COMPACTLY GENERATED RELATIVE STABLE CATEGORIES

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Abstract. Let G be a finite group. The stable module category of G has been applied extensively in group representation theory. In particular, it has been used to great effect that it is a triangulated category which is compactly generated by the class of finitely generated modules.

Let H be a subgroup of G. It is possible to define a stable module category of G relative to H. This is also a triangulated category, but no non-trivial examples have been known where it was compactly generated. While the finitely generated modules are compact objects, they do not necessarily generate the category.

We show that the relative stable category is compactly generated if the group algebra of H has finite representation type. In characteristic p, this is equivalent to the Sylow p-subgroups of H being cyclic.

Let k be a field, G a finite group, and kG the group algebra of G. The stable module category $\mathrm{StMod}(kG)$ is defined as $\mathrm{Mod}(kG)$ modulo morphisms which factor through a projective module.

The stable module category is a triangulated category [5]. It is compactly generated by the class of finitely generated modules, and hence amenable to methods such as Brown Representability and Thomason Localization [10]. This was used to great effect in Rickard's classical work [11] which introduced so-called idempotent modules; these have played a key role in group representation theory ever since.

Let H be a subgroup of G. In their paper [4], Carlson, Peng, and Wheeler considered the H-projective kG-modules which are the direct summands of modules induced from kH to kG, and they defined the relative stable module category $\mathrm{StMod}_H(kG)$ as $\mathrm{Mod}(kG)$ modulo morphisms which factor through an H-projective module.

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One of the useful properties of $\text{StMod}_H(kG)$ is that it, too, is a triangulated category. However, it has not been known whether it was compactly generated, and that is the motivation for this paper. We will show:

Theorem. *Let k be a field and let G be a finite group with a subgroup H. If kH has finite representation type, then $\text{StMod}_H(kG)$ is compactly generated.*

Let us make two remarks to put this into context.

Remark 1. It is not possible to prove the theorem naively, by showing that $\text{StMod}_H(kG)$ is compactly generated by the class of finitely generated modules. These modules are compact objects of $\text{StMod}_H(kG)$, but they do not necessarily generate the category.

To see why, let $\{q_\alpha\}$ be a filtered system in $\text{Mod}(kG)$ of H-projective modules, and let $q = \operatorname{colim} q_\alpha$ be the colimit. If m is a finitely generated kG-module, then each homomorphism $m \to q$ factors through a q_α, so each such homomorphism is 0 in $\text{StMod}_H(kG)$. If $\text{StMod}_H(kG)$ were generated by the class of finitely generated modules, then it would follow that q was isomorphic to 0 in $\text{StMod}_H(kG)$; that is, q would be H-projective. However, it is not true that filtered colimits of H-projective modules are again H-projective; indeed, this fails already for $G = C_2 \times C_2$ and $H = C_2$ although the technical proof of this would take us outside the scope of this short note.

Remark 2. Suppose that the field k has prime characteristic p. Then the group algebra kH has finite representation type if and only if the Sylow p-subgroups of H are cyclic; see [2, thm. VI.3.3].

Let us introduce some notation. Throughout, k, G, and H will be used in the sense of the above theorem.

All modules are left-modules unless they are explicitly called right-modules. The class of H-projective kG-modules will be denoted H-Proj, and $\text{K}(H\text{-Proj})$ is the homotopy category of complexes of modules from H-Proj; it is a triangulated category. We will let $\text{Tate}_H(kG)$ denote the collection of complexes Q of modules from H-Proj for which the restriction $\text{Res}^G_H(Q)$ to H is split exact. We may think of $\text{Tate}_H(kG)$ either as a triangulated subcategory of $\text{K}(H\text{-Proj})$, or as a full subcategory of $\text{C}(H\text{-Proj})$, the category of complexes of modules from H-Proj and chain maps.

