

# SPARSENESS OF T-STRUCTURES AND NEGATIVE CALABI–YAU DIMENSION IN TRIANGULATED CATEGORIES GENERATED BY A SPHERICAL OBJECT

THORSTEN HOLM, PETER JØRGENSEN, AND DONG YANG

ABSTRACT. Let  $k$  be an algebraically closed field and let  $\mathsf{T}$  be the  $k$ -linear algebraic triangulated category generated by a  $w$ -spherical object for an integer  $w$ . For certain values of  $w$  this category is classical. For instance, if  $w = 0$  then it is the compact derived category of the dual numbers over  $k$ .

As main results of the paper we show that for  $w \leq 0$ , the category  $\mathsf{T}$  has no non-trivial t-structures, but does have one family of non-trivial co-t-structures, whereas for  $w \geq 1$  the opposite statement holds.

Moreover, without any claim to originality, we observe that for  $w \leq -1$ , the category  $\mathsf{T}$  is a candidate to have negative Calabi–Yau dimension since  $\Sigma^w$  is the unique power of the suspension functor which is a Serre functor.

## 0. INTRODUCTION

Let  $k$  be an algebraically closed field,  $w$  an integer, and let  $\mathsf{T}$  be a  $k$ -linear algebraic triangulated category which is idempotent complete and classically generated by a  $w$ -spherical object.

The categories  $\mathsf{T}$ , examined initially in [10] for  $w \geq 2$ , have recently been of considerable interest, see [5], [7], [13], [15], and [18]. The purpose of this paper is twofold.

First, we show the following main result.

**Theorem A.** *If  $w \leq 0$ , then  $\mathsf{T}$  has no non-trivial t-structures. It has one family of non-trivial co-t-structures, all of which are (de)suspensions of a canonical one.*

*If  $w \geq 1$ , then  $\mathsf{T}$  has no non-trivial co-t-structures. It has one family of non-trivial t-structures, all of which are (de)suspensions of a canonical one.*

For  $w \leq 0$  this is a particularly clean instance of Bondarko’s remark [4, rmk. 4.3.4.4] that there are sometimes “more” co-t-structures than t-structures in a triangulated category. Note that the case  $w = 2$  is originally due to Ng [15, thms. 4.1 and 4.2].

Secondly, without any claim to originality, we observe that if  $w \leq -1$  then  $\mathsf{T}$  is a candidate for having negative Calabi–Yau dimension, although there does not yet appear to be a universally accepted definition of this concept. Namely,

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the  $w$ 'th power of the suspension functor,  $\Sigma^w$ , is a Serre functor for  $\mathbb{T}$ , and  $\Sigma^w$  is the *only* power of the suspension which is a Serre functor. For  $w \geq 2$  this is contained in [10, prop. 6.5]. For a general  $w$  it is well known to the experts; we show an easy proof in Proposition 1.8.

The proof of Theorem A occupies Section 4 while Sections 1 to 3 are preparatory. Let us end the introduction by giving some background and explaining the terms used above.

### 0.a. What is $\mathbb{T}$ ?

For certain small values of  $w$ , the category  $\mathbb{T}$  is well known in different guises: For  $w = 0$  it is  $\mathbf{D}^c(k[X]/(X^2))$ , the compact derived category of the dual numbers. For  $w = 1$  it is  $\mathbf{D}^f(k[[X]])$ , the derived category of complexes with bounded finite length homology over the formal power series ring. And for  $w = 2$  it is the cluster category of type  $A_\infty$ , see [7]. For  $w$  negative,  $\mathbb{T}$  is less classical.

In general,  $\mathbb{T}$  is determined up to triangulated equivalence by the properties stated in the first paragraph of the paper by [13, thm. 2.1]. We briefly explain these properties:

A triangulated category is algebraic if it is the stable category of a Frobenius category; see [6, sec. 9].

An additive category  $\mathbf{A}$  is idempotent complete if, for each idempotent  $e$  in an endomorphism ring  $\mathbf{A}(a, a)$ , we have  $e = \iota\pi$  where  $\iota$  and  $\pi$  are the inclusion and projection of a direct summand of  $a$ . Note that  $\mathbf{A}(-, -)$  is shorthand for  $\mathrm{Hom}_{\mathbf{A}}(-, -)$ .

A  $w$ -spherical object  $s$  in a  $k$ -linear triangulated category  $\mathbf{S}$  is defined by having graded endomorphism algebra  $\mathbf{S}(s, \Sigma^*s)$  isomorphic to  $k[X]/(X^2)$  with  $X$  placed in cohomological degree  $w$ .

A triangulated category  $\mathbf{S}$  is classically generated by an object  $s$  if each object in  $\mathbf{S}$  can be built from  $s$  using finitely many (de)suspensions, distinguished triangles, and direct summands.

### 0.b. t-structures and co-t-structures.

To explain these, we first introduce the more fundamental notion of a torsion pair in a triangulated category due to Iyama and Yoshino [9, def. 2.2].

If  $\mathbf{S}$  is a triangulated category, then a torsion pair in  $\mathbf{S}$  is a pair  $(\mathbf{M}, \mathbf{N})$  of full subcategories closed under direct sums and summands, satisfying that  $\mathbf{S}(\mathbf{M}, \mathbf{N}) = 0$  and that  $\mathbf{S} = \mathbf{M} * \mathbf{N}$  where  $\mathbf{M} * \mathbf{N}$  stands for the class of objects  $s$  appearing in distinguished triangles  $m \rightarrow s \rightarrow n$  with  $m \in \mathbf{M}$ ,  $n \in \mathbf{N}$ .

