

ON THE RELATION BETWEEN CLUSTER AND CLASSICAL TILTING

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ABSTRACT. Let D be a triangulated category with a cluster tilting subcategory \mathcal{U} . The quotient category D/\mathcal{U} is abelian; suppose that it has finite global dimension.

We show that projection from D to D/\mathcal{U} sends cluster tilting subcategories of D to support tilting subcategories of D/\mathcal{U} , and that, in turn, support tilting subcategories of D/\mathcal{U} can be lifted uniquely to weak cluster tilting subcategories of D .

0. INTRODUCTION

Classical tilting is a major subject in the representation theory of finite dimensional algebras. According to the historical remarks in [1, chp. VI], classical tilting theory goes back to the study of reflection functors by Bernstein, Gelfand, and Ponomarev in [5] and by Auslander, Platzeck, and Reiten in [4]. It was later axiomatized by Brenner and Butler in [7] and by Happel and Ringel in [13], and is now one of the mainstays of representation theory.

Let Q be a finite quiver without loops and cycles and consider the module category $\text{mod } kQ$ of the path algebra kQ . The principal notion of classical tilting theory is that of a tilting module T in $\text{mod } kQ$. Such a module satisfies $\text{Ext}_{kQ}^1(T, T) = 0$ and permits an exact sequence $0 \rightarrow kQ \rightarrow T^0 \rightarrow T^1 \rightarrow 0$ where the T^i are in $\text{add } T$, the category of direct summands of (finite) direct sums of copies of T . In this situation, $A = \text{End}_{kQ}(T)^\circ$ is called a tilted algebra.

Cluster tilting is a recent, important development in tilting theory where tilting modules are replaced by so-called cluster tilting objects; see [10] or the surveys in [8] and [22]. These objects typically live in the cluster category \mathcal{C} which is the orbit category $D^f(kQ)/\tau^{-1}\Sigma$,

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where $D^f(kQ)$ is the finite derived category of kQ while τ and Σ are the Auslander-Reiten translation and the suspension functor of $D^f(kQ)$. The category \mathcal{C} is triangulated, and a cluster tilting object U in \mathcal{C} is defined by satisfying

$$u \in \text{add } U \Leftrightarrow \mathcal{C}(U, \Sigma u) = 0$$

and

$$u \in \text{add } U \Leftrightarrow \mathcal{C}(u, \Sigma U) = 0$$

for u in \mathcal{C} . In this situation, $A = \text{End}_{\mathcal{C}}(U)^\circ$ is called a cluster tilted algebra.

For any vertex which is a sink or source of Q , classical tilting theory permits the construction of a tilting module whose tilted algebra has quiver Q' given by inverting the arrows of Q incident to the sink or source. One of the exciting new aspects of cluster tilting theory is that, in a sense, it permits the extension of this to arbitrary vertices of Q ; see [10, sec. 4].

A result by Ingalls and Thomas throws light on the relation between cluster and classical tilting. The following precise statement is part of the main theorem of [15] which also introduced the concept of support tilting modules.

Theorem A (Ingalls and Thomas). *Let Q be a finite quiver without loops and cycles and let \mathcal{C} be the cluster category of type Q over an algebraically closed field k .*

Then there is a bijection between the isomorphism classes of basic cluster tilting objects of \mathcal{C} and the isomorphism classes of basic support tilting modules in $\text{mod } kQ$.

As the name suggest, a support tilting module T in $\text{mod } kQ$ is a module which is tilting on its support: It satisfies $\text{Ext}_{kQ}^1(T, T) = 0$ and is a tilting module for the algebra $kQ/\text{ann } T$ which turns out to be the path algebra of the support of T in Q ; see [15, prop. 2.5 and lem. 2.6].

To prove this theorem, Ingalls and Thomas view $\text{mod } kQ$ as a subcategory of \mathcal{C} by means of the canonical functors

$$\text{mod } kQ \rightarrow D^f(kQ) \rightarrow D^f(kQ)/\tau^{-1}\Sigma = \mathcal{C}. \quad (1)$$

The existence of these relies on the definition of \mathcal{C} as an orbit category of $D^f(kQ)$. However, there is a dual viewpoint which can be generalized to other triangulated categories, whereby $\text{mod } kQ$ is a quotient category of \mathcal{C} . Namely, kQ can be viewed as a module over itself and hence also as an object of \mathcal{C} . As such, it is the ‘‘canonical’’ cluster tilting object of \mathcal{C} , and the quotient category $\mathcal{C}/\text{add } kQ$ is equivalent to $\text{mod } kQ$.

Under this viewpoint, the theorem states a relation between the cluster tilting objects of the triangulated category \mathcal{C} and the support tilting objects of the abelian quotient category $\mathcal{C}/\text{add } kQ$.

The results of this paper provide similar relations in a general setup between a triangulated category \mathcal{D} and the abelian quotient category \mathcal{D}/\mathcal{U} , where \mathcal{U} is a cluster tilting subcategory (see Definition 1.2). It was proved by König and Zhu that \mathcal{D}/\mathcal{U} is indeed abelian; see [20].

Suppose that \mathcal{D} satisfies the technical conditions of Setup 1.1 below, and assume that \mathcal{D}/\mathcal{U} has finite global dimension. Our first main result is the following.

Theorem B. *Let \mathcal{V} be a cluster tilting subcategory of \mathcal{D} . Then the image $\underline{\mathcal{V}}$ in \mathcal{D}/\mathcal{U} is a support tilting subcategory of \mathcal{D}/\mathcal{U} .*

From the Serre functor S and the suspension functor Σ of \mathcal{D} can be constructed the autoequivalence $S\Sigma^{-2}$ of \mathcal{D} . It induces an autoequivalence of \mathcal{D}/\mathcal{U} which we also denote $S\Sigma^{-2}$. Observe that the notion of a cluster tilting subcategory can be relaxed by dropping the precovering and preenveloping conditions; this gives the notion of a weak cluster tilting subcategory (see Definition 1.2). Our second main result is the following.

Theorem C. *Assume that each object of \mathcal{D}/\mathcal{U} has finite length. Let $\underline{\mathcal{W}}$ be a support tilting subcategory of \mathcal{D}/\mathcal{U} with $S\Sigma^{-2}\underline{\mathcal{W}} = \underline{\mathcal{W}}$. Then there is a unique subcategory \mathcal{X} of \mathcal{D} which is weak cluster tilting and whose image $\underline{\mathcal{X}}$ in \mathcal{D}/\mathcal{U} satisfies $\underline{\mathcal{X}} = \underline{\mathcal{W}}$.*

The assumption $S\Sigma^{-2}\underline{\mathcal{W}} = \underline{\mathcal{W}}$ is reasonable in the context: In good cases, \mathcal{X} is not just weak cluster tilting but cluster tilting, and then $S\Sigma^{-2}\mathcal{X} = \mathcal{X}$ by [20, prop. 4.7.3] which forces $S\Sigma^{-2}\underline{\mathcal{W}} = \underline{\mathcal{W}}$.

The assumption that each object of \mathcal{D}/\mathcal{U} has finite length is equivalent to \mathcal{U} being locally finite in the sense that, for each indecomposable object u of \mathcal{U} , there are only finitely many isomorphism classes of indecomposable objects u' of \mathcal{U} with $\mathcal{U}(u', u) \neq 0$ or $\mathcal{U}(u, u') \neq 0$. To see this, first note that [20, prop. 4.7(3)] implies that \mathcal{U} is locally finite under the weaker condition that, for each indecomposable object u of \mathcal{U} , there are only finitely many isomorphism classes of indecomposable objects u' of \mathcal{U} with $\mathcal{U}(u', u) \neq 0$. Then combine the equivalence $\mathcal{D}/\mathcal{U} \simeq \text{mod } \Sigma^{-1}\mathcal{U}$ of [20, cor. 4.4] with a selection of results from [2] (section III.2, proposition, part b.iii) and [3] (propositions 1.11, 2.1(c), and theorem 2.12).