Remark 3. If X is in $\mathrm{Tate}_H(kG)$ then X is exact and splits into short exact sequences

$$0 \to \mathrm{Z}^n(X) \to X^n \to \mathrm{Z}^{n+1}(X) \to 0$$

which become split exact upon restriction to H. In particular, it is easy to show that $\mathrm{Z}^n(X) \to X^n$ is an H-Proj-preenvelope and $X^n \to \mathrm{Z}^{n+1}(X)$ is an H-Proj-precover.

Several variants of the following result are well known, see for instance [3, thm. 2.3] and [6, thm. 9.6.4].

Proposition 4. *View* $\mathrm{Tate}_H(kG)$ *as a triangulated subcategory of the category* $\mathrm{K}(H\text{-Proj})$. *There is an equivalence of triangulated categories*

$$\mathrm{Tate}_H(kG) \simeq \mathrm{StMod}_H(kG)$$

given by $X \mapsto \mathrm{Z}^0(X)$.

Proof. We can clearly view Z^0 as a functor $\mathrm{C}(H\text{-Proj}) \to \mathrm{Mod}(kG)$. Viewing $\mathrm{Tate}_H(kG)$ as a full subcategory of $\mathrm{C}(H\text{-Proj})$, we hence have a functor

$$\mathrm{Z}^0 : \mathrm{Tate}_H(kG) \to \mathrm{Mod}(kG). \tag{1}$$

Let $X \to Y$ be a chain map of complexes from $\mathrm{Tate}_H(kG)$. There is an induced homomorphism $\mathrm{Z}^0(X) \to \mathrm{Z}^0(Y)$. The chain map is null homotopic if and only if the induced homomorphism factors through a module from H-Proj, that is, if and only if the induced homomorphism becomes 0 in $\mathrm{StMod}_H(kG)$. This holds by a lifting argument using Remark 3; cf. [3, proof of lem. 2.2].

Hence the functor in equation (1) induces a faithful functor

$$\mathrm{Z}^0 : \mathrm{Tate}_H(kG) \to \mathrm{StMod}_H(kG) \tag{2}$$

where $\mathrm{Tate}_H(kG)$ is now viewed as a triangulated subcategory of the category $\mathrm{K}(H\text{-Proj})$. The functor in equation (2) can be shown to be triangulated by a standard argument.

Observe that X can be viewed as a relative Tate resolution of $\mathrm{Z}^0(X)$. Hence the functor (2) is full, since any homomorphism of modules can be lifted to the relative Tate resolutions; this is again a lifting argument using Remark 3.

To conclude that the functor (2) is an equivalence of categories, all that is now needed is to see that it is essentially surjective. But each kG-module m has a relative Tate resolution X, so indeed, $m \cong \mathrm{Z}^0(X)$ for some X. Note that we can construct such an X by splicing a left-H-Proj-resolution and a right-H-Proj-resolution of m. These resolutions become split exact upon restriction to H because this is true for H-Proj-precovers and -preenvelopes. □

Definition 5. If kH has finite representation type, then y will denote the direct sum of its indecomposable finitely generated modules, and $x = \operatorname{Ind}_H^G(y)$ the induced module over kG.

Remark 6. In the case of the definition, note that H-Proj $= \operatorname{Add}(x)$. Moreover, by projectivization, $\operatorname{Add}(x)$ is equivalent to $\operatorname{Proj}(\Gamma^{\mathrm{o}})$, the category of projective right-modules over the endomorphism algebra $\Gamma = \operatorname{End}_{kG}(x)$.

Note also that x can be viewed as a complex concentrated in degree zero. As such, it is in $\mathrm{K}(H\text{-Proj})$.

Lemma 7. *We have*

$$\begin{aligned}\operatorname{Tate}_H(kG) &= x^{\perp}\\ &= \{\, Q \in \mathrm{K}(H\text{-Proj}) \mid \operatorname{Hom}_{\mathrm{K}(H\text{-Proj})}(\Sigma^n x, Q) = 0 \text{ for each } n \,\}\end{aligned}$$

in $\mathrm{K}(H\text{-Proj})$.

