

Calabi-Yau categories and Poincaré duality spaces

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Abstract. The singular cochain complex of a topological space is a classical object. It is a Differential Graded algebra which has been studied intensively with a range of methods, not least within rational homotopy theory.

More recently, the tools of Auslander-Reiten theory have also been applied to the singular cochain complex. One of the highlights is that by these methods, each Poincaré duality space gives rise to a Calabi-Yau category. This paper is a review of the theory.

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1. Introduction

Finite dimensional algebras over a field are classical, well studied mathematical objects. Their representation theory is a particularly large and active area which has inspired a number of powerful mathematical techniques, not least Auslander-Reiten theory which is a beautiful and effective set of tools and ideas. See Appendix B and the references listed there for an introduction.

It seems reasonable to look for applications of Auslander-Reiten (AR) theory to areas outside representation theory. Specifically, let X be a topological space. The singular cochain complex $C^*(X; k)$ with coefficients in a field k of characteristic 0 is a Differential Graded algebra which has been studied intensively, in particular in rational homotopy theory, see [6]. For an introduction to Differential Graded (DG) homological algebra, see Appendix A and the references listed there. The singular cohomology $H^*(X; k)$ is defined as the cohomology of the complex $C^*(X; k)$; it is a graded algebra. Now let X be simply connected with $\dim_k H^*(X; k) < \infty$; then $C^*(X; k)$ is quasi-isomorphic to a DG algebra R with $\dim_k R < \infty$, and it is natural to try to apply AR theory to R . This was the subject of [11], [12], and [20], and the object of this paper is to review the results of those papers.

Among the highlights is Theorem 6.4 from which comes the title of the paper. Consider the derived category of DG left- R -modules, $D(R)$, which is equivalent to $D(C^*(X; k))$ since the two DG algebras are quasi-isomorphic. The latter category

has the full subcategory $D^c(C^*(X; k))$ consisting of compact DG modules; these play the role of finitely generated representations. Theorem 6.4 now says that if k has characteristic 0, then

$$\begin{aligned} D^c(C^*(X; k)) \text{ is an } n\text{-Calabi-Yau category} & \quad (1.1) \\ \Leftrightarrow X \text{ has } n\text{-dimensional Poincaré duality over } k. \end{aligned}$$

Let me briefly explain the terminology. A triangulated category T , such as for instance $D^c(C^*(X; k))$, is called n -Calabi-Yau if n is the smallest non-negative integer for which Σ^n , the n th power of the suspension functor, is a Serre functor, that is, permits natural isomorphisms

$$\mathrm{Hom}_k(\mathrm{Hom}_T(M, N), k) \cong \mathrm{Hom}_T(N, \Sigma^n M).$$

The topological space X is said to have n -dimensional Poincaré duality over k if there is an isomorphism

$$\mathrm{Hom}_k(H^*(X; k), k) \cong \Sigma^n H^*(X; k)$$

of graded left- $H^*(X; k)$ -modules.

Examples of n -Calabi-Yau categories are higher cluster categories, see [15, sec. 4], and examples of spaces with n -dimensional Poincaré duality are compact n -dimensional manifolds. Equation (1.1) provides a link between the currently popular theory of Calabi-Yau categories and algebraic topology. It also gives a new class of examples of Calabi-Yau categories which, so far, typically have been exemplified by higher cluster categories. The new categories appear to behave very differently from higher cluster categories, cf. Section 7, Problem 7.8.

A number of other results are also obtained, not least on the structure of the AR quiver of $D^c(C^*(X; k))$ which, for a space with Poincaré duality, consists of copies of the repetitive quiver $\mathbb{Z}A_\infty$, see Theorem 6.5.

In a speculative vein, the theory presented here ties in with the version of non-commutative geometry in which a DG algebra, or more generally a DG category, is viewed as a non-commutative scheme. The idea is to think of the derived category of the DG algebra or DG category as being the derived category of quasi-coherent sheaves on a non-commutative scheme (which does not actually exist). There appear so far to be no published references for this viewpoint which has been brought forward by Drinfeld and Kontsevich, but it does seem to call for a detailed study of the derived categories of DG algebras and DG categories. Auslander-Reiten theory is an obvious tool to try, and [11], [12], and [20] along with this paper can, perhaps, be viewed as a first, modest step.

As indicated, the paper is a review. The results were known previously, the main references being [11], [12], and [20]; more details of the origin of individual results are given in the introductions to the sections. There is no claim to originality, except that some of the proofs are new. It is also the first time this material has appeared together.

Most of the paper is phrased in terms of the DG algebra R rather than $C^*(X; k)$, see Setup 2.1. This is merely a notational convenience: R and $C^*(X; k)$ are quasi-isomorphic, so have equivalent derived categories. Hence, all results about the

derived category of R also hold for the derived category of $C^*(X; k)$. The paper is organized as follows:

Some background on DG homological algebra and AR theory is collected in two appendices, A and B.

Section 2 gives some preliminary results on cochain DG algebras and their DG modules. The main result is Theorem 2.7 which gives a number of alternative descriptions of when R is a so-called Gorenstein DG algebra. The importance of this condition is that $C^*(X; k)$ is Gorenstein precisely when X has Poincaré duality.