Theorem C is related to results by Fu-Liu [11] and Smith [25] who show that tilting objects lift to cluster tilting objects. Our theorem is more general since it considers support tilting subcategories, not just tilting objects, and provides a unique lift.

However, note that Fu-Liu and Smith do not impose the condition that D/\mathcal{U} has finite global dimension, and this makes it natural to think that there may be a version of Theorem C without this condition. Unfortunately, our proof relies on formulae for Ext groups in D/\mathcal{U} in terms of data in D , and we can presently only prove such formulae when certain homological dimensions are finite. In practice, this forces us to assume that D/\mathcal{U} has finite global dimension. Note that Theorem B fails without finite global dimension of D/\mathcal{U} , as shown in Section 4.c.

The paper is organized as follows: Section 1 prepares the ground by proving the mentioned formulae for Ext groups in D/\mathcal{U} (Proposition 1.5); this should be of independent interest. Section 2 proves Theorem B (see Theorem 2.2), and Section 3 proves Theorem C (see Theorem 3.5). Section 4 considers some examples: Derived categories of path algebras, the category of type A_∞ studied in [14], and a counterexample to show that Theorem B fails without finite global dimension.

Remark 0.1. We will follow a common abuse of terminology by saying that subcategories are equal when we really mean that they have the same essential closure, that is, intersect the same set of isomorphism classes in the ambient category. For instance, the equation $S\Sigma^{-2}\underline{\mathcal{U}} = \underline{\mathcal{U}}$ in Theorem C must be read according to this remark.

1. EXT GROUPS IN AN ABELIAN QUOTIENT OF A TRIANGULATED CATEGORY

This section gives some background on the abelian quotient category D/\mathcal{U} . The main item is Proposition 1.5 which, under certain conditions, gives formulae for the Ext groups of D/\mathcal{U} in terms of data in the triangulated category D .

Setup 1.1. In the rest of the paper, k is an algebraically closed field and D is a skeletally small k -linear triangulated category with finite dimensional Hom spaces and split idempotents which has Serre functor S .

We denote by \mathcal{U} a cluster tilting subcategory of D .

We refer to [21, sec. I.1] for background on Serre functors, but wish to recall the following definitions; cf. [9], [16], [17], [18], and [19].

Definition 1.2. A full subcategory \mathcal{V} of \mathbf{D} is called *weak cluster tilting* if it satisfies

$$v \in \mathcal{V} \Leftrightarrow \mathbf{D}(\mathcal{V}, \Sigma v) = 0$$

and

$$v \in \mathcal{V} \Leftrightarrow \mathbf{D}(v, \Sigma \mathcal{V}) = 0.$$

A weak cluster tilting subcategory is called *cluster tilting* if it is precovering and preenveloping.

Remark 1.3. Recall that \mathcal{V} is called *precovering* if each object x of \mathbf{D} has a \mathcal{V} -precover, that is, a morphism $v \rightarrow x$ with v in \mathcal{V} through which any other morphism $v' \rightarrow x$ with v' in \mathcal{V} factors. Dually, \mathcal{V} is called *preenveloping* if each object x of \mathbf{D} has a \mathcal{V} -preenvelope, that is, a morphism $x \rightarrow v$ with v in \mathcal{V} through which any other morphism $x \rightarrow v'$ with v' in \mathcal{V} factors.

The category \mathbf{D} may be locally finite in the sense that, for each indecomposable object x of \mathbf{D} , there are only finitely many isomorphism classes of indecomposable objects x' in \mathbf{D} which satisfy $\mathbf{D}(x', x) \neq 0$ or $\mathbf{D}(x, x') \neq 0$. If this is the case, then each full subcategory of \mathbf{D} which is closed under direct sums and direct summands is precovering and preenveloping, and hence the notions of weak cluster tilting subcategory and cluster tilting subcategory are synonymous.

Our distinction between weak cluster tilting and cluster tilting subcategories is not standard, but it is useful for this paper.

Note that precovering classes are often referred to as *contravariantly finite* and preenveloping classes as *covariantly finite*.

Remark 1.4. The quotient category \mathbf{D}/\mathcal{U} has the same objects as \mathbf{D} , and its Hom spaces are obtained from those of \mathbf{D} upon dividing by the morphisms which factor through an object of \mathcal{U} . The projection functor $\mathbf{D} \rightarrow \mathbf{D}/\mathcal{U}$ will be denoted by $x \mapsto \underline{x}$. The space of morphisms $x \rightarrow y$ which factor through an object of \mathcal{U} will be denoted $\mathcal{U}(x, y)$, so

$$(\mathbf{D}/\mathcal{U})(\underline{x}, \underline{y}) = \mathbf{D}(x, y) / \mathcal{U}(x, y).$$

The category \mathbf{D} is Krull-Schmidt by [23, p. 52]. By [20, lem. 2.1] so is \mathbf{D}/\mathcal{U} , and the projection functor $\mathbf{D} \rightarrow \mathbf{D}/\mathcal{U}$ induces a bijective correspondence between the isomorphism classes of indecomposable objects of \mathbf{D}/\mathcal{U} and the isomorphism classes of indecomposable objects of \mathbf{D} which are outside \mathcal{U} .

By [20, thm. 3.3, prop. 4.2, and thm. 4.3], the category \mathbf{D}/\mathcal{U} is abelian with enough projective and injective objects. Its projectives are the

objects isomorphic to objects in $\underline{\Sigma^{-1}\mathcal{U}}$ and its injectives are the objects isomorphic to objects in $\underline{\Sigma\mathcal{U}}$.

By [20, cor. 4.4], there is an equivalence $\mathbf{D}/\mathcal{U} \simeq \mathbf{mod} \Sigma^{-1}\mathcal{U}$. The right hand side is clearly equivalent to $\mathbf{mod} \mathcal{U}$, so we have $\mathbf{D}/\mathcal{U} \simeq \mathbf{mod} \mathcal{U}$.

Let $\Sigma^{-1}u$ be in $\Sigma^{-1}\mathcal{U}$ and x in \mathbf{D} . It is a useful observation that since we have $\mathbf{D}(\Sigma^{-1}\mathcal{U}, \mathcal{U}) = 0$, there is an isomorphism

$$\mathbf{D}(\Sigma^{-1}u, x) \cong (\mathbf{D}/\mathcal{U})(\underline{\Sigma^{-1}u}, \underline{x}).$$

Let $x \rightarrow y \rightarrow z \rightarrow$ be a distinguished triangle in \mathbf{D} . The composition of two consecutive morphisms in a distinguished triangle is zero and remains so on projecting to \mathbf{D}/\mathcal{U} , so there is an induced sequence $\underline{x} \rightarrow \underline{y} \rightarrow \underline{z}$ in \mathbf{D}/\mathcal{U} . This is an exact sequence. To see so, it is enough to check that it becomes exact under the functor $(\mathbf{D}/\mathcal{U})(\underline{p}, -)$ when \underline{p} is projective in \mathbf{D}/\mathcal{U} . We can assume $p = \Sigma^{-1}u$ for a u in \mathcal{U} , so we must show that

$$(\mathbf{D}/\mathcal{U})(\underline{\Sigma^{-1}u}, \underline{x}) \rightarrow (\mathbf{D}/\mathcal{U})(\underline{\Sigma^{-1}u}, \underline{y}) \rightarrow (\mathbf{D}/\mathcal{U})(\underline{\Sigma^{-1}u}, \underline{z})$$

is exact. By the above this is just

$$\mathbf{D}(\Sigma^{-1}u, x) \rightarrow \mathbf{D}(\Sigma^{-1}u, y) \rightarrow \mathbf{D}(\Sigma^{-1}u, z)$$

which is indeed exact.

