, it follows from (i) that $k \overset{\mathbb{L}}{\otimes}_R M$ is non-zero in $D(k)$. Then $\inf(k \overset{\mathbb{L}}{\otimes}_R M) \leq \sup(k \overset{\mathbb{L}}{\otimes}_R M)$ by equation (1.b). By Proposition 1.5 and equation (1.d), this reads $\inf M \leq \text{pcd } M$. \square

2. A CONSTRUCTION

This section starts to set up the DG module adaptation of Serre's ideas from the proof of [11, thm. 10, p. 217]. The main item is Construction

2.2 which approximates a DG module by (de)suspensions of the DG algebra R .

Lemma 2.1. *Let M be in $\mathbf{D}^+(R)$ and let i be an integer with $i \leq \inf M$. There is a distinguished triangle in $\mathbf{D}(R)$,*

$$\Sigma^{-i}R^{(\beta)} \rightarrow M \rightarrow M'' \rightarrow,$$

which satisfies the following. Recall that $\beta^i(M)$ is the i th Betti number of M , cf. equation (1.c).

- (i) $\beta = \beta^i(M)$.
- (ii) $\beta^j(M) = \beta^j(M'')$ for each $j \geq i + 1$.
- (iii) $\inf M'' \geq i + 1$; in particular, M'' is in $\mathbf{D}^+(R)$.

If M is locally finite then $\beta < \infty$ and M'' is also locally finite.

Proof. The distinguished triangle can be constructed like (1.f) in the proof of Proposition 1.5, by picking cycles z_α in M^i such that the corresponding cohomology classes form a k -basis for $H^i(M)$, then defining $\Sigma^{-i}R^{(\beta)} \rightarrow M$ by sending the elements $\Sigma^{-i}1_R$ to the z_α , and finally completing to a distinguished triangle.

Property (iii) follows from the long exact cohomology sequence. Tensoring the distinguished triangle with k , property (ii) is immediate and property (i) follows by using Proposition 1.5.

If M is locally finite, then there are only finitely many cycles z_α , so $\beta < \infty$. The long exact cohomology sequence then shows that M'' is also locally finite. \square

Construction 2.2. Let M be non-zero in $\mathbf{D}^+(R)$ and write $i = \inf M$. Observe that i is an integer. Set $M\langle i \rangle = M$ and let $u \geq i$ be an integer. By iterating Lemma 2.1, we can construct a sequence of distinguished triangles in $\mathbf{D}(R)$,

$$\begin{array}{ccccccc} \Sigma^{-i}R^{(\beta^i)} & \rightarrow & M\langle i \rangle & \rightarrow & M\langle i + 1 \rangle & \rightarrow, \\ \Sigma^{-i-1}R^{(\beta^{i+1})} & \rightarrow & M\langle i + 1 \rangle & \rightarrow & M\langle i + 2 \rangle & \rightarrow, \\ & & \vdots & & & \\ \Sigma^{-u+1}R^{(\beta^{u-1})} & \rightarrow & M\langle u - 1 \rangle & \rightarrow & M\langle u \rangle & \rightarrow, \\ \Sigma^{-u}R^{(\beta^u)} & \rightarrow & M\langle u \rangle & \rightarrow & M\langle u + 1 \rangle & \rightarrow, \end{array}$$

where

- (i) $\beta^j = \beta^j(M)$ for each j .

(ii) $\beta^j(M) = \beta^j(M\langle\ell\rangle)$ for each $j \geq \ell$.

(iii) $\inf M\langle\ell\rangle \geq \ell$ for each ℓ . In particular, each $M\langle\ell\rangle$ is in $D^+(R)$.

If M is locally finite, then so is each $M\langle\ell\rangle$, and then each $\beta^j = \beta^j(M) = \beta^j(M\langle j\rangle)$ is finite.

Proposition 2.3. (i) *If M is in $D^c(R)$, then it is locally finite and belongs to $D^+(R)$.*

(ii) *Let M be locally finite in $D^+(R)$. Then*

$$M \text{ is in } D^c(R) \Leftrightarrow \text{pcd } M < \infty.$$

Proof. Let M be in $D^c(R)$, that is, M is finitely built from R in $D(R)$. Since R is locally finite and belongs to $D^+(R)$, the same holds for M . This proves (i).