Section 3 studies the existence of AR triangles in the category $D^c(R)$, which turns out to be equivalent to R being Gorenstein by Theorem 3.4.

Section 4 considers the local structure of the AR quiver Γ of $D^c(R)$. If R is Gorenstein with $\dim_k HR \geq 2$, then Theorem 4.10 shows that each component of Γ is isomorphic to $\mathbb{Z}A_\infty$.

Section 5 reports on work by Karsten Schmidt. It looks at the global structure of Γ where the results are so far less conclusive. If $\dim_k HR = 2$, then Γ has precisely $d - 1$ components isomorphic to $\mathbb{Z}A_\infty$, where $d = \sup\{i \mid H^i R \neq 0\}$. On the other hand, if R is Gorenstein with $\dim_k HR \geq 3$, then Γ has infinitely many components, and if $\dim_k H^e R \geq 2$ for some e , then it is even possible to find families of distinct components indexed by projective manifolds, and these manifolds can be of arbitrarily high dimension.

Section 6 makes explicit the highlights of the theory for the algebras $C^*(X; k)$.

Section 7 is a list of open problems.

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2. Cochain Differential Graded algebras

This section provides some results on cochain Differential Graded (DG) algebras, not least on the ones which are Gorenstein. The results first appeared in [11], except Lemma 2.5 which is [7, lem. 1.5] and Theorem 2.8 which is [20, cor. 3.12].

For background and terminology on DG algebras and their derived categories, see Appendix A.

Setup 2.1. *In Sections 2 through 7, k is a field and R is a DG algebra over k which has the form*

$$\cdots \rightarrow 0 \rightarrow k \rightarrow 0 \rightarrow R^2 \rightarrow R^3 \rightarrow \cdots,$$

that is, $R^{<0} = 0$, $R^0 = k$, and $R^1 = 0$. It will be assumed that $\dim_k R < \infty$, and throughout,

$$d = \sup R$$

where \sup is as in Definition A.3.

Note that either $d = 0$, in which case R is quasi-isomorphic to k , or $d \geq 2$.

Remark 2.2. If X is a simply connected topological space with $\dim_k H^*(X; k) < \infty$ and k has characteristic 0, then $C^*(X; k)$ is quasi-isomorphic to a DG algebra R satisfying the conditions of Setup 2.1 by [6, exa. 6, p. 146]. This means that the derived categories of $C^*(X; k)$ and R are equivalent, and hence, all results about the derived category of R also hold for the derived category of $C^*(X; k)$.

The highlights of the theory will be made explicit for $C^*(X; k)$ in Section 6.

Proposition 2.3. *The full subcategory $D^c(R)$ of compact objects of the derived category $D(R)$ is contained in $D^f(R)$, the full subcategory of $D(R)$ of objects with $\dim_k HM < \infty$.*

Proof. The DG module ${}_R R$ is in $D^f(R)$ by assumption, and $D^c(R)$ consists of the DG modules which are finitely built from it, cf. Definition A.6, so it follows that $D^c(R)$ is contained in $D^f(R)$. \square

Proposition 2.4. *The triangulated categories $D^f(R)$ and $D^c(R)$ have finite dimensional Hom spaces and split idempotents.*

Consequently, $D^f(R)$ and $D^c(R)$ are Krull-Schmidt categories.

Proof. If M is in $D^f(R)$, then it is finitely built from ${}_R k$ in $D(R)$, see Remark A.10. So to see that $D^f(R)$ has finite dimensional Hom spaces, it is enough to see that $\text{Hom}_{D(R)}(\Sigma^i k, k)$ is finite dimensional for each i , where Σ denotes the suspension functor of $D(R)$.

Let F be a minimal semi-free resolution of ${}_R(\Sigma^i k)$; then

$$\begin{aligned} \text{Hom}_{D(R)}(\Sigma^i k, k) &\stackrel{(a)}{\cong} H^0 \text{RHom}_R(\Sigma^i k, k) \\ &\stackrel{(b)}{\cong} H^0 \text{Hom}_R(F, k) \\ &\stackrel{(c)}{\cong} \text{Hom}_{R^\natural}(F^\natural, k^\natural)^0 \\ &= (*) \end{aligned}$$

where (a) is by Definition A.7 and (b) and (c) are by Lemma A.13, (2) and (5). However, Lemma A.13(3) says that $F^\natural \cong \bigoplus_{j < i} \Sigma^j (R^\natural)^{(\beta_j)}$ with the β_j finite, where superscript (β) indicates the direct sum of β copies of the module, and so

$$(*) \cong k^{(\beta_0)}.$$

This is finite dimensional.

Since $D^c(R)$ is contained in $D^f(R)$ by Proposition 2.3, it follows that $D^c(R)$ also has finite dimensional Hom spaces.

Idempotents split in both $D^f(R)$ and $D^c(R)$ since by [4, prop. 3.2] they split already in $D(R)$ because this is a triangulated category with set indexed coproducts.