By repeatedly “turning” the distinguished triangle, it is possible to obtain a long sequence in \mathbf{D} in which each four term part is a distinguished triangle. This induces a long exact sequence in \mathbf{D}/\mathcal{U} .

By [20, prop. 4.7.3], the autoequivalence $S\Sigma^{-2}$ of \mathbf{D} satisfies $S\Sigma^{-2}\mathcal{U} = \mathcal{U}$. Hence $S\Sigma^{-2}$ induces an autoequivalence of \mathbf{D}/\mathcal{U} which, by abuse of notation, will also be denoted $S\Sigma^{-2}$.

In the following result, recall that $\mathcal{U}(x, \Sigma y)$ is the space of morphisms $x \rightarrow \Sigma y$ in \mathbf{D} which factor through an object from \mathcal{U} .

Proposition 1.5. *Let x and y be in \mathbf{D} .*

- (i) *If x has no direct summands from \mathcal{U} and \underline{x} has finite projective dimension in \mathbf{D}/\mathcal{U} , then*

$$\mathrm{Ext}_{\mathbf{D}/\mathcal{U}}^1(\underline{x}, \underline{y}) \cong \mathcal{U}(x, \Sigma y).$$

- (ii) *If y has no direct summands from \mathcal{U} and \underline{y} has finite injective dimension in \mathbf{D}/\mathcal{U} , then*

$$\mathrm{Ext}_{\mathbf{D}/\mathcal{U}}^1(\underline{x}, \underline{y}) \cong \mathcal{U}(\Sigma^{-1}x, y).$$

Proof. We will only prove (i) since (ii) can be established by the dual argument.

Since \underline{x} has finite projective dimension in \mathbf{D}/\mathcal{U} , its projective dimension is at most one, see [20, thm. 4.3] and [19, 2.1, cor.].

By [20, lem. 3.2.1], there is a distinguished triangle

$$\Sigma^{-1}u_1 \xrightarrow{\alpha} \Sigma^{-1}u_0 \longrightarrow x \longrightarrow$$

in \mathbf{D} where the u_i are in \mathcal{U} . Turning the triangle gives a sequence

$$\Sigma^{-2}u_0 \xrightarrow{\gamma} \Sigma^{-1}x \xrightarrow{\beta} \Sigma^{-1}u_1 \xrightarrow{\alpha} \Sigma^{-1}u_0 \longrightarrow x \longrightarrow u_1 \quad (2)$$

which by Remark 1.4 induces a long exact sequence in \mathbf{D}/\mathcal{U} ,

$$\underline{\Sigma^{-1}x} \xrightarrow{\underline{\beta}} \underline{\Sigma^{-1}u_1} \xrightarrow{\underline{\alpha}} \underline{\Sigma^{-1}u_0} \longrightarrow \underline{x} \longrightarrow \underline{u_1}.$$

In \mathbf{D}/\mathcal{U} the object $\underline{u_1}$ is isomorphic to 0, so the penultimate morphism is an epimorphism onto \underline{x} . The object $\underline{\Sigma^{-1}u_0}$ is projective and \underline{x} has projective dimension at most one, so the image \underline{p} of $\underline{\alpha}$ is projective and so $\underline{\alpha}$ viewed as a morphism to \underline{p} is a split epimorphism. Hence the kernel \underline{q} of $\underline{\alpha}$ is a direct summand of $\underline{\Sigma^{-1}u_1}$, and since $\underline{\Sigma^{-1}u_1}$ is projective so is \underline{q} . But \underline{q} is also the image of $\underline{\beta}$, and so $\underline{\beta}$ viewed as a morphism to \underline{q} is a split epimorphism. Hence the kernel \underline{z} of $\underline{\beta}$ is a direct summand of $\underline{\Sigma^{-1}x}$.

Putting together this information, the exact sequence is isomorphic to

$$\underline{z} \oplus \underline{q} \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \underline{q} \oplus \underline{p} \xrightarrow{\begin{pmatrix} 0 & \text{mono} \end{pmatrix}} \underline{\Sigma^{-1}u_0} \longrightarrow \underline{x} \longrightarrow 0.$$

In particular we have $\underline{\Sigma^{-1}x} \cong \underline{z} \oplus \underline{q}$ in \mathbf{D}/\mathcal{U} . But x has no direct summands from \mathcal{U} so $\underline{\Sigma^{-1}x}$ has no direct summands from $\underline{\Sigma^{-1}\mathcal{U}}$; that is, $\underline{\Sigma^{-1}x}$ has no projective direct summands so $\underline{q} \cong 0$. Hence the exact sequence is isomorphic to

$$\underline{z} \xrightarrow{0} \underline{p} \xrightarrow{\underline{\alpha}} \underline{\Sigma^{-1}u_0} \longrightarrow \underline{x} \longrightarrow 0.$$

It follows that

$$\begin{aligned} \text{Ext}_{\mathbf{D}/\mathcal{U}}^1(\underline{x}, \underline{y}) &\cong \text{Coker } (\mathbf{D}/\mathcal{U})(\underline{\alpha}, \underline{y}) \\ &\stackrel{(a)}{=} \text{Coker } \mathbf{D}(\alpha, y) \\ &\stackrel{(b)}{=} \text{Ker } \mathbf{D}(\gamma, y) \\ &= (*) \end{aligned}$$

where (a) is by Remark 1.4 because α is a morphism in $\Sigma^{-1}\mathcal{U}$ and (b) is by equation (2). But the kernel $(*)$ consists of the morphisms $\Sigma^{-1}x \rightarrow y$ which factor through β , and it is easy to check that these are precisely the morphisms which factor through some object of $\Sigma^{-1}\mathcal{U}$ whence

$$(*) = (\Sigma^{-1}\mathcal{U})(\Sigma^{-1}x, y) \cong \mathcal{U}(x, \Sigma y).$$

□

2. PROJECTING A CLUSTER TILTING SUBCATEGORY

This section proves Theorem B from the Introduction; see Theorem 2.2.

The following is a straightforward abstraction of the notion of support tilting modules from [15].

Definition 2.1. To say that \mathcal{S} is a *support tilting subcategory* of an abelian category \mathbf{A} means that \mathcal{S} is a full subcategory which

- is closed under (finite) direct sums and direct summands;
- is precovering and preenveloping;
- satisfies $\text{Ext}_{\mathbf{A}}^2(\mathcal{S}, -) = 0$;
- satisfies $\text{Ext}_{\mathbf{A}}^1(\mathcal{S}, \mathcal{S}) = 0$;
- satisfies that if y is a subquotient of an object from \mathcal{S} for which we have $\text{Ext}_{\mathbf{A}}^1(\mathcal{S}, y) = 0$, then y is a quotient of an object from \mathcal{S} .