A torsion pair  $(\mathbf{M}, \mathbf{N})$  is called a t-structure if  $\Sigma\mathbf{M} \subseteq \mathbf{M}$ , and a co-t-structure if  $\Sigma^{-1}\mathbf{M} \subseteq \mathbf{M}$ . In each case, the structure is called trivial if it is  $(\mathbf{S}, 0)$  or  $(0, \mathbf{S})$  and non-trivial otherwise.

This is not how t-structures and co-t-structures were first defined by Beilinson, Bernstein, and Deligne in [3, def. 1.3.1], respectively by Bondarko and Pauksztello

in [4, def. 1.1.1] and [16, def. 2.4], but it is an economical way to present them and to highlight their dual natures.

t-structures have become classical objects of homological algebra while co-t-structures were introduced more recently. They both enable one to “slice” objects of a triangulated category into simpler bits and they are the subject of vigorous research.

### 0.c. Silting subcategories.

We are grateful to Changjian Fu for the following observation:  $\Sigma^w$  is a Serre functor of  $\mathbb{T}$ . In the terminology of [1] this means that  $\mathbb{T}$  is  $w$ -Calabi–Yau. Moreover,  $\mathbb{T}$  is generated by a  $w$ -spherical object  $s$ ; in particular, for  $w \leq -1$  we have  $\mathbb{T}(s, \Sigma^{>0}s) = 0$ . In the terminology of [1], this means that  $s$  is a silting object.

So for  $w \leq -1$ , the category  $\mathbb{T}$  is  $w$ -Calabi–Yau with the silting subcategory  $\text{add}(s)$ . The existence of a category with these properties was left as a question at the end of [1, sec. 2.1].

It is not hard to check directly that for  $w \leq -1$ , the basic silting objects in  $\mathbb{T}$  are precisely the (de)suspensions of  $s$ . This also follows from [1, thm. 2.26].

## 1. BASIC PROPERTIES OF $\mathbb{T}$

None of the material of this section is original, but not all of it is given explicitly in the original references [5], [10], and [13]. We give a brief, explicit presentation to facilitate the rest of the paper.

**Remark 1.1.** The category  $\mathbb{T}$  is Krull-Schmidt by [17, p. 52]. Namely, it is idempotent complete by assumption, and it has finite dimensional Hom spaces because each object is finitely built from a  $w$ -spherical object  $s$  which in particular satisfies  $\dim_k \mathbb{T}(s, \Sigma^i s) < \infty$  for each  $i$ .

We need to compute inside  $\mathbb{T}$ . Hence a concrete model is more useful than an abstract characterisation. Let us redefine  $\mathbb{T}$  as such a model.

**Definition 1.2.** Set  $d = w - 1$  and consider the polynomial ring  $k[T]$  as a Differential Graded (DG) algebra with  $T$  in homological degree  $d$  and zero differential. We denote this DG algebra by  $A$ .

Consider  $\mathbb{D}(A)$ , the derived category of DG left- $A$ -modules, and let  $\mathbb{T}$  be  $\langle k \rangle$ , the thick subcategory generated by the trivial DG module  $k = A/(T)$  where  $(T)$  is the DG ideal generated by  $T$ .

This is how  $\mathbb{T}$  will be defined for the rest of the paper, except in the proof of Proposition 1.8. It is compatible with the previous definition of  $\mathbb{T}$  by the following result.

**Lemma 1.3.** *The category  $\mathbb{T} = \langle k \rangle$  is a  $k$ -linear algebraic triangulated category which is idempotent complete and classically generated by the  $w$ -spherical object  $k$ .*

*Proof.* The only part which is not clear is that  $k$  is  $w$ -spherical. But there is a distinguished triangle

$$\Sigma^d A \xrightarrow{\cdot T} A \longrightarrow k \quad (1)$$

in  $\mathbf{D}(A)$ , induced by the corresponding short exact sequence of DG modules. Applying  $\mathbf{R}\mathrm{Hom}_A(-, k)$  gives another distinguished triangle whose long exact homology sequence shows that  $k$  is a  $w$ -spherical object of  $\mathbf{D}(A)$ .  $\square$

**Remark 1.4.** The distinguished triangle (1) also shows that  $k$  is a compact object of  $\mathbf{D}(A)$ , so  $\mathbb{T}$  is even a subcategory of the compact derived category  $\mathbf{D}^c(A)$ .

**Definition 1.5.** For each  $r \geq 0$ , the element  $T^{r+1}$  of  $A$  generates a DG ideal  $(T^{r+1})$ . Consider the quotient  $X_r = A/(T^{r+1})$  as a DG left- $A$ -module.

**Remark 1.6.** There is a distinguished triangle

$$\Sigma^{(r+1)d} A \xrightarrow{\cdot T^{r+1}} A \longrightarrow X_r$$

in  $\mathbf{D}^c(A)$ , induced by the corresponding short exact sequence of DG modules.