By [19, p. 52], both $D^f(R)$ and $D^c(R)$ are Krull-Schmidt categories. \square

Recall from Definition A.3 the notion of inf of a DG module, and from Definition A.1 that R^o is the opposite DG algebra of R and that DG left- R^o -modules can be viewed as DG right- R -modules.

Lemma 2.5. *Let M be in $D^f(R^o)$ and let N be in $D^f(R)$. Then*

$$\inf(M \otimes_R^L N) = \inf M + \inf N.$$

Proof. If M or N is isomorphic to zero then the equation reads $\infty = \infty$, so let me assume not. Then $i = \inf M$ and $j = \inf N$ are integers.

Lemma A.11(1) says that M can be replaced with a quasi-isomorphic DG module which satisfies $M^\ell = 0$ for $\ell < i$.

Lemma A.13(3) says that N has a semi-free resolution F which satisfies that $F^\natural \cong \bigoplus_{\ell \leq -j} \Sigma^\ell(R^\natural)^{(\beta_\ell)}$, and it follows that $(M \otimes_R F)^\natural \cong \bigoplus_{\ell \leq -j} \Sigma^\ell(M^\natural)^{(\beta_\ell)}$.

Since $M^\ell = 0$ for $\ell < i$, this implies that $(M \otimes_R F)^\ell = 0$ for $\ell < i + j$. In particular, $\inf(M \otimes_R F) \geq i + j$ whence

$$\inf(M \otimes_R^L N) \geq i + j = \inf M + \inf N. \quad (2.1)$$

Conversely, to give a morphism of DG left- R -modules $\Sigma^{-j}R \rightarrow N$ is the same thing as to give the image z of $\Sigma^{-j}(1_R)$, and z is a cycle in N^j . Since $H^j(\Sigma^{-j}R) \cong H^0R \cong k$, the induced map $H^j(\Sigma^{-j}R) \rightarrow H^jN$ is just the map $k \rightarrow H^jN$ sending 1_k to the cohomology class of z . Hence, picking cycles z_α whose cohomology classes form a k -basis of H^jN and constructing a morphism $\Sigma^{-j}R^{(\beta)} \rightarrow N$ by sending the elements $\Sigma^{-j}(1_R)$ to the z_α gives that the induced map $H^j(\Sigma^{-j}R^{(\beta)}) \rightarrow H^jN$ is an isomorphism. Complete to a distinguished triangle

$$\Sigma^{-j}R^{(\beta)} \rightarrow N \rightarrow N'' \rightarrow; \quad (2.2)$$

since $H^{j+1}(\Sigma^{-j}R^{(\beta)}) \cong H^1(R^{(\beta)}) = 0$, the long exact cohomology sequence shows

$$\inf N'' \geq j + 1. \quad (2.3)$$

Tensoring the distinguished triangle (2.2) with M gives

$$\Sigma^{-j}M^{(\beta)} \rightarrow M \otimes_R^L N \rightarrow M \otimes_R^L N'' \rightarrow$$

and the long exact cohomology sequence of this contains

$$H^{i+j-1}(M \otimes_R^L N'') \rightarrow H^{i+j}(\Sigma^{-j}M^{(\beta)}) \rightarrow H^{i+j}(M \otimes_R^L N). \quad (2.4)$$

The inequality (2.1) can be applied to M and N'' ; because of the inequality (2.3), this gives $\inf(M \otimes_R^L N'') \geq i + j + 1$ so the first term of the exact sequence (2.4)

is zero. The second term is $H^{i+j}(\Sigma^{-j}M^{(\beta)}) \cong H^i(M^{(\beta)})$ which is non-zero since $i = \inf M$. It follows that the third term is non-zero, so

$$\inf(M \overset{L}{\otimes}_R N) \leq i + j = \inf M + \inf N.$$

Combining with the inequality (2.1) proves the lemma. \square

Definition 2.6. The DG algebra R is said to be Gorenstein if it satisfies the equivalent conditions of the following theorem.

In the theorem, recall from Definition A.9 that $D(-) = \text{Hom}_k(-, k)$.

Theorem 2.7. *The following conditions are equivalent.*

(1) *There are isomorphisms of k -vector spaces*

$$\text{Ext}_R^i(k, R) \cong \begin{cases} k & \text{for } i = d, \\ 0 & \text{otherwise} \end{cases} \cong \text{Ext}_{R^\circ}^i(k, R).$$

(2) *There are isomorphisms of graded HR-modules*

$$\text{HR}(\text{DHR}) \cong_{\text{HR}}(\Sigma^d \text{HR}) \quad \text{and} \quad (\text{DHR})_{\text{HR}} \cong (\Sigma^d \text{HR})_{\text{HR}}.$$

(3) *There are isomorphisms*

$${}_R(\text{DR}) \cong {}_R(\Sigma^d R) \text{ in } D(R) \quad \text{and} \quad (\text{DR})_R \cong (\Sigma^d R)_R \text{ in } D(R^\circ).$$

(4) $\dim_k \text{Ext}_R(k, R) < \infty$ and $\dim_k \text{Ext}_{R^\circ}(k, R) < \infty$.

(5) ${}_R(\text{DR})$ is in $D^c(R)$ and $(\text{DR})_R$ is in $D^c(R^\circ)$.