Moreover, we have $\sup(k \overset{L}{\otimes}_R R) = \sup k = 0 < \infty$ so we must also have $\sup(k \overset{L}{\otimes}_R M) < \infty$, that is, $\text{pcd } M < \infty$. This proves (ii), implication \Rightarrow .

(ii), implication \Leftarrow : If M is zero then it is certainly in $D^c(R)$, so assume that M is non-zero in $D^+(R)$.

Then $\inf M$ is an integer and $\inf M \leq \text{pcd } M$ by Lemma 1.6(ii). On the other hand, $\text{pcd } M < \infty$, so

$$p = \text{pcd } M \text{ is an integer.}$$

But $\text{pcd } M = \sup(k \overset{L}{\otimes}_R M)$ so $\beta^j(M) = \dim_k H^j(k \overset{L}{\otimes}_R M) = 0$ for $j \geq p + 1$. In Construction 2.2, by part (ii) this implies

$$\beta^j(M\langle p + 1\rangle) = 0 \text{ for } j \geq p + 1.$$

However, part (iii) of the construction says $\inf M\langle p + 1\rangle \geq p + 1$, so

$$\inf(k \overset{L}{\otimes}_R M\langle p + 1\rangle) = \inf M\langle p + 1\rangle \geq p + 1$$

by Proposition 1.5, that is

$$\beta^j(M\langle p + 1\rangle) = \dim_k H^j(k \overset{L}{\otimes}_R M\langle p + 1\rangle) = 0 \text{ for } j < p + 1.$$

Altogether, $\beta^j(M\langle p + 1\rangle) = 0$ for each j . That is, each cohomology group of $k \overset{L}{\otimes}_R M\langle p + 1\rangle$ is zero and so $k \overset{L}{\otimes}_R M\langle p + 1\rangle$ is itself zero. Lemma 1.6(i) hence gives

$$M\langle p + 1\rangle \cong 0.$$

But now the distinguished triangles in Construction 2.2, starting with $\Sigma^{-p}R^{(\beta_p)} \rightarrow M\langle p \rangle \rightarrow M\langle p+1 \rangle \rightarrow$ and running backwards to the first one, $\Sigma^{-i}R^{(\beta_i)} \rightarrow M \rightarrow M\langle i+1 \rangle \rightarrow$, show that M is finitely built from R since each β^j is finite. That is, M is in $\mathbf{D}^c(R)$. \square

Remark 2.4. Let M be non-zero in $\mathbf{D}^c(R)$. Proposition 2.3 implies that M is locally finite in $\mathbf{D}^+(R)$ with $\text{pcd } M < \infty$. The proof of the proposition actually gives a bit more: First,

$$p = \text{pcd } M \text{ is an integer.}$$

Secondly, in Construction 2.2, we have

$$M\langle p+1 \rangle \cong 0 \tag{2.a}$$

in $\mathbf{D}(R)$.

Combining this isomorphism with the distinguished triangle

$$\Sigma^{-p}R^{(\beta_p)} \rightarrow M\langle p \rangle \rightarrow M\langle p+1 \rangle \rightarrow$$

shows

$$M\langle p \rangle \cong \Sigma^{-p}R^{(\beta_p(M))} \tag{2.b}$$

in $\mathbf{D}(R)$.

3. INEQUALITIES

This section continues to set up the DG module adaptation of Serre's ideas from the proof of [11, thm. 10, p. 217]. The main items are Lemmas 3.5 and 3.6 which use Construction 2.2 to prove some termwise inequalities of power series.

Setup 3.1. Let

$$F : \mathbf{D}(R) \rightarrow \mathbf{Mod}(k)$$

be a k -linear homological functor which respects coproducts. For M in $\mathbf{D}(R)$, we set

$$f_M(t) = \sum_{\ell} \dim_k F(\Sigma^{\ell}M)t^{\ell}.$$

Remark 3.2. In the generality of Setup 3.1, the expression $f_M(t)$ may not belong to any reasonable set. Some of the coefficients may be infinite, and there may be non-zero coefficients in arbitrarily high positive and negative degrees at the same time.

However, we will see that there are circumstances in which $f_M(t)$ is a Laurent series.