Theorem 2.2. *Assume that \mathbf{D}/\mathcal{U} has finite global dimension.*

Let \mathcal{V} be a cluster tilting subcategory of \mathbf{D} . Then the image $\underline{\mathcal{V}}$ is a support tilting subcategory of \mathbf{D}/\mathcal{U} .

Proof. Since \mathcal{V} is cluster tilting, it is closed under direct sums and direct summands, as follows from Definition 1.2. Hence $\underline{\mathcal{V}}$ is closed under direct sums and direct summands.

Moreover, $\underline{\mathcal{V}}$ -precovers and $\underline{\mathcal{V}}$ -preenvelopes are induced by \mathcal{V} -precovers and \mathcal{V} -preenvelopes, so $\underline{\mathcal{V}}$ is precovering and preenveloping.

The objects of $\underline{\mathcal{V}}$ have finite projective dimension since \mathbf{D}/\mathcal{U} has finite global dimension, so each object of $\underline{\mathcal{V}}$ has projective dimension at most one by [20, thm. 4.3] and [19, 2.1, cor.]. Hence the condition $\text{Ext}_{\mathbf{D}/\mathcal{U}}^2(\underline{\mathcal{V}}, -) = 0$ is satisfied.

For \underline{v} and \underline{v}' in $\underline{\mathcal{V}}$, let us prove $\text{Ext}_{\mathbf{D}/\mathcal{U}}^1(\underline{v}, \underline{v}') = 0$. We can discard any direct summands of \underline{v} which are in \mathcal{U} since they do not make any

difference to the isomorphism class of \underline{v} . But \underline{v} has finite projective dimension in \mathbf{D}/\mathcal{U} since that category has finite global dimension, so $\mathrm{Ext}_{\mathbf{D}/\mathcal{U}}^1(\underline{v}, \underline{v}') \cong \mathcal{U}(v, \Sigma v')$ by Proposition 1.5(i), and here the right hand side is zero since it is a subspace of $\mathbf{D}(v, \Sigma v')$ which is zero because \mathcal{V} is cluster tilting.

Finally, let \underline{y} be a subquotient of \underline{v} in \mathbf{D}/\mathcal{U} where \underline{v} is in $\underline{\mathcal{V}}$, and suppose $\mathrm{Ext}_{\mathbf{D}/\mathcal{U}}^1(\underline{\mathcal{V}}, \underline{y}) = 0$. Let us prove that \underline{y} is a quotient of an object from $\underline{\mathcal{V}}$.

We can discard any direct summands of \underline{y} which are in \mathcal{U} . Moreover, \underline{y} has finite injective dimension because \mathbf{D}/\mathcal{U} has finite global dimension. It follows by Proposition 1.5(ii) that

$$\mathcal{U}(\Sigma^{-1}v', y) \cong \mathrm{Ext}_{\mathbf{D}/\mathcal{U}}^1(\underline{v}', \underline{y}) = 0 \quad (3)$$

for each v' in \mathcal{V} .

For \underline{y} to be a subquotient of \underline{v} means that we have an epimorphism and a monomorphism $\underline{v} \twoheadrightarrow \underline{t} \hookrightarrow \underline{y}$. Lift these two morphisms to \mathbf{D} and complete to distinguished triangles. Since the morphisms in \mathbf{D}/\mathcal{U} are, respectively, an epimorphism and a monomorphism, [20, thm. 2.3] implies that the other morphisms in the distinguished triangles factor as follows,

$$\begin{array}{ccccc} v & \xrightarrow{\sigma} & t & \xrightarrow{\tau} & c \longrightarrow \\ & & & \searrow & \nearrow \\ & & & u & \end{array}$$

and

$$\begin{array}{ccccc} k & \xrightarrow{\kappa} & y & \xrightarrow{\gamma} & t \longrightarrow \\ & \searrow & & \nearrow & \\ & & u' & & \end{array},$$

with u and u' in \mathcal{U} .

For v' in \mathcal{V} , the image of

$$\mathbf{D}(\Sigma^{-1}v', \mu') : \mathbf{D}(\Sigma^{-1}v', u') \rightarrow \mathbf{D}(\Sigma^{-1}v', y)$$

is a subset of $\mathcal{U}(\Sigma^{-1}v', y)$ which is zero by equation (3). So we have $\mathbf{D}(\Sigma^{-1}v', \mu') = 0$ and by Serre duality $\mathbf{D}(\mu', S\Sigma^{-1}v') = 0$ where S is the Serre functor of \mathbf{D} . But [20, prop. 4.7] implies that $S\Sigma^{-1}\mathcal{V} = \Sigma\mathcal{V}$, so it follows that

$$\mathbf{D}(\mu', \Sigma v'') = 0 \quad (4)$$

for each v'' in \mathcal{V} .

Now use [20, lem. 3.2.1] to construct a distinguished triangle in \mathbf{D} ,

$$v' \xrightarrow{\sigma'} y \xrightarrow{\beta} \Sigma v'' \longrightarrow ,$$

with v' and v'' in \mathcal{V} . We will show that $\underline{\sigma'} : \underline{v'} \rightarrow \underline{y}$ is an epimorphism in \mathbf{D}/\mathcal{U} whence \underline{y} is a quotient of the object $\underline{v'}$ from \mathcal{V} . Combining the three distinguished triangles we have constructed gives the solid arrows in the following commutative diagram,

$$\begin{array}{ccccc}
 & & k & & \\
 & & \swarrow & & \\
 & & u' & & \\
 & & \searrow & & \\
 & & \mu' & & \\
 v' & \xrightarrow{\sigma'} & y & \xrightarrow{\beta} & \Sigma v'' \longrightarrow \\
 & & \downarrow \gamma & \nearrow \theta & \uparrow \chi \\
 v & \xrightarrow{\sigma} & t & \xrightarrow{\tau} & c \longrightarrow \\
 & & \downarrow & \searrow u & \nearrow \\
 & & & &
 \end{array}$$

Here $\mathbf{D}(\mu', \Sigma v'') = 0$ by equation (4), so in particular $\beta\mu' = 0$. It follows that $\beta\kappa = 0$. Hence θ exists with $\theta\gamma = \beta$, but $\theta\sigma = 0$ since $\mathbf{D}(\mathcal{V}, \Sigma\mathcal{V}) = 0$ so finally, χ exists with $\chi\tau = \theta$.

That is, $\beta = \chi\tau\gamma$, but τ factors through u so β also factors through u . By [20, thm. 2.3], it follows that $\underline{\sigma'}$ is an epimorphism as desired. \square

3. LIFTING A SUPPORT TILTING SUBCATEGORY

This section proves Theorem C from the Introduction; see Theorem 3.5.

Remark 3.1. In this section, we will often consider a special way of lifting a full subcategory from \mathbf{D}/\mathcal{U} to \mathbf{D} .

Namely, consider a full subcategory of \mathbf{D}/\mathcal{U} which is closed under direct sums and direct summands. We can and will assume that it has the form $\underline{\mathcal{W}}$ where \mathcal{W} is a full subcategory of \mathbf{D} which is closed under

direct sums and direct summands and consists of objects without direct summands from \mathcal{U} . Note that there is a bijective correspondence between isomorphism classes of indecomposable objects of \mathcal{W} and of $\underline{\mathcal{W}}$.

A lifting of $\underline{\mathcal{W}}$ to \mathbf{D} is a subcategory \mathcal{X} of \mathbf{D} with $\underline{\mathcal{X}} = \underline{\mathcal{W}}$. Obviously, \mathcal{W} is a lifting of $\underline{\mathcal{W}}$ to \mathbf{D} , and any other lifting which is a full subcategory closed under direct sums and direct summands has the form

$$\mathcal{X} = \text{add}(\mathcal{W} \cup \mathcal{T})$$

where \mathcal{T} is contained in \mathcal{U} .