Proof. (1) \Rightarrow (3). Let F be a minimal semi-free resolution of ${}_R(\text{DR})$. Then

$$\text{Ext}_{R^\circ}(k, R) \stackrel{(a)}{\cong} \text{Ext}_R(\text{DR}, k) = H(\text{RHom}_R(\text{DR}, k)) \stackrel{(b)}{\cong} \text{Hom}_{R^\natural}(F^\natural, k^\natural) \quad (2.5)$$

where (a) is by duality and (b) is by Lemma A.13, (2) and (5). If the second isomorphism in (1) holds, then this implies $F^\natural \cong \Sigma^d R^\natural$. But then there is clearly only a single step in the semi-free filtration of F , whence $F \cong {}_R(\Sigma^d R)$ so ${}_R(\text{DR}) \cong {}_R(\Sigma^d R)$, proving the first isomorphism in (3). Likewise, the first isomorphism in (1) implies the second isomorphism in (3).

(3) \Rightarrow (2). This follows by taking cohomology.

(2) \Rightarrow (1). This follows from the Eilenberg-Moore spectral sequence

$$E_2^{pq} = \text{Ext}_{\text{HR}}^p(k, \text{HR})^q \Rightarrow \text{Ext}_R^{p+q}(k, R)$$

which exists by [5, 1.3(2)], and the corresponding spectral sequence over R° .

(4) \Leftrightarrow (5). Lemma A.13(3) says that the semi-free resolution F of ${}_R(\text{DR})$ has $F^\natural \cong \bigoplus_i \Sigma^i(R^\natural)^{(\beta_i)}$, and Equation (2.5) shows that $\dim_k \text{Ext}_{R^\circ}(k, R)$ is the number of direct summands $\Sigma^i R^\natural$. By Lemma A.13(4), this number is finite if and

only if ${}_R(DR)$ is in $D^c(R)$, so the second condition in (4) is equivalent to the first condition in (5) and vice versa.

(1) \Rightarrow (4) is clear.

(4) \Rightarrow (1). When (4) holds, so does (5) by the previous part of the proof; hence ${}_R(DR)$ is finitely built from ${}_R R$. Then the canonical morphism

$$\mathrm{RHom}_R(k, R) \otimes_R^L DR \rightarrow \mathrm{RHom}_R(k, R \otimes_R^L DR)$$

is an isomorphism, because it clearly is if DR is replaced with R . That is,

$$\mathrm{RHom}_R(k, R) \otimes_R^L DR \cong k. \quad (2.6)$$

Since (4) holds, $\mathrm{RHom}_R(k, R)$ is in $D^f(R^\circ)$, so Lemma 2.5 applies to the tensor product and gives

$$\inf \mathrm{RHom}_R(k, R) + \inf DR = \inf k = 0$$

which amounts to

$$\inf \mathrm{RHom}_R(k, R) = d.$$

On the other hand, adjointness gives the first of the next isomorphisms,

$$\mathrm{RHom}_k((DR) \otimes_R^L k, k) \cong \mathrm{RHom}_R(k, \mathrm{RHom}_k(DR, k)) \cong \mathrm{RHom}_R(k, R),$$

and so

$$\begin{aligned} \sup \mathrm{RHom}_R(k, R) &= \sup \mathrm{RHom}_k((DR) \otimes_R^L k, k) \\ &= -\inf((DR) \otimes_R^L k) \\ &\stackrel{(c)}{=} -\inf DR - \inf k \\ &= d \end{aligned}$$

where (c) is by Lemma 2.5 again. Hence the cohomology of $\mathrm{RHom}_R(k, R)$ is concentrated in degree d , and it is not hard to show that hence

$$\mathrm{RHom}_R(k, R) \cong (\Sigma^{-d} k^{(\beta)})_R$$

for some β . Inserting this into Equation (2.6) shows $\beta = 1$, so

$$\mathrm{RHom}_R(k, R) \cong (\Sigma^{-d} k)_R.$$

This is equivalent to the first isomorphism in (1), and the second one follows by a symmetric argument. \square

Theorem 2.8. *If $\dim_k HR \geq 2$, then ${}_R k$ is not in $D^c(R)$.*

Proof. Recall from Definition A.3 the notion of amplitude of a DG module. There is an amplitude inequality $\text{amp}(M \overset{\mathbb{L}}{\otimes}_R N) \geq \text{amp} M$ for M in $\mathbf{D}^f(R^\circ)$ and N in $\mathbf{D}^c(R)$. This was first stated in [20, prop. 3.11]; see [7, cor. 4.4] for an alternative proof.

If ${}_R k$ were in $\mathbf{D}^c(R)$, then this would give $\text{amp}(R \overset{\mathbb{L}}{\otimes}_R k) \geq \text{amp} R$, that is, $0 \geq \text{amp} R$ contradicting $\dim_k HR \geq 2$ whereby R must (also) have cohomology in a degree different from 0. \square

3. Auslander-Reiten triangles over Differential Graded algebras

In this section, it is proved that the compact derived category $\mathbf{D}^c(R)$ has Auslander-Reiten (AR) triangles if and only if R is a Gorenstein DG algebra. In this case, a formula is found for the AR translation of $\mathbf{D}^c(R)$. These results first appeared in [11].