Notation 3.3. Termwise inequalities \leq of coefficients between expressions like $f_M(t)$ make sense; they will be denoted by \preceq .

Likewise, it makes sense to add such expressions, and to multiply them by a number or a power of t .

Finally, the *degree* of an expression like $f_M(t)$ is defined by

$$\deg\left(\sum_{\ell} f_{\ell} t^{\ell}\right) = \sup\{\ell \mid f_{\ell} \neq 0\}.$$

Lemma 3.4. *Let M be in $\mathbf{D}(R)$.*

- (i) $f_{\Sigma^j M}(t) = t^{-j} f_M(t)$.
- (ii) $f_{M^{(\beta)}}(t) = \beta f_M(t)$.
- (iii) *If*

$$M' \rightarrow M \rightarrow M'' \rightarrow$$

is a distinguished triangle in $\mathbf{D}(R)$, then there is a termwise inequality

$$f_M(t) \preceq f_{M'}(t) + f_{M''}(t).$$

Proof. Parts (i) and (ii) are clear. In part (iii), the distinguished triangle gives a long exact sequence consisting of pieces

$$F(\Sigma^{\ell} M') \rightarrow F(\Sigma^{\ell} M) \rightarrow F(\Sigma^{\ell} M''),$$

whence

$$\dim_k F(\Sigma^{\ell} M) \leq \dim_k F(\Sigma^{\ell} M') + \dim_k F(\Sigma^{\ell} M'')$$

and the lemma follows. \square

Lemma 3.5. *Let M be non-zero in $\mathbf{D}^+(R)$. Write $i = \inf M$, let $u \geq i$ be an integer, and consider Construction 2.2. There are termwise inequalities*

- (i) $f_M(t) \preceq (\beta^i(M)t^i + \cdots + \beta^u(M)t^u)f_R(t) + f_{M^{(u+1)}}(t)$,
- (ii) $f_{M^{(u+1)}}(t) \preceq f_M(t) + t^{-1}(\beta^i(M)t^i + \cdots + \beta^u(M)t^u)f_R(t)$.

Proof. This follows by applying Lemma 3.4 successively to the distinguished triangles of Construction 2.2. For instance, (i) can be proved

as follows,

$$\begin{aligned}
f_M(t) &= f_{M\langle i \rangle}(t) \\
&\preceq f_{\Sigma^{-i}R^{(\beta^i)}}(t) + f_{M\langle i+1 \rangle}(t) \\
&= \beta^i(M)t^i f_R(t) + f_{M\langle i+1 \rangle}(t) \\
&\preceq \beta^i(M)t^i f_R(t) + f_{\Sigma^{-i-1}R^{(\beta^{i+1})}}(t) + f_{M\langle i+2 \rangle}(t) \\
&= \beta^i(M)t^i f_R(t) + \beta^{i+1}(M)t^{i+1} f_R(t) + f_{M\langle i+2 \rangle}(t) \\
&\preceq \dots \\
&= (\beta^i(M)t^i + \dots + \beta^u(M)t^u) f_R(t) + f_{M\langle u+1 \rangle}(t),
\end{aligned}$$

and (ii) is proved by similar manipulations. \square

Lemma 3.6. *Let M be non-zero in $D^c(R)$ and write $i = \inf M$ and $p = \text{pcd } M$.*

Then i and p are integers with $i \leq p$, we have $\beta^p(M) \neq 0$, and there are termwise inequalities

- (i) $f_M(t) \preceq (\beta^i(M)t^i + \dots + \beta^p(M)t^p) f_R(t)$,
- (ii) $\beta^p(M)t^p f_R(t) \preceq f_M(t) + t^{-1}(\beta^i(M)t^i + \dots + \beta^{p-1}(M)t^{p-1}) f_R(t)$.

Proof. Proposition 2.3(i) says that M is in $D^+(R)$, and since M is non-zero it follows that $i = \inf M$ is an integer. Remark 2.4 says that $p = \text{pcd } M$ is an integer. Lemma 1.6(ii) says $i \leq p$. Since $p = \sup(k \overset{L}{\otimes}_R M)$, it is clear that $\beta^p(M) = \dim_k H^p(k \overset{L}{\otimes}_R M) \neq 0$.