We wish to consider the specific choice

$$\mathcal{T} = \{ u \in \mathcal{U} \mid \mathbf{D}(\mathcal{W}, \Sigma u) = 0 \}$$

since the resulting \mathcal{X} has the property described in the following lemma.

Lemma 3.2. *If it is possible to lift $\underline{\mathcal{W}}$ to a weak cluster tilting subcategory \mathcal{X}' of \mathbf{D} , then $\mathcal{X}' = \mathcal{X}$.*

Proof. Suppose that \mathcal{X}' exists. Since \mathcal{X}' is a lifting of $\underline{\mathcal{W}}$, we have $\mathcal{X}' = \text{add}(\mathcal{W} \cup \mathcal{T}')$ for a \mathcal{T}' which is contained in \mathcal{U} . We can take \mathcal{T}' to be closed under direct sums and direct summands.

On one hand, if an indecomposable u from \mathcal{U} has $\mathbf{D}(\mathcal{W}, \Sigma u) = 0$, then $\mathbf{D}(\mathcal{X}', \Sigma u) = 0$ since \mathcal{T}' is contained in \mathcal{U} , and consequently u is in \mathcal{X}' and so must be in \mathcal{T}' . On the other hand, if an indecomposable u from \mathcal{U} has $\mathbf{D}(\mathcal{W}, \Sigma u) \neq 0$, then $\mathbf{D}(\mathcal{X}', \Sigma u) \neq 0$, and consequently u is not in \mathcal{X}' and so cannot be in \mathcal{T}' . Hence $\mathcal{T}' = \mathcal{T}$ and $\mathcal{X}' = \mathcal{X}$. \square

Lemma 3.3. *Let $\underline{\mathcal{W}}$ and \mathcal{W} be as in Remark 3.1, and assume that each object of $\underline{\mathcal{W}}$ has finite projective dimension, that $\text{Ext}_{\mathbf{D}/\mathcal{U}}^1(\underline{\mathcal{W}}, \underline{\mathcal{W}}) = 0$, and that $S\Sigma^{-2}\underline{\mathcal{W}} = \underline{\mathcal{W}}$. Then $\mathbf{D}(\mathcal{W}, \Sigma\mathcal{W}) = 0$.*

Proof. Let w and w' be objects of \mathcal{W} . Since objects of \mathcal{W} have no direct summands from \mathcal{U} , the condition $S\Sigma^{-2}\underline{\mathcal{W}} = \underline{\mathcal{W}}$ implies $S\Sigma^{-2}\mathcal{W} = \mathcal{W}$ whence

$$\Sigma^2 w' \cong S\tilde{w} \tag{5}$$

for an object \tilde{w} in \mathcal{W} .

By [20, lem. 3.2.1] there is a distinguished triangle

$$u^1 \rightarrow \Sigma w \rightarrow \Sigma u^0 \rightarrow$$

with the u^i in \mathcal{U} . This induces an exact sequence

$$\mathbf{D}(\tilde{w}, u^1) \xrightarrow{\alpha} \mathbf{D}(\tilde{w}, \Sigma w) \xrightarrow{\beta} \mathbf{D}(\tilde{w}, \Sigma u^0),$$

and it is easy to check that the image of α is $\mathcal{U}(\tilde{w}, \Sigma w)$ which by Proposition 1.5(i) is $\text{Ext}_{\mathbf{D}/\mathcal{U}}^1(\tilde{w}, \underline{w})$ since \tilde{w} has no direct summands from \mathcal{U} and since \tilde{w} has finite projective dimension because it is in $\underline{\mathcal{W}}$. By assumption this Ext is zero, so β is injective.

Using the Serre functor S and k -linear duality $(-)^{\vee} = \text{Hom}_k(-, k)$ along with equation (5), we can rewrite β as follows,

$$\begin{array}{ccc}
\mathbf{D}(\tilde{w}, \Sigma w) & \xrightarrow{\beta} & \mathbf{D}(\tilde{w}, \Sigma u^0) \\
\cong \downarrow & & \downarrow \cong \\
\mathbf{D}(\Sigma w, S\tilde{w})^{\vee} & \longrightarrow & \mathbf{D}(\Sigma u^0, S\tilde{w})^{\vee} \\
\cong \downarrow & & \downarrow \cong \\
\mathbf{D}(\Sigma w, \Sigma^2 w')^{\vee} & \longrightarrow & \mathbf{D}(\Sigma u^0, \Sigma^2 w')^{\vee} \\
\cong \downarrow & & \downarrow \cong \\
\mathbf{D}(w, \Sigma w')^{\vee} & \longrightarrow & \mathbf{D}(u^0, \Sigma w')^{\vee},
\end{array}$$

and since these maps are injective, the dual $\mathbf{D}(u^0, \Sigma w') \rightarrow \mathbf{D}(w, \Sigma w')$ of the last map is surjective. It is easy to see that the image of this map is $\mathcal{U}(w, \Sigma w')$, so we have

$$\mathbf{D}(w, \Sigma w') = \mathcal{U}(w, \Sigma w') = (*).$$

But

$$(*) \cong \text{Ext}_{\mathbf{D}/\mathcal{U}}^1(\underline{w}, \underline{w}')$$

by Proposition 1.5(i). By assumption this Ext is zero, so $\mathbf{D}(w, \Sigma w') = 0$ as claimed. \square

Lemma 3.4. *Assume that \mathbf{D}/\mathcal{U} has finite global dimension and that each object of \mathbf{D}/\mathcal{U} has finite length.*

Let $\underline{\mathcal{W}}$ be a full subcategory of \mathbf{D}/\mathcal{U} which is closed under direct sums and direct summands, and assume $\text{Ext}_{\mathbf{D}/\mathcal{U}}^1(\underline{\mathcal{W}}, \underline{\mathcal{W}}) = 0$.

Let \underline{a} be an object of \mathbf{D}/\mathcal{U} for which the following implication holds when \underline{i} is an injective object of \mathbf{D}/\mathcal{U} :

$$(\mathbf{D}/\mathcal{U})(\underline{a}, \underline{i}) \neq 0 \Rightarrow \text{there is a } \underline{w} \text{ in } \underline{\mathcal{W}} \text{ such that } (\mathbf{D}/\mathcal{U})(\underline{w}, \underline{i}) \neq 0.$$

Then \underline{a} is a subquotient in \mathbf{D}/\mathcal{U} of an object from $\underline{\mathcal{W}}$.

Proof. It is easy to check that, since \mathbf{D}/\mathcal{U} has enough injectives and all its objects have finite length, \mathbf{D}/\mathcal{U} has injective envelopes. Let $e(\underline{t})$ be the injective envelope of a simple object \underline{t} . It is also easy to check

that \underline{t} appears in the composition series of an object \underline{a} if and only if $(\mathbf{D}/\mathcal{U})(\underline{a}, e(\underline{t})) \neq 0$.

Now let the simple object \underline{t} be in the composition series of the object \underline{a} . Then $(\mathbf{D}/\mathcal{U})(\underline{a}, e(\underline{t})) \neq 0$ whence, by the assumption of the lemma, $(\mathbf{D}/\mathcal{U})(\underline{w}, e(\underline{t})) \neq 0$ for some \underline{w} in $\underline{\mathcal{W}}$. This in turn means that \underline{t} appears in the composition series of \underline{w} , so \underline{t} is a subquotient of an object of $\underline{\mathcal{W}}$.