For background on AR theory, see Appendix B.

In the following proposition, note that $DR \overset{\mathbb{L}}{\otimes}_R P$ inherits a left- R -structure from the DG bi- R -module DR so $DR \overset{\mathbb{L}}{\otimes}_R P$ is in $\mathbf{D}(R)$; see Definition A.7.

Proposition 3.1. *Let P be an indecomposable object of $\mathbf{D}^c(R)$. There is an AR triangle in $\mathbf{D}^f(R)$,*

$$\Sigma^{-1}(DR \overset{\mathbb{L}}{\otimes}_R P) \rightarrow N \rightarrow P \rightarrow .$$

Proof. Since P is finitely built from ${}_R R$, there is a natural equivalence

$$\mathbf{D}(\text{Hom}_{\mathbf{D}(R)}(P, -)) \simeq \text{Hom}_{\mathbf{D}(R)}(-, DR \overset{\mathbb{L}}{\otimes}_R P), \quad (3.1)$$

since there is clearly such an equivalence if P is replaced with ${}_R R$. By [16, prop. 4.2], this means that the AR triangle of the present proposition exists in $\mathbf{D}(R)$.

To complete the proof, observe that the triangle is in fact in $\mathbf{D}^f(R)$: The object P is in $\mathbf{D}^c(R)$, so it is in $\mathbf{D}^f(R)$ by Proposition 2.3. Since R_R is in $\mathbf{D}^f(R^\circ)$, the dual ${}_R(DR)$ is in $\mathbf{D}^f(R)$, and since P is finitely built from R , it follows that $DR \overset{\mathbb{L}}{\otimes}_R P$ is also in $\mathbf{D}^f(R)$. Finally, N is in $\mathbf{D}^f(R)$ by the long exact cohomology sequence. \square

Proposition 3.2. *An AR triangle in $\mathbf{D}^c(R)$ is also an AR triangle in $\mathbf{D}^f(R)$.*

Proof. By [16, lem. 4.3], each object in $\mathbf{D}^c(R)$ is a pure injective object of $\mathbf{D}(R)$. Hence by [16, prop. 3.2], each AR triangle in $\mathbf{D}^c(R)$ is an AR triangle in $\mathbf{D}(R)$, and in particular in $\mathbf{D}^f(R)$. \square

Proposition 3.3. (1) *$\mathbf{D}^c(R)$ has right AR triangles if and only if ${}_R(DR)$ is in $\mathbf{D}^c(R)$.*

(2) *$\mathbf{D}^c(R)$ has left AR triangles if and only if $(DR)_R$ is in $\mathbf{D}^c(R^\circ)$.*

Proof. (1). Suppose that $D^c(R)$ has right AR triangles. The object ${}_R R$ of $D^c(R)$ has endomorphism ring k which is local, so there is an AR triangle $M \rightarrow N \rightarrow {}_R R \rightarrow$ in $D^c(R)$. By Proposition 3.2, it is even an AR triangle in $D^f(R)$. On the other hand, Proposition 3.1 gives that there is also an AR triangle $\Sigma^{-1}({}_R(DR)) \rightarrow N' \rightarrow {}_R R \rightarrow$ in $D^f(R)$, and since the right hand terms of the two AR triangles are isomorphic, so are the left hand terms, $M \cong \Sigma^{-1}({}_R(DR))$. But M is in $D^c(R)$, so it follows that $\Sigma^{-1}({}_R(DR))$ and hence ${}_R(DR)$ is in $D^c(R)$.

Conversely, suppose that ${}_R(DR)$ is in $D^c(R)$. Given P in $D^c(R)$, Proposition 3.1 gives an AR triangle

$$\Sigma^{-1}(DR \otimes_R^L P) \rightarrow N \rightarrow P \rightarrow$$

in $D^f(R)$. Since ${}_R(DR)$ is in $D^c(R)$, it is finitely built from ${}_R R$. The same is true for P , and so $DR \otimes_R^L P$ is also finitely built from ${}_R R$, that is, it is in $D^c(R)$. It follows that both outer terms of the AR triangle are in $D^c(R)$, and then so is N . That is, the AR triangle is in $D^c(R)$, so it is an AR triangle in that category.

(2). The functors $\mathrm{RHom}_R(-, R)$ and $\mathrm{RHom}_{R^\circ}(-, R)$ are quasi-inverse dualities between $D^c(R)$ and $D^c(R^\circ)$, so $D^c(R)$ has left AR triangles if and only if $D^c(R^\circ)$ has right AR triangles. By the right module version of part (1), this happens if and only if $(DR)_R$ is in $D^c(R^\circ)$. \square

Theorem 3.4. *The following conditions are equivalent.*

- (1) $D^c(R)$ has AR triangles.
- (2) $D^c(R^\circ)$ has AR triangles.
- (3) R is Gorenstein.

Proof. By Theorem 2.7(5), condition (3) is equivalent to having that ${}_R(DR)$ is in $D^c(R)$ and $(DR)_R$ is in $D^c(R^\circ)$. This is equivalent to condition (1) by Proposition 3.3, and it is equivalent to condition (2) by the right module version of Proposition 3.3. \square

Remark 3.5. Assume the situation of Theorem 3.4.