Consider Construction 2.2 for M . By Remark 2.4, equations (2.a) and (2.b), we have $M\langle p \rangle \cong \Sigma^{-p}R^{(\beta^p(M))}$ and $M\langle p+1 \rangle \cong 0$. Inserting this into the inequalities of Lemma 3.5 gives the inequalities of the present proposition. \square

As an immediate application, consider the following lemma.

Lemma 3.7. *Let M be in $D^c(R)$. If $f_R(t)$ is a Laurent series in t^{-1} then so is $f_M(t)$, and*

$$\deg f_M(t) = \deg f_R(t) + \text{pcd } M.$$

Proof. If M is zero then $f_M(t) = 0$ is trivially a Laurent series in t^{-1} , and the equation of the lemma reads $-\infty = -\infty$ so the lemma holds.

Suppose that M is non-zero in $D^c(R)$. Since $f_R(t)$ is a Laurent series in t^{-1} , Lemma 3.6(i) implies that so is $f_M(t)$ since each $\beta^j(M)$ is finite, cf. Proposition 2.3(i) and Construction 2.2.

If $f_R(t)$ has all coefficients equal to zero then Lemma 3.6(i) forces $f_M(t)$ to have all coefficients equal to zero, and the equation of the lemma reads $-\infty = -\infty$ so the lemma holds.

Suppose that not all coefficients of $f_R(t)$ are equal to zero. Then Lemma 3.6(i) implies

$$\deg f_M(t) \leq \deg f_R(t) + p = \deg f_R(t) + \text{pcd } M.$$

On the other hand, consider the inequality of Lemma 3.6(ii). The left hand side contains a non-zero monomial of degree $\deg f_R(t) + p$. The right hand side consists of two terms, and the second one, $t^{-1}(\beta^i(M)t^i + \cdots + \beta^{p-1}(M)t^{p-1})f_R(t)$, consists of monomials of degree $< \deg f_R(t) + p$. Hence the first term, $f_M(t)$, must contain a non-zero monomial of degree $\deg f_R(t) + p$ whence

$$\deg f_M(t) \geq \deg f_R(t) + p = \deg f_R(t) + \text{pcd } M.$$

Combining the displayed inequalities gives the desired equation. \square

4. MAIN RESULTS

This section shows the cochain Amplitude Inequality, Auslander-Buchsbaum Equality, and Gap Theorem for Bass numbers in Theorems 4.4, 4.7, and 4.11.

Setup 4.1. From now on, we will only consider a special form of F and f from Setup 3.1. Namely, let P be in $\mathbf{D}(R^\circ)$ and set

$$F(-) = \mathbf{H}^0(P \overset{\mathbf{L}}{\otimes}_R -).$$

This means that

$$f_M(t) = \sum_{\ell} \dim_k \mathbf{H}^{\ell}(P \overset{\mathbf{L}}{\otimes}_R M) t^{\ell} \quad (4.a)$$

and in particular

$$f_R(t) = \sum_{\ell} \dim_k \mathbf{H}^{\ell}(P) t^{\ell}. \quad (4.b)$$

Proposition 4.2. *Let M be in $\mathbf{D}^c(R)$, and let P be locally finite in $\mathbf{D}^-(R^\circ)$.*

Then

$$\sup(P \overset{\mathbf{L}}{\otimes}_R M) = \sup P + \text{pcd } M.$$

Proof. The expression $f_R(t)$ is given by equation (4.b) and it is a Laurent series in t^{-1} because P is locally finite in $\mathbf{D}^-(R^\circ)$. Lemma 3.7 gives

$$\deg f_M(t) = \deg f_R(t) + \text{pcd } M. \quad (4.c)$$

However, equations (4.a) and (4.b) imply

$$\deg f_M(t) = \sup(P \overset{\mathbf{L}}{\otimes}_R M)$$

and

$$\deg f_R(t) = \sup P,$$

and inserting this into equation (4.c) proves the proposition. \square

Proposition 4.3. *Let M be in $\mathbf{D}^c(R)$. Let P be locally finite in $\mathbf{D}(R^\circ)$ and suppose $\text{amp } P < \infty$. Then*

$$\text{amp}(P \overset{\mathbf{L}}{\otimes}_R M) = \text{amp } P + \text{pcd } M - \inf M.$$