**Proposition 1.7.** *The indecomposable objects of  $\mathbb{T}$  are precisely the (de)suspensions of the objects  $X_r$ .*

*Proof.* Note that  $\mathbf{D}^c(A) = \langle A \rangle$  and that  $\mathrm{Hom}_{\mathbf{D}^c(A)}(A, \Sigma^* A)$  is isomorphic to  $k[T]$  as a graded algebra, where  $T$  is still in homological degree  $d$ . Since  $\mathbf{gr}(k[T]^{\mathrm{op}})$ , the abelian category of finitely generated graded right- $k[T]$ -modules and graded homomorphisms, is hereditary, [13, thm. 3.6] says that the functor

$$\mathrm{Hom}_{\mathbf{D}^c(A)}(A, \Sigma^*(-)) = \mathbf{H}^*(-)$$

induces a bijection between the isomorphism classes of indecomposable objects of  $\mathbf{D}^c(A)$  and  $\mathbf{gr}(k[T]^{\mathrm{op}})$ . This has the following consequences.

If  $w \neq 1$  then  $d \neq 0$ . Then up to isomorphism, the indecomposable objects of  $\mathbf{gr}(k[T]^{\mathrm{op}})$  are precisely the graded shifts of the graded modules  $k[T]$  and  $k[T]/(T^{r+1})$  for  $r \geq 0$ . So up to isomorphism, the indecomposable objects of  $\mathbf{D}^c(A)$  are the (de)suspensions of  $A$  and the objects  $X_r$  for  $r \geq 0$ .

Of these objects, precisely the  $X_r$  are in  $\mathbb{T}$ , so up to isomorphism the indecomposable objects of  $\mathbb{T}$  are the (de)suspensions of the objects  $X_r$  for  $r \geq 0$ .

If  $w = 1$  then  $d = 0$  so  $A$  and  $k[T]$  are concentrated in degree 0. A graded right- $k[T]$ -module is the direct sum of its graded components, and it follows that the indecomposable objects of  $\mathbf{gr}(k[T]^{\mathrm{op}})$  are the indecomposable ungraded right- $k[T]$ -modules placed in a single graded degree. But up to isomorphism, these are  $k[T]$  and  $k[T]/(f(T))$  where  $f(T)$  is a power of an irreducible, hence first degree, polynomial. So up to isomorphism, the indecomposable objects of  $\mathbf{D}^c(A)$  are the (de)suspensions of the objects  $A$  and  $A/(f(T))$  viewed in  $\mathbf{D}^c(A)$ .

Again, of these objects, precisely the  $X_r$  are in  $\mathbb{T}$ , so up to isomorphism the indecomposable objects of  $\mathbb{T}$  are the (de)suspensions of the objects  $X_r$  for  $r \geq 0$ .  $\square$

It is not hard to see that  $A$  is the  $w$ -Calabi–Yau completion of  $k$  in the sense of [11, 4.1]. As a consequence,  $\mathbb{T} = \langle k \rangle$  has Serre functor  $S = \Sigma^w$ . Here we give a direct proof of this fact.

**Proposition 1.8.** *The category  $\mathbb{T}$  has Serre functor  $S = \Sigma^w$ , and this is the only power of the suspension which is a Serre functor.*

*Proof.* For this proof only, it is convenient to use another model for  $\mathbb{T}$ . Consider the dual numbers  $k[U]/(U^2)$  and view them as a DG algebra with  $U$  placed in cohomological degree  $w$  and zero differential. Denoting this DG algebra by  $B$ , it is immediate that  $B$  is a  $w$ -spherical object of  $\mathbf{D}(B)$ , the derived category of DG left- $B$ -modules, and so the thick subcategory  $\langle B \rangle$  generated by  $B$  is equivalent to  $\mathbb{T}$ . This is the model we will use. In fact,  $\langle B \rangle$  is equal to the compact derived category  $\mathbf{D}^c(B)$ .

For  $X, Y \in \mathbf{D}^c(B)$  we have the following natural isomorphisms where  $D(-) = \mathrm{Hom}_k(-, k)$ .

$$\begin{aligned} D \mathrm{RHom}_B(Y, DB \overset{\mathrm{L}}{\otimes}_B X) &\stackrel{(a)}{\cong} D(\mathrm{RHom}_B(Y, DB) \overset{\mathrm{L}}{\otimes}_B X) \\ &\stackrel{(b)}{\cong} \mathrm{RHom}_{B^{\mathrm{op}}}( \mathrm{RHom}_B(Y, DB), DX) \\ &\stackrel{(c)}{\cong} \mathrm{RHom}_{B^{\mathrm{op}}}(DY, DX) \\ &\stackrel{(d)}{\cong} \mathrm{RHom}_B(X, Y). \end{aligned}$$

Here (a) holds for  $X = B$  and hence for the given  $X$  because it is finitely built from  $B$ . The isomorphisms (b) and (c) are by adjointness of  $\overset{\mathrm{L}}{\otimes}$  and  $\mathrm{RHom}$ . And (d) is duality.