Since $D^c(R)$ has AR triangles, [16, thm. 4.4] and Equation (3.1) imply that

$$S(-) = DR \otimes_R^L - \tag{3.2}$$

is a Serre functor of $D^c(R)$, cf. Definition B.9. So the AR translation τ of $D^c(R)$ extends to the autoequivalence

$$\Sigma^{-1}(DR \otimes_R^L -) \tag{3.3}$$

of $D^c(R)$, cf. Theorem B.10. A quasi-inverse equivalence is

$$\Sigma \mathrm{RHom}_{R^\circ}(DR, R) \otimes_R^L -;$$

these two expressions can also be viewed as quasi-inverse autoequivalences of $D(R)$.

If X is an indecomposable object of $D^c(R)$ then there are AR triangles in $D^c(R)$,

$$\Sigma^{-1}(DR \overset{L}{\otimes}_R X) \rightarrow Y \rightarrow X \rightarrow$$

and

$$X \rightarrow Y' \rightarrow \Sigma \operatorname{RHom}_{R^c}(DR, R) \overset{L}{\otimes}_R X \rightarrow .$$

Combining Equation (3.3) with Theorem 2.7(3) which says $(DR)_R \cong (\Sigma^d R)_R$ gives

$$\mathrm{H}(\tau(-)) \cong \mathrm{H}(\Sigma^{d-1}(-)) \quad (3.4)$$

as graded k -vector spaces.

4. The Auslander-Reiten quiver of a Differential Graded algebra: Local structure

This section considers the AR quiver Γ of the compact derived category $D^c(R)$. When R is Gorenstein with $\dim_k HR \geq 2$, it is proved that each component of Γ is isomorphic to $\mathbb{Z}A_\infty$ as a translation quiver. The results first appeared in [12]; the methods of Karsten Schmidt [20] have permitted some technical assumptions to be removed.

Setup 4.1. *In this section, R will be Gorenstein with $\dim_k HR \geq 2$.*

The category $D^c(R)$ has AR triangles by Theorem 3.4, and ${}_R k$ is not in $D^c(R)$ by Theorem 2.8.

The AR quiver $\Gamma(D^c(R))$ will be abbreviated to Γ .

Then Γ with the AR translation τ is a stable translation quiver by Proposition B.8. By C will be denoted a component of the translation quiver Γ .

Lemma 4.2. (1) *No positive power τ^p of the AR translation τ has a fixed point in Γ .*

(2) *Γ has no loops.*

Proof. (1). Remark 3.5 says $\tau(M) = \Sigma^{-1}(DR \overset{L}{\otimes}_R M)$. Lemma 2.5 implies

$$\inf \tau(M) = 1 + \inf DR + \inf M = 1 - d + \inf M.$$

Since d is either 0 or ≥ 2 , it follows that each positive power $\tau^p(M)$ has \inf different from $\inf M$, so no positive power is isomorphic to M .

(2). The existence of a loop $[M] \rightarrow [M]$ would mean the existence of an irreducible morphism $M \rightarrow M$ in $D^c(R)$. Such a morphism would be in the radical of the finite dimensional algebra $\operatorname{Hom}_{D^c(R)}(M, M)$, and hence some power would be zero. Mimicking the proof of [1, lem. VII.2.5] now shows $\tau(M) = M$, but this contradicts part (1). \square

A reference for the graph theoretical terminology of the following proposition is [3, sec. 4.15]. A salient fact is that when T is a directed tree, then the vertices of the repetitive quiver $\mathbb{Z}T$ have the form (p, t) where p is an integer, t is a vertex of T . The translation of the stable translation quiver $\mathbb{Z}T$ is determined by $\tau(p, t) = (p + 1, t)$.

Proposition 4.3. *There exist a directed tree T and an admissible group of automorphisms Π of $\mathbb{Z}T$ so that $C \cong \mathbb{Z}T/\Pi$ as stable translation quivers.*

Proof. Since τ extends to an autoequivalence of $D^c(R)$ by Remark 3.5, the AR translation is an automorphism of Γ so restricts to an automorphism of C . By definition, C has no multiple arrows, and by Lemma 4.2(2), it has no loops. Hence the proposition follows from the Riedtmann Structure Theorem, see [3, thm. 4.15.6]. \square

To show that $T = A_\infty$ and that Π acts trivially, the following definitions are useful.

Definition 4.4. Define a function on the objects of $D(R)$ by

$$\varphi(M) = \dim_k \text{Ext}_R(M, k).$$

By abuse of notation, the induced function on the vertices of the AR quiver Γ is also denoted by φ .