But \underline{a} is a successive extension of the simple objects in its composition series, so \underline{a} is a successive extension of subquotients of objects of $\underline{\mathcal{W}}$. The method used in the proof of [15, lem. 2.4] shows that the class of subquotients of objects from $\underline{\mathcal{W}}$ is closed under extensions, so it follows that \underline{a} is a subquotient of an object from $\underline{\mathcal{W}}$. \square

Theorem 3.5. *Assume that \mathbf{D}/\mathcal{U} has finite global dimension and that each object of \mathbf{D}/\mathcal{U} has finite length.*

Let $\underline{\mathcal{W}}$ be a support tilting subcategory of \mathbf{D}/\mathcal{U} with $S\Sigma^{-2}\underline{\mathcal{W}} = \underline{\mathcal{W}}$. Then the category \mathcal{X} from Remark 3.1 is the unique weak cluster tilting subcategory of \mathbf{D} which is a lifting of $\underline{\mathcal{W}}$.

Proof. Remark 3.1 says that \mathcal{X} is a lifting of $\underline{\mathcal{W}}$ to \mathbf{D} , and Lemma 3.2 says that if \mathcal{X}' is a lifting which is weak cluster tilting then $\mathcal{X}' = \mathcal{X}$. So we just need to show that \mathcal{X} is indeed weak cluster tilting; that is,

$$\begin{aligned} x \in \mathcal{X} &\Leftrightarrow \mathbf{D}(\mathcal{X}, \Sigma x) = 0, \\ x \in \mathcal{X} &\Leftrightarrow \mathbf{D}(x, \Sigma \mathcal{X}) = 0. \end{aligned}$$

Since we have assumed that the objects of \mathcal{W} have no direct summands from \mathcal{U} , the condition $S\Sigma^{-2}\underline{\mathcal{W}} = \underline{\mathcal{W}}$ implies $S\Sigma^{-2}\mathcal{W} = \mathcal{W}$.

The implications \Rightarrow . It is enough to show $\mathbf{D}(x, \Sigma y) = 0$ for indecomposable objects x and y of \mathcal{X} . Recall the construction from Remark 3.1; in particular $\mathcal{X} = \text{add}(\mathcal{W} \cup \mathcal{T})$ so we may assume that each of x and y is in \mathcal{W} or \mathcal{T} .

If x and y are in \mathcal{W} , then Lemma 3.3 gives $\mathbf{D}(x, \Sigma y) = 0$.

If x and y are in \mathcal{T} , then they are in particular in \mathcal{U} whence $\mathbf{D}(x, \Sigma y) = 0$.

If x is in \mathcal{W} and y is in \mathcal{T} , then $\mathbf{D}(x, \Sigma y) = 0$ by the definition of \mathcal{T} in Remark 3.1.

Finally, if x is in \mathcal{T} and y is in \mathcal{W} , then $y \cong S\Sigma^{-2}w$ for a w in \mathcal{W} since $S\Sigma^{-2}\mathcal{W} = \mathcal{W}$. So

$$\begin{aligned} \mathrm{D}(x, \Sigma y) &\cong \mathrm{D}(x, \Sigma S\Sigma^{-2}w) \\ &\cong \mathrm{D}(x, S\Sigma^{-1}w) \\ &\cong \mathrm{D}(\Sigma^{-1}w, x)^\vee \\ &\cong \mathrm{D}(w, \Sigma x)^\vee, \end{aligned}$$

and the right hand side is zero by the definition of \mathcal{T} .

The implications \Leftarrow . We know $S\Sigma^{-2}\mathcal{W} = \mathcal{W}$, and $S\Sigma^{-2}\mathcal{U} = \mathcal{U}$ by [20, prop. 4.7]. It follows that $S\Sigma^{-2}\mathcal{T} = \mathcal{T}$, and hence $S\Sigma^{-2}\mathcal{X} = \mathcal{X}$. So

$$\begin{aligned} \mathrm{D}(x, \Sigma \mathcal{X}) = 0 &\Leftrightarrow \mathrm{D}(x, \Sigma S\Sigma^{-2}\mathcal{X}) = 0 & (6) \\ &\Leftrightarrow \mathrm{D}(x, S\Sigma^{-1}\mathcal{X}) = 0 \\ &\Leftrightarrow \mathrm{D}(\Sigma^{-1}\mathcal{X}, x)^\vee = 0 \\ &\Leftrightarrow \mathrm{D}(\mathcal{X}, \Sigma x)^\vee = 0 \\ &\Leftrightarrow \mathrm{D}(\mathcal{X}, \Sigma x) = 0, \end{aligned}$$

and it is sufficient to prove the first implication \Leftarrow . So let x be an indecomposable object of D with $\mathrm{D}(\mathcal{X}, \Sigma x) = 0$; in particular

$$\mathrm{D}(\mathcal{W}, \Sigma x) = 0. \quad (7)$$

If x is in \mathcal{U} then (7) says that x is in \mathcal{T} and so x is in \mathcal{X} .

Suppose that x is not in \mathcal{U} ; then \underline{x} is non-zero and indecomposable in D/\mathcal{U} . By Proposition 1.5(i), equation (7) implies $\mathrm{Ext}_{\mathrm{D}/\mathcal{U}}^1(\underline{\mathcal{W}}, \underline{x}) = 0$ since the objects of \mathcal{W} have no direct summands from \mathcal{U} and since the objects of $\underline{\mathcal{W}}$ have finite projective dimension.

Let \underline{i} be an injective object of D/\mathcal{U} and suppose that $(\mathrm{D}/\mathcal{U})(\underline{x}, \underline{i}) \neq 0$. Then $\mathrm{D}(x, i) \neq 0$. By [20, prop. 4.2], we can suppose $i = \Sigma u$ for a u in \mathcal{U} . So we have $\mathrm{D}(x, \Sigma u) \neq 0$, and since $\mathrm{D}(x, \Sigma \mathcal{X}) = 0$ by equation (6), this forces u to have a direct summand in \mathcal{U} outside \mathcal{T} . Then there exists a w in \mathcal{W} with $\mathrm{D}(w, \Sigma u) \neq 0$, but this implies $(\mathrm{D}/\mathcal{U})(\underline{w}, \underline{\Sigma u}) \neq 0$, that is, $(\mathrm{D}/\mathcal{U})(\underline{w}, \underline{i}) \neq 0$. We have shown

$$(\mathrm{D}/\mathcal{U})(\underline{x}, \underline{i}) \neq 0 \Rightarrow \text{there is a } \underline{w} \text{ in } \underline{\mathcal{W}} \text{ such that } (\mathrm{D}/\mathcal{U})(\underline{w}, \underline{i}) \neq 0.$$

It follows from Lemma 3.4 that \underline{x} is a subquotient of an object from $\underline{\mathcal{W}}$. But we already know $\mathrm{Ext}_{\mathrm{D}/\mathcal{U}}^1(\underline{\mathcal{W}}, \underline{x}) = 0$, and since $\underline{\mathcal{W}}$ is support tilting it follows that \underline{x} is a quotient of an object from $\underline{\mathcal{W}}$.