Proof. Proposition 4.2 says

$$\sup(P \overset{\mathbf{L}}{\otimes}_R M) = \sup P + \text{pcd } M.$$

Subtracting the equation of Proposition 1.5 gives

$$\sup(P \overset{\mathbf{L}}{\otimes}_R M) - \inf(P \overset{\mathbf{L}}{\otimes}_R M) = \sup P + \text{pcd } M - \inf P - \inf M,$$

and this is the equation of the present theorem. \square

Theorem 4.4 (Amplitude Inequality). *Let M be non-zero in $\mathbf{D}^c(R)$. Let P be locally finite in $\mathbf{D}(R^\circ)$ and suppose $\text{amp } P < \infty$. Then*

$$\text{amp}(P \overset{\mathbf{L}}{\otimes}_R M) \geq \text{amp } P.$$

Proof. Combine Proposition 4.3 with Lemma 1.6(ii). \square

Proposition 4.5. *Assume that there is a P which is non-zero in $\mathbf{D}^c(R^\circ)$ and satisfies $\sup P < \infty$. Set*

$$d = \text{pcd } P - \sup P.$$

If M is in $\mathbf{D}^c(R)$ and satisfies $\sup M < \infty$, then

$$\text{pcd } M = \sup M + d.$$

Proof. Proposition 2.3(i) gives that M and P are locally finite. They are also in \mathbf{D}^- because they have finite supremum. Hence Proposition 4.2 says

$$\sup(P \overset{\mathbf{L}}{\otimes}_R M) = \sup P + \text{pcd } M,$$

and Proposition 4.2 with M and P interchanged says

$$\sup(P \overset{\mathbf{L}}{\otimes}_R M) = \sup M + \text{pcd } P.$$

The two right hand sides must be equal,

$$\sup P + \text{pcd } M = \sup M + \text{pcd } P,$$

and rearranging terms proves the proposition. \square

Question 4.6. For which DG algebras R does there exist a DG module like P ? For such DG algebras, the invariant $d = \text{pcd } P - \sup P$ appears to be interesting, and it would be useful to find a formula expressing it directly in terms of R .

The following theorem considers two easy, special cases of Proposition 4.5 which can reasonably be termed Auslander-Buchsbaum Equalities.

Theorem 4.7 (Auslander-Buchsbaum Equalities). *Let M be in $\mathbf{D}^c(R)$.*

(i) *If R has $\sup R < \infty$, then*

$$\text{pcd } M = \sup M - \sup R.$$

(ii) *If $\sup M < \infty$ and k is in $\mathbf{D}^c(R)$, then*

$$\text{pcd } M = \sup M + \text{pcd } k.$$

Proof. Both parts follow from Proposition 4.5, by using R and k in place of P . In part (i), note that when M is in $\mathbf{D}^c(R)$, it is finitely built from R , so $\sup R < \infty$ implies $\sup M < \infty$. \square

Proposition 4.8. *Let M be locally finite in $\mathbf{D}^+(R)$. Let P be locally finite and non-zero in $\mathbf{D}(R^0)$ and suppose $\text{amp } P < \infty$.*

Let $g \geq \text{amp } P$. If the Betti numbers of M have a gap of length g in the sense that there is a j such that

$$\beta^\ell(M) \begin{cases} \neq 0 & \text{for } \ell = j, \\ = 0 & \text{for } j + 1 \leq \ell \leq j + g, \\ \neq 0 & \text{for } \ell = j + g + 1, \end{cases}$$

then

$$\text{amp}(P \overset{\mathbf{L}}{\otimes}_R M) \geq g + 1.$$

Proof. Since $\beta^j(M) \neq 0$, it is clear that M is non-zero in $D^+(R)$. By (de)suspending, we can suppose $\inf M = \inf P = 0$. Write

$$s = \sup P;$$

then $s = \text{amp } P$ and we have the assumption

$$g \geq s.$$

Proposition 1.5 gives $\inf(P \overset{\mathbb{L}}{\otimes}_R M) = \inf P + \inf M = 0$, so $H^0(P \overset{\mathbb{L}}{\otimes}_R M) \neq 0$. To show the present proposition, we need to prove $H^\ell(P \overset{\mathbb{L}}{\otimes}_R M) \neq 0$ for some $\ell \geq g + 1$, so let us assume