Label the AR quiver Γ by assigning to the arrow $[M] \xrightarrow{\mu} [N]$ the label (α_μ, β_μ) , where α_μ is the multiplicity of M as a direct summand of Y in the AR triangle

$$\tau N \rightarrow Y \rightarrow N \rightarrow$$

and β_μ is the multiplicity of N as a direct summand of X in the AR triangle

$$M \rightarrow X \rightarrow \tau^{-1}M \rightarrow .$$

The vertices of $\mathbb{Z}T$ have the form (p, t) where p is an integer, t a vertex of T , so each vertex t of T gives a vertex $(0, t)$ of $\mathbb{Z}T$ and hence a vertex $\Pi(0, t)$ of $\mathbb{Z}T/\Pi$, that is, of C . Similarly, an arrow $t \rightarrow t'$ in T gives an arrow $\Pi(0, t) \rightarrow \Pi(0, t')$ of C . Hence the function φ and the labelling (α, β) on Γ induce a function and a labelling on T . These will be denoted by f and (a, b) .

Lemma 4.5. *The function φ and the labelling (α, β) have the following properties.*

- (1) *If F is a minimal semi-free resolution of M with $F^\natural \cong \bigoplus_i \Sigma^i (R^\natural)^{(\beta_i)}$, then $\varphi(M)$ is equal to the number of direct summands $\Sigma^i R^\natural$ in F^\natural .*
- (2) $\varphi(\tau N) = \varphi(N)$.
- (3) *If $\tau N \rightarrow Y \rightarrow N \rightarrow$ is an AR triangle in $D^c(R)$, then $\varphi(Y) = \varphi(\tau N) + \varphi(N)$.*
- (4) *If there is an arrow $[M] \xrightarrow{\mu} [N]$ in Γ then there is a corresponding arrow $\tau[N] \xrightarrow{\nu} [M]$, and $(\alpha_\nu, \beta_\nu) = (\beta_\mu, \alpha_\mu)$.*

- (5) If there is an arrow $[M] \xrightarrow{\mu} [N]$ in Γ then there is also an arrow $\tau[M] \xrightarrow{\tau(\mu)} \tau[N]$, and $(\alpha_{\tau(\mu)}, \beta_{\tau(\mu)}) = (\alpha_\mu, \beta_\mu)$.
- (6) $\sum_{\mu: [M] \rightarrow [N]} \alpha_\mu \varphi(M) = \varphi(\tau N) + \varphi(N)$, where the sum is over all arrows in Γ with target $[N]$.

Proof. (1). It holds that

$$\varphi(M) = \dim_k \mathbf{H}(\mathrm{RHom}_R(M, k)) \stackrel{(c)}{=} \dim_k \mathbf{H}(\mathrm{Hom}_R(F, k)) \stackrel{(d)}{=} \dim_k \mathrm{Hom}_{R^\natural}(F^\natural, k^\natural),$$

where (c) and (d) are by Lemma A.13, parts (2) and (5). The right hand side is clearly equal to the number of direct summands $\Sigma^i R^\natural$ in F^\natural .

(2). It holds that

$$\begin{aligned} \varphi(\tau N) &\stackrel{(a)}{=} \dim_k \mathbf{H}(\mathrm{RHom}_R(\Sigma^{-1}(\mathrm{DR} \overset{\mathbf{L}}{\otimes}_R N), k)) \\ &= \dim_k \mathbf{H}(\mathrm{RHom}_R(N, \Sigma \mathrm{RHom}_R(\mathrm{DR}, k))) \\ &\stackrel{(b)}{=} \dim_k \mathbf{H}(\mathrm{RHom}_R(N, \Sigma^{1-d}k)) \\ &= \dim_k \mathbf{H}(\mathrm{RHom}_R(N, k)) \\ &= \varphi(N), \end{aligned}$$

where (a) is by Remark 3.5 and (b) follows from Theorem 2.7(3).

(3). The AR triangle of the lemma induces a long exact sequence consisting of pieces

$$\mathrm{Ext}_R^i(N, k) \rightarrow \mathrm{Ext}_R^i(Y, k) \rightarrow \mathrm{Ext}_R^i(\tau N, k),$$

and the claim will follow if the connecting maps are zero.

Indeed, the AR triangle is also an AR triangle in $\mathbf{D}^f(R)$ by Proposition 3.2. A morphism $\tau N \rightarrow {}_R(\Sigma^i k)$ in $\mathbf{D}^f(R)$ cannot be a split monomorphism since τN is in $\mathbf{D}^c(R)$ while ${}_R(\Sigma^i k)$ is not, cf. Setup 4.1. It follows that each such morphism factors through $\tau N \rightarrow Y$ whence the composition $\Sigma^{-1}N \rightarrow \tau N \rightarrow \Sigma^i k$ is zero. Hence the connecting morphism $\mathrm{Ext}_R^i(\tau N, k) \rightarrow \mathrm{Ext}_R^{i+1}(N, k)$ is zero as desired.

(4). Let

$$\tau N \rightarrow Y \rightarrow N \rightarrow \tag{4.1}$$

be an AR triangle in $\mathbf{D}^c(R)$. By the definition of the labelling of Γ , the multiplicity of M as a direct summand of Y is equal to both β_ν and α_μ , so $\beta_\nu = \alpha_\mu$. A similar argument shows $\alpha_\nu = \beta_\mu$, so $(\alpha_\nu, \beta_\nu) = (\beta_\mu, \alpha_\mu)$.