Consequently, each $\underline{\mathcal{W}}$ -precover of \underline{x} is an epimorphism. Pick a precover and complete to a short exact sequence

$$0 \rightarrow \underline{\ell} \rightarrow \underline{w} \rightarrow \underline{x} \rightarrow 0 \quad (8)$$

which we will show to be split in order to prove that x is in \mathcal{W} . The long exact Ext sequence implies that $\text{Ext}_{\mathbf{D}/\mathcal{U}}^1(\underline{\mathcal{W}}, \underline{\ell}) = 0$, so since $\underline{\ell}$ is a subobject and in particular a subquotient of \underline{w} , the support tilting property of $\underline{\mathcal{W}}$ shows that $\underline{\ell}$ is a quotient of an object from $\underline{\mathcal{W}}$,

$$0 \rightarrow \underline{\ell}' \rightarrow \underline{w}' \rightarrow \underline{\ell} \rightarrow 0.$$

Now, our assumption is that $\mathbf{D}(\mathcal{X}, \Sigma x) = 0$, and by equation (6) this implies $\mathbf{D}(x, \Sigma \mathcal{X}) = 0$ and in particular $\mathbf{D}(x, \Sigma \mathcal{W}) = 0$. By Proposition 1.5(i), it follows that $\text{Ext}_{\mathbf{D}/\mathcal{U}}^1(\underline{x}, \underline{\mathcal{W}}) = 0$ because x has no direct summands from \mathcal{U} while \underline{x} has finite projective dimension since \mathbf{D}/\mathcal{U} has finite global dimension. So in particular $\text{Ext}_{\mathbf{D}/\mathcal{U}}^1(\underline{x}, \underline{w}') = 0$, and since the projective dimension of \underline{x} is at most one by [20, thm. 4.3] and [19, 2.1, cor.], the long exact Ext sequence then implies $\text{Ext}_{\mathbf{D}/\mathcal{U}}^1(\underline{x}, \underline{\ell}) = 0$. Hence the exact sequence (8) is split, and since \underline{w} is in $\underline{\mathcal{W}}$ it follows that \underline{x} is isomorphic to an object of $\underline{\mathcal{W}}$. But then the indecomposable x is isomorphic to an object of \mathcal{W} since x is outside \mathcal{U} , and hence x is in \mathcal{X} . \square

Remark 3.6. In the following proposition and in Section 4 we will consider a bijective correspondence between cluster tilting subcategories and support tilting subcategories.

Tacitly, the correspondence is in fact between equivalence classes of such subcategories, the equivalence relation being that subcategories with the same essential closure are equivalent; cp. Remark 0.1.

Proposition 3.7. *Assume that \mathbf{D}/\mathcal{U} has finite global dimension and that each object of \mathbf{D}/\mathcal{U} has finite length.*

Suppose that the following condition is satisfied: If $\underline{\mathcal{W}}$ is a support tilting subcategory of \mathbf{D}/\mathcal{U} with $S\Sigma^{-2}\underline{\mathcal{W}} = \underline{\mathcal{W}}$, then the weak cluster tilting subcategory \mathcal{X} of Remark 3.1 and Theorem 3.5 is precovering and preenveloping, and hence cluster tilting.

Then the projection functor $\mathbf{D} \rightarrow \mathbf{D}/\mathcal{U}$ induces a bijection between the cluster tilting subcategories of \mathbf{D} and the support tilting subcategories of \mathbf{D}/\mathcal{U} which are equal to their image under $S\Sigma^{-2}$.

Proof. The operation $\mathcal{V} \xrightarrow{\pi} \underline{\mathcal{V}}$ induced by the projection functor sends full subcategories of \mathbf{D} to full subcategories of \mathbf{D}/\mathcal{U} . By Theorem 2.2, it sends cluster tilting subcategories to support tilting subcategories.

Cluster tilting subcategories are equal to their image under $S\Sigma^{-2}$ by [20, thm. 4.7.3], so the support tilting subcategories arising from this are too.

The operation $\underline{\mathcal{W}} \xrightarrow{\lambda} \mathcal{X}$ of Remark 3.1 sends full subcategories of \mathbf{D}/\mathcal{U} to full subcategories of \mathbf{D} . By Theorem 3.5 and the assumption of the present proposition, it sends support tilting subcategories which are equal to their image under $S\Sigma^{-2}$ to cluster tilting subcategories.

Let \mathcal{V} be cluster tilting in \mathbf{D} . Then \mathcal{V} and $\lambda\pi(\mathcal{V})$ are both liftings of $\pi(\mathcal{V}) = \underline{\mathcal{V}}$ to \mathbf{D} , and they are both cluster tilting and so in particular weak cluster tilting. Hence $\lambda\pi(\mathcal{V}) = \mathcal{V}$ by Theorem 3.5.

Let $\underline{\mathcal{W}}$ be support tilting in \mathbf{D}/\mathcal{U} with $S\Sigma^{-2}\underline{\mathcal{W}} = \underline{\mathcal{W}}$. Then $\mathcal{X} = \lambda(\underline{\mathcal{W}})$ is a lifting of $\underline{\mathcal{W}}$ to \mathbf{D} , that is, $\pi\lambda(\underline{\mathcal{W}}) = \underline{\mathcal{W}}$.

This shows that π and λ are mutually inverse maps between the set of cluster tilting subcategories of \mathbf{D} and the set of support tilting subcategories of \mathbf{D}/\mathcal{U} which are equal to their image under $S\Sigma^{-2}$, and the proposition follows. \square

Remark 3.8. The situation of the proposition occurs in practice, as we will see in one of the examples of the next section. It would be interesting to find a simple criterion which guarantees that we are in this situation.

4. EXAMPLES

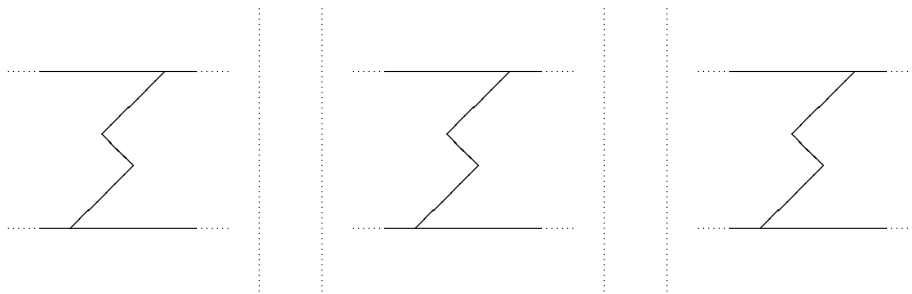
Let Q be a finite quiver without loops or cycles and let \mathbf{D} be the cluster category of type Q over k , cf. [9]. It is not hard to show that Proposition 3.7 implies Theorem A by Ingalls and Thomas. However, we can also handle other examples.

4.a. Derived categories. Let Q be a finite quiver without loops and cycles and set \mathbf{D} equal to $\mathbf{D}^f(kQ)$, the finite derived category of the path algebra kQ . Consider kQ itself as an object of \mathbf{D} and set \mathcal{U} equal to add of the orbit of kQ under $S\Sigma^{-2}$; cf. [20, 4.5.2].

The conditions of Setup 1.1 are satisfied: The Hom spaces of \mathbf{D} are finite dimensional by an explicit computation with projective resolutions. Idempotents in \mathbf{D} split because, by [6, prop. 3.2], they do so in $\mathbf{D}(kQ)$, the derived category of all complexes. And there is a Serre functor by [12, 3.6] and [21, thm. I.2.4].

Consider the module category $\mathbf{mod} kQ$. Its Auslander-Reiten quiver (AR quiver) Γ typically consists of a preprojective component of the

form $\mathbb{N}Q$, a regular component, and a preinjective component which is the mirror image of the preprojective component. The AR quiver of \mathbf{D} is obtained by taking a countable number of copies of Γ and gluing them together, preinjective components to preprojective components; cf. [12]. It typically looks as follows, where the zig zags indicate the subcategory \mathcal{U} .