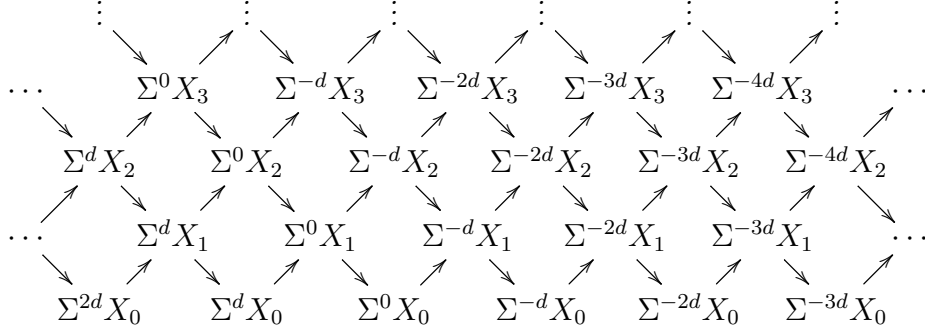
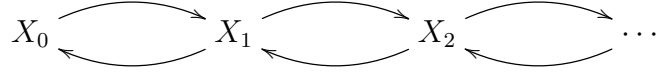
Taking zeroth homology of the above formula shows that  $DB \overset{\mathrm{L}}{\otimes}_B -$  is a right Serre functor of  $\mathbf{D}^c(B)$ . But direct computation shows  $DB \cong \Sigma^w B$  as DG  $B$ -bimodules, so  $\Sigma^w$  is a right Serre functor. Since it is an equivalence of categories, it is even a Serre functor.

Finally, no other power of  $\Sigma$  is a Serre functor of  $\mathbf{D}^c(B)$ : If  $\Sigma^i$  is a Serre functor then  $\Sigma^i \simeq \Sigma^w$  whence  $\Sigma^{i-w} \simeq \mathrm{id}$ . This implies  $i = w$  since already  $\Sigma^{i-w} B \cong B$  implies  $i = w$  as one sees by taking homology.  $\square$

**Remark 1.9.** The AR translation of  $\mathbb{T}$  is  $\tau = \Sigma^{-1}S = \Sigma^{w-1} = \Sigma^d$ .

**Proposition 1.10.** (i) *If  $w \neq 1$  then the AR quiver of  $\mathbb{T}$  consists of  $|d|$  copies of  $\mathbb{Z}A_\infty$ . One copy is shown in Figure 1 and the others are obtained by applying  $\Sigma, \Sigma^2, \dots, \Sigma^{|d|-1}$ .*

(ii) *If  $w = 1$  then the AR quiver of  $\mathbb{T}$  consists of countably many homogeneous tubes. One tube is shown in Figure 2 and the others are obtained by applying all non-zero powers of  $\Sigma$ .*

FIGURE 1. A component of the AR quiver for  $w \neq 1$ FIGURE 2. A component of the AR quiver for  $w = 1$ 

*Proof.* For  $w \geq 2$  this is [10, thm. 8.13].

For  $w$  general, the shape of the AR quiver is given in [5, sec. 3.3]. For  $w \leq 0$ , to see that the  $|d|$  copies of  $\mathbb{Z}A_\infty$  look as claimed, one can compute the AR triangles of  $\mathbb{T}$  by methods similar to those of [10, sec. 8].

Finally, for  $w = 1$  we have  $d = 0$ . The AR translation is  $\tau = \Sigma^0 = \text{id}$  by Remark 1.9, so for each  $X_r$  there is an AR triangle  $X_r \rightarrow Y \rightarrow X_r$ . The long exact homology sequence shows that if  $r = 0$  then  $Y = X_1$  and if  $r \geq 1$  then  $Y = X_{r-1} \oplus X_{r+1}$ . Hence the homogeneous tube in Figure 2 is a component of the AR quiver as claimed. For each  $i$ , applying  $\Sigma^i$  to Figure 2 gives a component of the AR quiver. The components obtained in this fashion contain all indecomposable objects of  $\mathbb{T}$  so form the whole AR quiver.  $\square$

## 2. MORPHISMS IN $\mathbb{T}$

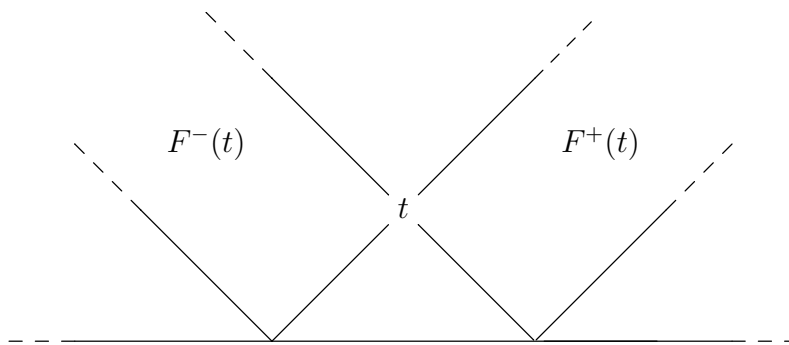
This section computes the Hom spaces between indecomposable objects in the category  $\mathbb{T}$ .

**Definition 2.1.** Suppose that  $w \neq 1$  so the AR quiver of  $\mathbb{T}$  consists of copies of  $\mathbb{Z}A_\infty$  by Proposition 1.10(i). Let  $t \in \mathbb{T}$  be an indecomposable object. Figure 3 defines two sets  $F^\pm(t)$  consisting of indecomposable objects in the same component of the AR quiver as  $t$ . Each set can be described as a rectangle stretching off to infinity in one direction; it consists of the objects inside the indicated boundaries including the ones on the boundaries. In particular we have  $t \in F^\pm(t)$ .

**Proposition 2.2.** *Suppose that  $w \neq 0, 1$ . Let  $t, u$  be indecomposable objects in  $\mathbb{T}$ . Then*

$$\dim_k \mathbb{T}(t, u) = \begin{cases} 1 & \text{for } u \in F^+(t) \cup F^-(St), \\ 0 & \text{otherwise,} \end{cases}$$

where  $S = \Sigma^w$  is the Serre functor of  $\mathbb{T}$ ; see Proposition 1.8.