$$H^{\geq g+1}(P \overset{\mathbb{L}}{\otimes}_R M) = 0$$

and show a contradiction. Proposition 1.5 implies

$$\beta^\ell(M) = \dim_k H^\ell(k \overset{\mathbb{L}}{\otimes}_R M) = 0 \text{ for } \ell < 0,$$

so the integer j from the proposition satisfies $j \geq 0$. Hence in particular

$$H^{\geq j+g+1}(P \overset{\mathbb{L}}{\otimes}_R M) = 0. \quad (4.d)$$

Inserting (4.a) and (4.b) into the inequality of Lemma 3.5(i) gives

$$\begin{aligned} & \sum_{\ell} \dim_k H^\ell(P \overset{\mathbb{L}}{\otimes}_R M) t^\ell \\ & \preccurlyeq (\beta^0(M)t^0 + \cdots + \beta^u(M)t^u) \sum_{\ell} \dim_k H^\ell(P) t^\ell \\ & \quad + \sum_{\ell} \dim_k H^\ell(P \overset{\mathbb{L}}{\otimes}_R M \langle u+1 \rangle) t^\ell \end{aligned}$$

where $u \geq 0$ is an integer and $M \langle u+1 \rangle$ is defined by Construction 2.2. We have $\beta^\ell(M) = 0$ for $j+1 \leq \ell \leq j+g$ while $\sum_{\ell} \dim_k H^\ell(P) t^\ell$ has terms only of degree $0, \dots, s$, so the first term on the right hand side is zero in degree ℓ for $j+s+1 \leq \ell \leq j+g$. And $\inf M \langle u+1 \rangle \geq u+1$ by Construction 2.2(iii) so Proposition 1.5 implies $\inf(P \overset{\mathbb{L}}{\otimes}_R M \langle u+1 \rangle) \geq u+1$, so by picking u large we can move the second term on the right hand side into large degrees and thereby ignore it. It follows that the left hand side is also zero in degree ℓ for $j+s+1 \leq \ell \leq j+g$; that is,

$$H^\ell(P \overset{\mathbb{L}}{\otimes}_R M) = 0 \text{ for } j+s+1 \leq \ell \leq j+g.$$

Combining with equation (4.d) shows

$$H^{\geq j+s+1}(P \overset{L}{\otimes}_R M) = 0. \quad (4.e)$$

Now insert (4.a) and (4.b) into Lemma 3.5(ii) with $u = j$,

$$\begin{aligned} \sum_{\ell} \dim_k H^{\ell}(P \overset{L}{\otimes}_R M\langle j+1 \rangle) t^{\ell} \\ \approx \sum_{\ell} \dim_k H^{\ell}(P \overset{L}{\otimes}_R M) t^{\ell} \\ + t^{-1}(\beta^0(M)t^0 + \dots + \beta^j(M)t^j) \sum_{\ell} \dim_k H^{\ell}(P) t^{\ell}. \end{aligned}$$

Again, $\sum_{\ell} \dim_k H^{\ell}(P) t^{\ell}$ only has terms of degree $0, \dots, s$, so on the right hand side, the second term is zero in degrees $\geq j+s$. In particular, it is zero in degrees $\geq j+s+1$, and since equation (4.e) implies that the same holds for the first term, it must also hold for the left hand side, that is,

$$H^{\geq j+s+1}(P \overset{L}{\otimes}_R M\langle j+1 \rangle) = 0. \quad (4.f)$$

Now, $\inf M\langle j+1 \rangle \geq j+1$ by Construction 2.2(iii), so Proposition 1.5 implies

$$\inf(k \overset{L}{\otimes}_R M\langle j+1 \rangle) \geq j+1.$$

And Construction 2.2(ii) says $\beta^{\ell}(M\langle j+1 \rangle) = \beta^{\ell}(M)$ for $\ell \geq j+1$, so $\beta^{\ell}(M) = 0$ for $j+1 \leq \ell \leq j+g$ gives

$$\beta^{\ell}(M\langle j+1 \rangle) = 0 \text{ for } j+1 \leq \ell \leq j+g,$$

that is,

$$H^{\ell}(k \overset{L}{\otimes}_R M\langle j+1 \rangle) = 0 \text{ for } j+1 \leq \ell \leq j+g,$$