The abelian quotient category $\mathbf{D}/\mathcal{U} \simeq \mathbf{mod} \mathcal{U}$ is the direct sum of countably many copies of $\mathbf{mod} kQ$, so it is clear that \mathbf{D}/\mathcal{U} has finite global dimension and that each of its objects has finite length.

Note that in the AR quiver of \mathbf{D} , the copies of Γ which are glued to obtain the quiver do not correspond to the copies of $\mathbf{mod} kQ$ whose direct sum is \mathbf{D}/\mathcal{U} . The former overlap with the vertices corresponding to \mathcal{U} , the latter correspond to their complement.

We claim that we are in the situation of Proposition 3.7, so the projection functor induces a bijection between the cluster tilting subcategories of \mathbf{D} and the support tilting subcategories of \mathbf{D}/\mathcal{U} which are equal to their image under $S\Sigma^{-2}$.

To see this, we must let $\underline{\mathcal{W}}$ be a support tilting subcategory of \mathbf{D}/\mathcal{U} with $S\Sigma^{-2}\underline{\mathcal{W}} = \underline{\mathcal{W}}$ and show that the lifted subcategory \mathcal{X} of Remark 3.1 and Theorem 3.5 is precovering and preenveloping. However, when $\underline{\mathcal{W}}$ is support tilting then its intersection with each copy of $\mathbf{mod} kQ$ inside \mathbf{D}/\mathcal{U} is a partial tilting subcategory, and so only contains finitely many isomorphism classes of indecomposable objects. This easily implies that \mathcal{W} only contains finitely many isomorphism classes corresponding to vertices in each of the copies of Γ which are glued to form the AR quiver of \mathbf{D} . As the same is the case for \mathcal{U} , it follows that it also holds for \mathcal{X} .

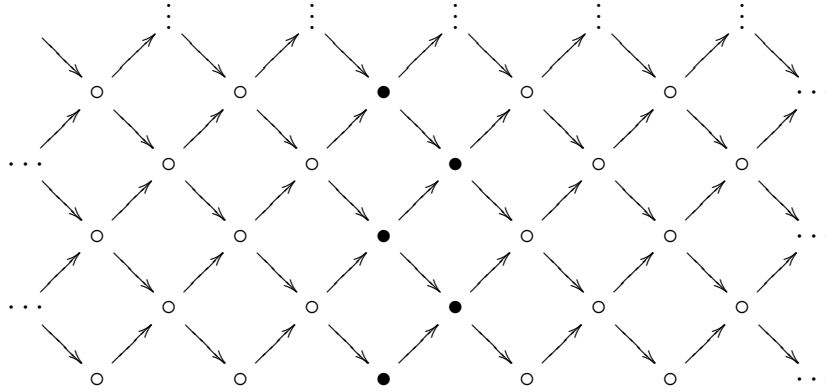
However, if d is an indecomposable object of \mathbf{D} , then the vertex of d sits in one of the copies of Γ . The only indecomposable objects of \mathbf{D} which have non-zero morphisms to and from d are the ones corresponding to vertices in that copy of Γ and the two neighbouring copies. But this

means that only finitely many isomorphism classes of indecomposable objects from \mathcal{X} have non-zero morphisms to and from d whence \mathcal{X} is precovering and preenveloping.

4.b. **A category of type A_∞ .** Let $R = k[X]$ be the polynomial algebra and view R as a DG algebra with zero differential and X placed in homological degree 1. Let \mathbf{D} be $\mathbf{D}^f(R)$, the derived category of DG R -modules with finite dimensional homology over k .

It was shown in [14] that \mathbf{D} can be viewed as a cluster category of type A_∞ . In particular, it was proved that its weak cluster tilting subcategories are in bijection with the set of maximal configurations of non-crossing arcs connecting non-neighbouring integers. It was also shown that not all weak cluster tilting subcategories are cluster tilting; indeed, a precise criterion was given to decide whether a maximal configuration of arcs determines a cluster tilting subcategory.

The category \mathbf{D} satisfies Setup 1.1 by [14]. It is 2-Calabi-Yau so the functor $S\Sigma^{-2}$ is equivalent to the identity. Its AR quiver is $\mathbb{Z}A_\infty$. Let \mathcal{U} be add of infinitely many indecomposable objects, the first few of which are indicated by solid dots in the following sketch of the AR quiver.



It was shown in [14] that \mathcal{U} is a cluster tilting subcategory of \mathbf{D} and that \mathcal{U} is equivalent to the path category of its quiver Q ,

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \dots$$

Accordingly, $\mathbf{D}/\mathcal{U} \simeq \mathbf{mod} \mathcal{U}$ is equivalent to $\mathbf{rep} Q$, the category of finitely presented representations of Q , which is hereditary by [21, sec. II.1]. Since Q is locally finite, each object of $\mathbf{rep} Q$ has finite length.

It follows that Theorems 2.2 and 3.5 both apply, so cluster tilting subcategories of \mathbf{D} project to support tilting subcategories of \mathbf{D}/\mathcal{U} , and

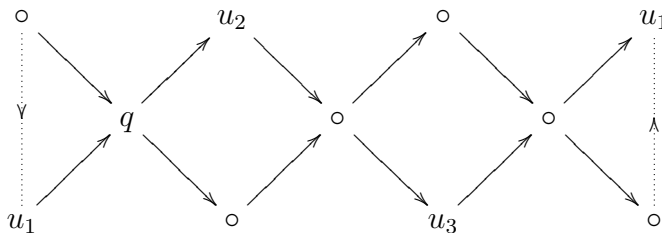
support tilting subcategories of \mathbf{D}/\mathcal{U} can be lifted uniquely to weak cluster tilting subcategories of \mathbf{D} .

In particular, any configuration of arcs which determines a cluster tilting subcategory of \mathbf{D} also gives rise to a support tilting subcategory of \mathbf{D}/\mathcal{U} , so we get an ample supply of such subcategories.

We do not know whether Proposition 3.7 applies to this situation. Support tilting subcategories of \mathbf{D}/\mathcal{U} lift to weak cluster tilting subcategories of \mathbf{D} , but not all such subcategories are cluster tilting. It would be interesting to determine whether or not Proposition 3.7 does apply.

4.c. A counterexample. We thank the referee for the following example which shows that Theorem B is false without the assumption that \mathbf{D}/\mathcal{U} has finite global dimension. The same situation can be found in [25, exa. 2.3].

Let Q be the quiver $\circ \longrightarrow \circ \longrightarrow \circ$ of type A_3 . The following figure from [24] and [25] shows the Auslander-Reiten quiver of the cluster category \mathbf{C} and the indecomposable summands u_i of a basic cluster tilting object U .



As indicated, the ends of the quiver have to be identified with opposite orientations.

The subcategory $\text{add } U$ is cluster tilting. The indecomposable projective objects of $\mathbf{C}/\text{add } U$ come from the indecomposable objects in $\Sigma^{-1} \text{add } U$. This means that the indecomposable object q in \mathbf{C} induces a non-projective indecomposable object in $\mathbf{C}/\text{add } U$, and since $\mathbf{C}/\text{add } U$ is Frobenius by [24], the induced object has infinite projective dimension.

Hence, the cluster tilting object $u_1 \oplus q \oplus u_2$ of \mathbf{C} induces an object of infinite projective dimension in $\mathbf{C}/\text{add } U$, so the induced object is not support tilting.

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