FIGURE 3. The regions  $F^\pm(t)$  for  $w \neq 1$ 

**Proposition 2.3.** *Suppose that  $w = 0$ . Let  $t, u$  be indecomposable objects in  $\mathbb{T}$ . Then*

$$\dim_k \mathbb{T}(t, u) = \begin{cases} 2 & \text{for } u = t, \\ 1 & \text{for } u \in (F^+(t) \cup F^-(t)) \setminus t, \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 2.4.** *Suppose that  $w = 1$ . Let  $u$  be an indecomposable object of  $\mathbb{T}$ . Then*

$$\dim_k \mathbb{T}(X_r, u) = \begin{cases} \min\{r, s\} + 1 & \text{for } u = X_s \text{ or } u = \Sigma X_s, \\ 0 & \text{for all other } u. \end{cases}$$

Note that in Proposition 2.2, the sets  $F^+(t)$  and  $F^-(St)$  are disjoint. For  $w \neq 2$ , they even sit in different components of the AR quiver, while for  $w = 2$  we have  $d = w - 1 = 1$  and the AR quiver has only one component. In Proposition 2.3, the sets  $F^+(t)$  and  $F^-(t)$  have intersection  $t$ . In this case  $w = 0$  so  $d = w - 1 = -1$  and the AR quiver has only one component.

*Proof* of Propositions 2.2, 2.3, and 2.4.

Propositions 2.2 and 2.3 give the dimensions of Hom spaces in a conceptual way using the regions  $F^\pm$ . Unfortunately we do not have a conceptual proof.

The proof we have is pedestrian: Applying  $\mathrm{RHom}_A(-, X_s)$  to the distinguished triangle from Remark 1.6 gives a new distinguished triangle whose long exact homology sequence contains

$$\mathrm{H}^{i-1}(X_s) \rightarrow \mathrm{H}^{i-(r+1)d-1}(X_s) \rightarrow \mathrm{H}^i \mathrm{RHom}_A(X_r, X_s) \rightarrow \mathrm{H}^i(X_s) \rightarrow \mathrm{H}^{i-(r+1)d}(X_s).$$

The middle term is isomorphic to  $\mathbb{T}(X_r, \Sigma^i X_s)$ . The DG module  $X_s$  is  $A/(T^{s+1})$ , so the four outer terms are easily computable. The first and last maps are induced by  $\cdot T^{r+1}$  and can also be computed. Hence the middle term can be determined.

For  $w \neq 1$ , combining the dimensions of Hom spaces with the detailed structure of the AR quiver as described by Proposition 1.10 proves Propositions 2.2 and 2.3, and for  $w = 1$  one gets Proposition 2.4 directly.  $\square$

## 3. T- AND CO-T-STRUCTURES

This section gives some easy properties of t- and co-t-structures. Lemmas 3.1 and 3.2 are valid in general triangulated categories.

Recall that if  $(\mathbf{X}, \mathbf{Y})$  is a t-structure then the heart is  $\mathbf{H} = \mathbf{X} \cap \Sigma\mathbf{Y}$ , and if  $(\mathbf{A}, \mathbf{B})$  is a co-t-structure then the co-heart is  $\mathbf{C} = \mathbf{A} \cap \Sigma^{-1}\mathbf{B}$ .

**Lemma 3.1.** *Let  $(\mathbf{X}, \mathbf{Y})$  be a t-structure and  $(\mathbf{A}, \mathbf{B})$  a co-t-structure with heart and co-heart  $\mathbf{H}$  and  $\mathbf{C}$ .*

- (i)  $\mathrm{Hom}(\mathbf{H}, \Sigma^{<0}\mathbf{H}) = 0$ .
- (ii)  $\mathrm{Hom}(\mathbf{C}, \Sigma^{>0}\mathbf{C}) = 0$ .
- (iii)  $\mathbf{X} = \Sigma\mathbf{X} \Leftrightarrow \mathbf{H} = 0$ .
- (iv)  $\mathbf{A} = \Sigma\mathbf{A} \Leftrightarrow \mathbf{C} = 0$ .

*Proof.* (i) We have  $\mathbf{H} \subseteq \mathbf{X}$  and  $\Sigma^{<0}\mathbf{H} \subseteq \Sigma^{<0}\Sigma\mathbf{Y} = \Sigma^{<0}\mathbf{Y} \subseteq \mathbf{Y}$ . The last  $\subseteq$  is a well known property of t-structures and follows from  $\Sigma^{\geq 0}\mathbf{X} \subseteq \mathbf{X}$  by taking right perpendicular categories; cf. [9, remark after def. 2.2] by which  $\mathbf{X}^\perp = \mathbf{Y}$ . Here  $\perp$  is as defined in [9, start of sec. 2]. But  $\mathrm{Hom}(\mathbf{X}, \mathbf{Y}) = 0$  so  $\mathrm{Hom}(\mathbf{H}, \Sigma^{<0}\mathbf{H}) = 0$  follows.

(ii) Dual to part (i).