Proof. Let Q be in $\mathrm{K}(H\text{-Proj})$. Then

$$\begin{aligned}\operatorname{Hom}_{\mathrm{K}(H\text{-Proj})}(\Sigma^n x, Q) &= \operatorname{Hom}_{\mathrm{K}(kG)}(\Sigma^n \operatorname{Ind}_H^G(y), Q)\\ &\cong \operatorname{Hom}_{\mathrm{K}(kH)}(\Sigma^n y, \operatorname{Res}_H^G(Q))\\ &= (*)\end{aligned}$$

by adjointness, since $\operatorname{Ind}_H^G(y) = kG \otimes_{kH} y$ while Res_H^G restricts kG-modules to kH-modules. If $(*)$ is 0 then so is

$$\operatorname{Hom}_{\mathrm{K}(kH)}(\Sigma^n m, \operatorname{Res}_H^G(Q))$$

for each m in $\operatorname{Mod}(kH)$, since $\operatorname{Mod}(kH)$ equals $\operatorname{Add}(y)$ by [1, cor. 4.8] because kH has finite representation type. But if this Hom is 0 for each m and each n, then $\operatorname{Res}_H^G(Q)$ is isomorphic to 0 in $\mathrm{K}(kH)$ by an easy argument, and it follows that $\operatorname{Res}_H^G(Q)$ is split exact; that is, Q is in $\operatorname{Tate}_H(kG)$. □

Proposition 8. *If kH has finite representation type, then* $\mathrm{K}(H\text{-Proj})$ *is compactly generated.*

Proof. By Remark 6 we have that H-Proj equals $\operatorname{Add}(x)$ and that $\operatorname{Add}(x)$ is equivalent to $\operatorname{Proj}(\Gamma^{\mathrm{o}})$ where $\Gamma = \operatorname{End}_{kG}(x)$. So $\mathrm{K}(H\text{-Proj})$ is equivalent to $\mathrm{K}(\operatorname{Proj}(\Gamma^{\mathrm{o}}))$.

However, Γ is a finite-dimensional algebra over k so it is artinian. In particular, it is coherent and each flat right-Γ-module is projective. Hence by [8, thm. 2.4], the category $\mathrm{K}(\mathrm{Proj}(\Gamma^{\circ}))$ is compactly generated, and the present proposition follows. □

Corollary 9. *If kH has finite representation type, then $\mathrm{Tate}_H(kG)$ is compactly generated.*

Proof. The category $\mathrm{K}(H\text{-Proj})$ is compactly generated by Proposition 8, and $\mathrm{Tate}_H(kG) = x^{\perp}$ by Lemma 7.

But x is a compact object of $\mathrm{K}(H\text{-Proj})$, as follows for instance from the formula

$$\mathrm{Hom}_{\mathrm{K}(H\text{-Proj})}(x, -) \simeq \mathrm{H}^0 \,\mathrm{Hom}_{kG}(x, -)$$

since x is finitely generated over kG.

So $\mathrm{Tate}_H(kG)$ is the right perpendicular category of a compact object, so it is compactly generated by [7, prop. 1.7(1)]. □

Finally we have:

Proof of Theorem from page 2. Combine Proposition 4 with Corollary 9. □

Remark 10. It is not clear that our methods can be used to compute a set of compact generators of $\mathrm{StMod}_H(kG)$.

To do so, we would need to find a set of compact generators of the category $\mathrm{Tate}_H(kG)$ and then use the equivalence Z^0. By unravelling the proof of [7, prop. 1.7(1)], it can be seen that the compact generators of $\mathrm{Tate}_H(kG)$ would come by taking a set of compact generators of $\mathrm{K}(H\text{-Proj})$ and applying the left adjoint to the inclusion of $\mathrm{Tate}_H(kG)$ into $\mathrm{K}(H\text{-Proj})$. This left adjoint is constructed by Neeman in [9], but the construction is infinite and does not obviously lend itself to concrete computations.

It would be interesting to find a procedure whereby a set of compact generators of $\mathrm{StMod}_H(kG)$ could be computed.

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