so we even have

$$\inf(k \overset{L}{\otimes}_R M\langle j+1 \rangle) \geq j+g+1,$$

that is, $\inf M\langle j+1 \rangle \geq j+g+1$ by Proposition 1.5 again. Hence

$$\inf(P \overset{L}{\otimes}_R M\langle j+1 \rangle) \geq j+g+1$$

by Proposition 1.5. However, $g \geq s$, so the only way this can be compatible with equation (4.f) is if we have

$$P \overset{L}{\otimes}_R M\langle j+1 \rangle \cong 0.$$

By Lemma 1.6(i) this means that $M\langle j+1 \rangle \cong 0$. And this gives

$$\beta^{j+g+1}(M) = \beta^{j+g+1}(M\langle j+1 \rangle) = 0,$$

which is the desired contradiction since we had assumed $\beta^{j+g+1}(M) \neq 0$. \square

Corollary 4.9 (Gap Theorem for Betti numbers). *Suppose $\sup R < \infty$. Let M be locally finite in $\mathbf{D}^+(R)$.*

Let $g \geq \sup R$. If the Betti numbers of M have a gap of length g in the sense that there is a j such that

$$\beta^\ell(M) \begin{cases} \neq 0 & \text{for } \ell = j, \\ = 0 & \text{for } j+1 \leq \ell \leq j+g, \\ \neq 0 & \text{for } \ell = j+g+1, \end{cases}$$

then

$$\text{amp } M \geq g+1.$$

Proof. This follows from Proposition 4.8 by using R in place of P . \square

Remark 4.10. Conversely, if $\sup R < \infty$ and M is locally finite in $\mathbf{D}^+(R)$ with $\text{amp } M \leq \sup R$, then the Betti numbers of M can have no gaps of length bigger than or equal to $\sup R$.

By evaluating Corollary 4.9 on the k -linear dual $\text{Hom}_k(M, k)$, we immediately get the following result in which the *Bass numbers* of a DG module are

$$\mu^j(M) = \dim_k H^j(\text{RHom}_R(k, M)).$$

Theorem 4.11 (Gap Theorem for Bass numbers). *Suppose $\sup R < \infty$. Let M be locally finite in $\mathbf{D}^-(R)$.*

Let $g \geq \sup R$. If the Bass numbers of M have a gap of length g in the sense that there is a j such that

$$\mu^\ell(M) \begin{cases} \neq 0 & \text{for } \ell = j, \\ = 0 & \text{for } j+1 \leq \ell \leq j+g, \\ \neq 0 & \text{for } \ell = j+g+1, \end{cases}$$

then

$$\text{amp } M \geq g+1.$$

Remark 4.12. Conversely, if $\sup R < \infty$ and M is locally finite in $\mathbf{D}^-(R)$ with $\text{amp } M \leq \sup R$, then the Bass numbers of M can have no gaps of length bigger than or equal to $\sup R$. In particular, the Bass

numbers of R itself can have no gaps of length bigger than or equal to $\sup R$.

5. TOPOLOGY

This section evaluates the Auslander-Buchsbaum Equality and the Gap Theorem for the singular cochain DG algebra of a topological space. The context will be a fibration of topological spaces, and we recover in Theorem 5.5 that homological dimension is additive on fibrations. Theorem 5.6 shows that a gap of length g in the Betti numbers of the fibre space implies that the total space has non-zero cohomology in a dimension $\geq g + 1$.

A reference for the algebraic topology of this section is [4].

Setup 5.1. Let

$$F \rightarrow X \rightarrow Y$$

be a fibration of topological spaces where $\dim_k H^j(X; k) < \infty$ and $\dim_k H^j(Y; k) < \infty$ for each j and where Y is simply connected.

Remark 5.2. Recall that the *singular cohomology* $H^j(Z; k)$ of a topological space Z is defined in terms of the *singular cochain complex* $C^*(Z; k)$ by

$$H^j(Z; k) = H^j(C^*(Z; k)).$$

The singular cochain complex is a DG algebra, and by [4, exa. 6, p. 146] the assumptions on the space Y mean that $C^*(Y; k)$ is quasi-isomorphic to a DG algebra which falls under Setup 1.1, so the results proved so far apply to it.