(iii)  $\Rightarrow$ : Suppose  $h \in \mathbf{H}$ . Then  $\Sigma^{-1}h \in \Sigma^{-1}\mathbf{H} \subseteq \Sigma^{-1}\mathbf{X} = \mathbf{X}$  so  $h = \Sigma(\Sigma^{-1}h) \in \Sigma\mathbf{X}$ . We also have  $h \in \mathbf{H} \subseteq \Sigma\mathbf{Y}$ . But  $\mathrm{Hom}(\Sigma\mathbf{X}, \Sigma\mathbf{Y}) = \mathrm{Hom}(\mathbf{X}, \mathbf{Y}) = 0$  so  $\mathrm{Hom}(h, h) = 0$  proving  $h = 0$ .

$\Leftarrow$ : For  $x \in \mathbf{X}$  consider the distinguished triangle

$$x' \rightarrow \Sigma^{-1}x \rightarrow y' \tag{2}$$

with  $x' \in \mathbf{X}$ ,  $y' \in \mathbf{Y}$  which exists because  $(\mathbf{X}, \mathbf{Y})$  is a torsion pair. It gives a distinguished triangle  $x \rightarrow \Sigma y' \rightarrow \Sigma^2 x'$  with  $x, \Sigma^2 x' \in \mathbf{X}$ . But  $\mathbf{X}$  is closed under extensions since it is equal to  ${}^\perp\mathbf{Y}$  by [9, remark after def. 2.2] again, so  $\Sigma y' \in \mathbf{X}$ .

We also have  $\Sigma y' \in \Sigma\mathbf{Y}$  so  $\Sigma y' \in \mathbf{H}$  and hence  $\Sigma y' = 0$ . But then the distinguished triangle (2) shows  $\Sigma^{-1}x \cong x' \in \mathbf{X}$ . Hence  $\Sigma^{-1}\mathbf{X} \subseteq \mathbf{X}$ , and since we also know  $\Sigma\mathbf{X} \subseteq \mathbf{X}$  it follows that  $\Sigma\mathbf{X} = \mathbf{X}$ .

(iv) Dual to part (iii). □

A torsion pair  $(\mathbf{M}, \mathbf{N})$  with  $\Sigma\mathbf{M} = \mathbf{M}$  (and consequently  $\Sigma\mathbf{N} = \mathbf{N}$ ) is called a stable t-structure; see [14, p. 468]. In this case,  $\mathbf{M}$  and  $\mathbf{N}$  are thick subcategories of  $\mathbf{T}$ .

**Lemma 3.2.** *If  $(\mathbf{M}, \mathbf{N})$  and  $(\mathbf{M}', \mathbf{N}')$  are torsion pairs with  $\mathbf{M} \subseteq \mathbf{M}'$  and  $\mathbf{N} \subseteq \mathbf{N}'$ , then  $(\mathbf{M}, \mathbf{N}) = (\mathbf{M}', \mathbf{N}')$ .*

*Proof.* The inclusion  $\mathbf{N} \subseteq \mathbf{N}'$  implies  ${}^\perp\mathbf{N} \supseteq {}^\perp\mathbf{N}'$ , but this reads  $\mathbf{M} \supseteq \mathbf{M}'$  by [9, remark after def. 2.2] so we learn  $\mathbf{M} = \mathbf{M}'$ . Hence also  $\mathbf{N} = \mathbf{M}^\perp = \mathbf{M}'^\perp = \mathbf{N}'$ . □

**Lemma 3.3.** *A stable t-structure in  $\mathbf{T}$  is trivial.*

*Proof.* Let  $(\mathbf{X}, \mathbf{Y})$  be a stable t-structure in  $\mathbb{T}$  with  $\mathbf{X} \neq 0$ . Then  $\mathbf{X}$  contains an indecomposable object  $x$ . But  $\mathbf{X}$  is a thick subcategory of  $\mathbb{T}$ , and it is easy to see from the AR quiver of  $\mathbb{T}$  that hence  $\mathbf{X} = \mathbb{T}$ .  $\square$

#### 4. PROOF OF THEOREM A

##### 4.a. Proof of Theorem A for t-structures, $w \leq -1$ .

Let  $(\mathbf{X}, \mathbf{Y})$  be a t-structure in  $\mathbb{T}$  with heart  $\mathbf{H} = \mathbf{X} \cap \Sigma\mathbf{Y}$  and let  $h \in \mathbf{H}$ . Serre duality gives

$$\mathrm{Hom}_k(\mathbb{T}(h, h), k) \cong \mathbb{T}(h, Sh) \cong \mathbb{T}(h, \Sigma^w h) = 0$$

where “= 0” is by Lemma 3.1(i) because  $w \leq -1$ . This implies  $h = 0$  so  $\mathbf{H} = 0$ . But then  $(\mathbf{X}, \mathbf{Y})$  is a stable t-structure by Lemma 3.1(iii) and hence trivial by Lemma 3.3.

##### 4.b. Proof of Theorem A for t-structures, $w = 0$ .

In this case we have  $d = w - 1 = -1$ . The AR quiver consists of  $|d| = 1$  copy of  $\mathbb{Z}A_\infty$  by Lemma 1.10(i); see Figure 1.

Assume that  $(\mathbf{X}, \mathbf{Y})$  is a non-trivial t-structure in  $\mathbb{T}$ . By Lemmas 3.3 and 3.1(iii) the heart  $\mathbf{H}$  is non-zero so contains an indecomposable object.

However, if  $t$  is an indecomposable object not on the base line of the AR quiver then  $\tau t \in F^-(t)$ ; see Figure 3. Hence  $\mathbb{T}(t, \tau t) \neq 0$  by Proposition 2.3, and by Remark 1.9 this reads  $\mathbb{T}(t, \Sigma^{-1}t) \neq 0$ . But  $\mathbb{T}(\mathbf{H}, \Sigma^{<0}\mathbf{H}) = 0$  by Lemma 3.1(i), so each indecomposable object in  $\mathbf{H}$  is forced to be on the base line of the AR quiver.

Moreover, if  $h \in \mathbf{H}$  is indecomposable then  $\mathbf{H}$  cannot contain another indecomposable object  $h'$ : Both objects would have to be on the base line of the AR quiver which has only one component, so we would have  $h' = \tau^i h$  for some  $i \neq 0$ , that is,  $h' = \Sigma^{-i}h$ . But this contradicts  $\mathbb{T}(\mathbf{H}, \Sigma^{<0}\mathbf{H}) = 0$ .

It follows that  $\mathbf{H} = \mathrm{add}(h)$  for an indecomposable object  $h$  on the base line of the AR quiver, and  $h = \Sigma^i X_0$  for some  $i$ . Direct computation shows that  $h$  is 0-spherical, so there is a non-zero, non-invertible morphism  $h \rightarrow h$ . But this morphism is easily verified not to have a kernel in  $\mathbf{H}$ , and this is a contradiction since the heart of a t-structure is abelian.

##### 4.c. Proof of Theorem A for t-structures, $w = 1$ .

Here the AR quiver of  $\mathbb{T}$  consists of countably many stable tubes as detailed in Proposition 1.10(ii).

By [5, sec. 3.1], an alternative model of  $\mathbb{T}$  is  $\mathrm{D}^f(k[[X]])$ , the derived category of complexes with bounded finite length homology over the ring  $k[[X]]$ . This shows that  $\mathbb{T}$  has a canonical t-structure.

Assume that  $(\mathbf{X}, \mathbf{Y})$  is a non-trivial t-structure in  $\mathbb{T}$ . In particular,  $\mathbf{X}$  is closed under extensions. The components of the AR quiver of  $\mathbb{T}$  are homogeneous tubes and the AR triangles of  $\mathbb{T}$  can be read off. The triangles imply that if  $\mathbf{X}$  contains

an indecomposable object  $t$  then it contains the whole component of  $t$ . So  $\mathbf{X}$  is equal to add of a collection of components of the AR quiver. Now let  $Q$  be a component such that  $Q \subseteq \mathbf{X}$  but  $\Sigma^{-1}Q, \Sigma^{-2}Q, \dots \not\subseteq \mathbf{X}$ . Such a  $Q$  exists because  $\mathbf{X}$  is closed under  $\Sigma$  and not equal to 0 or  $\mathbf{T}$ . It is then clear that

$$\mathbf{X} = \text{add}(Q \cup \Sigma Q \cup \dots).$$

The right hand side only depends on the component  $Q$  of the AR quiver, and since all other components have the form  $\Sigma^i Q$  (see Proposition 1.10(ii)), this implies that all non-trivial t-structures are (de)suspensions of each other, and hence (de)suspensions of the canonical t-structure.

#### 4.d. Proof of Theorem A for t-structures, $w \geq 2$ .

Recall that  $A$  is  $k[T]$  viewed as a DG algebra with  $T$  in homological degree  $d = w - 1$  and zero differential. Each object of  $\mathbf{T}$  is a direct sum of finitely many (de)suspensions of the objects  $X_r = A/(T^{r+1})$  by Remark 1.1 and Proposition 1.7. In particular, each object of  $\mathbf{T}$  is isomorphic to a DG module  $t$  which is finite dimensional over  $k$ .

Since  $w \geq 2$  we have  $d \geq 1$  which means that  $A$  is a chain DG algebra. So for each DG left- $A$ -module  $t$  there is a distinguished triangle  $t_{(\geq 0)} \rightarrow t \rightarrow t_{(<0)}$  in  $\mathbf{D}(A)$  induced by the following (vertical) short exact sequence of DG modules.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & t_2 & \longrightarrow & t_1 & \longrightarrow & \text{Ker } \partial_0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \parallel & & \parallel & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & t_2 & \longrightarrow & t_1 & \longrightarrow & t_0 & \xrightarrow{\partial_0} & t_{-1} & \longrightarrow & t_{-2} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \parallel & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & t_0 / \text{Ker } \partial_0 & \longrightarrow & t_{-1} & \longrightarrow & t_{-2} & \longrightarrow & \cdots \end{array}$$

Each of  $t_{(\geq 0)}$  and  $t_{(<0)}$  is also finite dimensional over  $k$  which implies that they can be built in finitely many steps from the DG module  $k$ ; that is, they belong to  $\langle k \rangle = \mathbf{T}$ . Hence the distinguished triangle is in  $\mathbf{T}$ , so  $(\mathbf{T}_{(\geq 0)}, \mathbf{T}_{(<0)})$  is a t-structure in  $\mathbf{T}$  where

$$\begin{aligned} \mathbf{T}_{(\geq 0)} &= \{ t \in \mathbf{T} \mid H_*(t) \text{ is in homological degrees } \geq 0 \}, \\ \mathbf{T}_{(<0)} &= \{ t \in \mathbf{T} \mid H_*(t) \text{ is in homological degrees } < 0 \}. \end{aligned} \quad (3)$$

We refer to this t-structure as canonical. Note that it was first constructed, in higher generality, in [8, thm. 1.3], and in the DG case in [2, lem. 2.2] and [12, lem. 5.2].