Moreover, the continuous map $X \rightarrow Y$ induces a morphism $C^*(Y; k) \rightarrow C^*(X; k)$ whereby $C^*(X; k)$ becomes a DG bi- $C^*(Y; k)$ -module which is locally finite and belongs to D^+ by the assumptions on X .

Notation 5.3. The dimensions $\dim_k H^j(Z; k)$ are called the *Betti numbers* of the topological space Z .

By

$$\text{hd } Z = \sup\{j \mid H^j(Z; k) \neq 0\} = \sup C^*(Z; k) \quad (5.a)$$

is denoted the *homological dimension* of Z ; it is a non-negative integer or ∞ .

By ΩZ is denoted the *Moore loop space* of Z .

Lemma 5.4. *We have*

$$\dim_k H^j(F; k) = \beta_{C^*(Y; k)}^j(C^*(X; k))$$

and $\text{hd } F = \text{pcd}_{C^*(Y; k)}(C^*(X; k))$.

Proof. We know that $C^*(F; k) \cong k \overset{L}{\otimes}_{C^*(Y; k)} C^*(X; k)$ in $\mathbf{D}(k)$ by [4, thm. 7.5]. Taking the dimension of the j 'th cohomology proves the displayed equation, and the other equation is an immediate consequence, cf. equations (1.d) and (5.a). \square

Theorem 5.5 (Additivity of homological dimension). *If $\text{hd } F < \infty$ and $\text{hd } Y < \infty$, then*

$$\text{hd } X = \text{hd } F + \text{hd } Y.$$

Proof. Let us apply Theorem 4.7 to the data

$$R = C^*(Y; k) \quad \text{and} \quad M = C^*(X; k).$$

Using Lemma 5.4 we have

$$\text{pcd}_R(M) = \text{pcd}_{C^*(Y; k)}(C^*(X; k)) = \text{hd } F < \infty.$$

By Proposition 2.3(ii), this says that M is in $\mathbf{D}^c(R)$. Moreover, equation (5.a) gives

$$\sup R = \sup C^*(Y; k) = \text{hd } Y < \infty.$$

This shows that Theorem 4.7(i) does apply, and evaluating its equation gives $\text{hd } F = \text{hd } X - \text{hd } Y$ as desired. \square

Theorem 5.6 (Gap). *If $\text{hd } Y < \infty$ and the Betti numbers of F have a gap of length $g \geq \text{hd } Y$ in the sense that there is a j such that*

$$H^\ell(F; k) \begin{cases} \neq 0 & \text{for } \ell = j, \\ = 0 & \text{for } j + 1 \leq \ell \leq j + g, \\ \neq 0 & \text{for } \ell = j + g + 1, \end{cases}$$

then

$$\text{hd } X \geq g + 1.$$

Proof. Let us apply Corollary 4.9 to the data

$$R = C^*(Y; k) \quad \text{and} \quad M = C^*(X; k).$$

Then $\sup R = \text{hd } Y$ is clear, $\beta^j(M) = \dim_k H^j(F; k)$ holds by Lemma 5.4, and $\text{amp } M = \text{hd } X$ is clear, so the theorem follows. \square

Example 5.7. Set $F = \Omega S^n$ for an $n \geq 3$ so the fibration from Setup 5.1 is

$$\Omega S^n \rightarrow X \rightarrow Y.$$

By [4, exa. 1, p. 234] we have

$$\dim_{\mathbb{Q}} H^{\ell}(\Omega S^n; \mathbb{Q}) = \begin{cases} 1 & \text{for } (n-1) \mid \ell, \\ 0 & \text{otherwise,} \end{cases}$$

so in particular, $\text{hd } \Omega S^n = \infty$ and the Betti numbers of ΩS^n have gaps of length $n-2$. So Theorem 5.6 says that $\text{hd } Y \leq n-2$ implies $\text{hd } X \geq n-1$.

Acknowledgement. We thank Kasper Andersen for an interesting discussion on fibrations, and for the suggestion of Example 5.7. We are grateful to Karsten Schmidt for communicating [10], and to the referee for pointing out [9].

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