It is easy to check that  $\mathbf{T}_{(\geq 0)}$  is the smallest subcategory of  $\mathbf{T}$  which contains  $X_0$  and is closed under  $\Sigma$ , extensions, and direct summands. Likewise,  $\mathbf{T}_{(<0)}$  is the smallest subcategory of  $\mathbf{T}$  which contains  $\Sigma^{-1}X_0$  and is closed under  $\Sigma^{-1}$ , extensions, and direct summands.

Now assume that  $(\mathbf{X}, \mathbf{Y})$  is a non-trivial t-structure in  $\mathbf{T}$ . By Lemmas 3.3 and 3.1(iii) the heart  $\mathbf{H}$  is non-zero so contains an indecomposable object  $h$ . We have  $\mathbf{T}(h, \Sigma^{<0}h) = 0$  by Lemma 3.1(i).

If  $t$  is an indecomposable object not on the base line of the AR quiver then  $\tau^{-1}t \in F^+(t)$ ; see Figure 3. Hence  $\mathbb{T}(t, \tau^{-1}t) \neq 0$  by Proposition 2.2, and by Remark 1.9 this reads  $\mathbb{T}(t, \Sigma^{-d}t) \neq 0$ . Hence  $h$  is forced to be on the base line of the AR quiver. Suspending or desuspending the t-structure, we can assume  $h = X_0$ .

We have  $h \in \mathbb{X}$  and  $h \in \Sigma\mathbb{Y}$  whence  $\Sigma^{-1}h \in \mathbb{Y}$ . That is,  $X_0 \in \mathbb{X}$  and  $\Sigma^{-1}X_0 \in \mathbb{Y}$ .

However,  $\mathbb{X}$  is closed under  $\Sigma$ , extensions, and direct summands, and since  $\mathbb{T}_{(\geq 0)}$  is the smallest subcategory of  $\mathbb{T}$  with these properties which contains  $X_0$ , we get  $\mathbb{T}_{(\geq 0)} \subseteq \mathbb{X}$ . Similarly,  $\mathbb{T}_{(< 0)} \subseteq \mathbb{Y}$ .

By Lemma 3.2 this forces  $(\mathbb{X}, \mathbb{Y}) = (\mathbb{T}_{(\geq 0)}, \mathbb{T}_{(< 0)})$ , and we have shown that as desired, up to (de)suspension, any non-trivial t-structure in  $\mathbb{T}$  is the canonical one.

#### 4.e. Proof of Theorem A for co-t-structures, $w \leq 0$ .

In the proof for t-structures,  $w \geq 2$ , we showed a canonical t-structure. Tweaking the method slightly in the present case produces a canonical co-t-structure. Each object of  $\mathbb{T}$  is still isomorphic to a DG module  $t$  which is finite dimensional over  $k$ . Since  $A$  is  $k[T]$  with  $T$  in homological degree  $d = w - 1$ , and since  $w \leq 0$  and  $d \leq -1$ , we have that  $A$  is a cochain DG algebra. So there is a distinguished triangle  $t_{\leq 0} \rightarrow t \rightarrow t_{> 0}$  in  $\mathbb{D}(A)$  where the subscripts indicate hard truncations in the relevant homological degrees. Each of  $t_{\leq 0}$  and  $t_{> 0}$  is also finite dimensional over  $k$  and is therefore in  $\mathbb{T}$ . Hence  $(\mathbb{T}_{(\leq 0)}, \mathbb{T}_{(> 0)})$  is a co-t-structure in  $\mathbb{T}$  where

$$\begin{aligned} \mathbb{T}_{(\leq 0)} &= \{ t \in \mathbb{T} \mid H_*(t) \text{ is in homological degrees } \leq 0 \}, \\ \mathbb{T}_{(> 0)} &= \{ t \in \mathbb{T} \mid H_*(t) \text{ is in homological degrees } > 0 \}. \end{aligned}$$

The rest of the proof is dual to the proof for t-structures,  $w \geq 2$ .

#### 4.f. Proof of Theorem A for co-t-structures, $w \geq 1$ .

This is dual to the proof for t-structures,  $w \leq -1$ .

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INSTITUT FÜR ALGEBRA, ZAHLENTHEORIE UND DISKRETE MATHEMATIK, FAKULTÄT FÜR  
 MATHEMATIK UND PHYSIK, LEIBNIZ UNIVERSITÄT HANNOVER, WELFENGARTEN 1, 30167  
 HANNOVER, GERMANY

*E-mail address:* [holm@math.uni-hannover.de](mailto:holm@math.uni-hannover.de)

*URL:* <http://www.iazd.uni-hannover.de/~tholm>

SCHOOL OF MATHEMATICS AND STATISTICS, NEWCASTLE UNIVERSITY, NEWCASTLE UPON  
 TYNE NE1 7RU, UNITED KINGDOM

*E-mail address:* [peter.jorgensen@ncl.ac.uk](mailto:peter.jorgensen@ncl.ac.uk)

*URL:* <http://www.staff.ncl.ac.uk/peter.jorgensen>

*E-mail address:* [dongyang2002@gmail.com](mailto:dongyang2002@gmail.com)