(5). This holds since the AR translation τ of $\mathbf{D}^c(R)$ is the restriction of an equivalence of categories by Remark 3.5.

(6). Consider the AR triangle (4.1). The object Y is a direct sum of copies of the indecomposable objects of $\mathbf{D}^c(R)$ which have irreducible morphisms to N , and the multiplicity of M as a direct summand of Y is α_μ where $[M] \xrightarrow{\mu} [N]$ is the arrow in Γ . Hence

$$\sum_{\mu: [M] \rightarrow [N]} \alpha_\mu \varphi(M) = \varphi(Y).$$

Now combine with part (3). □

Lemma 4.6. *Let $M\langle 0 \rangle, \dots, M\langle 2^p - 1 \rangle$ be indecomposable objects of $\mathcal{D}^c(R)$ with $\varphi(M\langle i \rangle) \leq \frac{p}{\dim_k R}$ for each i . If*

$$M\langle 2^p - 1 \rangle \rightarrow M\langle 2^p - 2 \rangle \rightarrow \dots \rightarrow M\langle 0 \rangle$$

are non-isomorphisms in $\mathcal{D}^c(R)$, then the composition is zero.

Proof. Let $F\langle i \rangle$ be a minimal semi-free resolution of $M\langle i \rangle$. Each $F\langle i \rangle$ must be indecomposable as a DG left- R -module, for if $F\langle i \rangle$ decomposed then it would do so into DG modules $F\langle i_\alpha \rangle$ with $\partial(F\langle i_\alpha \rangle) \subseteq R^{\geq 1} \cdot F\langle i_\alpha \rangle$, but this condition forces non-zero cohomology so the decomposition of $F\langle i \rangle$ as a DG module would induce a non-trivial decomposition of $M\langle i \rangle$ in $\mathcal{D}^c(R)$.

The morphisms in $\mathcal{D}^c(R)$ between the $M\langle i \rangle$ are represented by morphisms

$$F\langle 2^p - 1 \rangle \rightarrow F\langle 2^p - 2 \rangle \rightarrow \dots \rightarrow F\langle 0 \rangle \quad (4.2)$$

of DG left- R -modules. These cannot be bijections, since if they were, then the morphisms in $\mathcal{D}^c(R)$ between the $M\langle i \rangle$ would be isomorphisms.

Now note that if $F\langle i \rangle^\natural = \bigoplus_j \Sigma^j (R^\natural)^{(\beta_j)}$, then the direct sum has $\varphi(M\langle i \rangle)$ summands $\Sigma^j R^\natural$ by Lemma 4.5(1). Hence

$$\dim_k F\langle i \rangle = \varphi(M\langle i \rangle) \dim_k R \leq \frac{p}{\dim_k R} \dim_k R = p,$$

and it is not hard to mimick the proof of [3, lem. 4.14.1] to see that hence, the composition of the morphisms in Equation (4.2) is zero. This implies that the composition of the morphisms in the lemma is zero. \square

Lemma 4.7. *If $M\langle 0 \rangle$ is an indecomposable object of $\mathcal{D}^c(R)$ and $q \geq 0$ is an integer, then there exist indecomposable objects and irreducible morphisms in $\mathcal{D}^c(R)$,*

$$M\langle q \rangle \rightarrow M\langle q - 1 \rangle \rightarrow \dots \rightarrow M\langle 0 \rangle,$$

with non-zero composition.

Proof. Let me prove a stronger statement which implies the lemma: If $M\langle 0 \rangle$ is an indecomposable object of $\mathcal{D}^c(R)$ and $q \geq 0$ is an integer, then there exists

$${}_R(\Sigma^i k) \xrightarrow{\kappa_q} M\langle q \rangle \xrightarrow{\mu_q} M\langle q - 1 \rangle \xrightarrow{\mu_{q-1}} \dots \xrightarrow{\mu_1} M\langle 0 \rangle$$

where the $M\langle i \rangle$ are indecomposable objects of $\mathcal{D}^c(R)$ and the μ_i are irreducible morphisms in $\mathcal{D}^c(R)$, such that $\mu_1 \circ \dots \circ \mu_q \circ \kappa_q \neq 0$.

Using induction on q , first let $q = 0$. Let F be a minimal semi-free resolution of the dual $DM\langle 0 \rangle$. Then

$$\begin{aligned} \mathrm{H}(\mathrm{RHom}_R(k, M\langle 0 \rangle)) &\cong \mathrm{H}(\mathrm{RHom}_{R^\circ}(DM\langle 0 \rangle, k)) \\ &\stackrel{(a)}{\cong} \mathrm{H}(\mathrm{Hom}_{R^\circ}(F, k)) \\ &\stackrel{(b)}{\cong} \mathrm{Hom}_{(R^\circ)^\natural}(F^\natural, k^\natural) \\ &\stackrel{(c)}{\neq} 0. \end{aligned}$$

Here (a) and (b) are by Lemma A.13, parts (2) and (5). (c) is because $M\langle 0 \rangle$ is indecomposable hence has non-zero cohomology; this implies that $DM\langle 0 \rangle$ has non-zero cohomology, and then F is non-trivial semi-free whence F^{\natural} is a non-trivial graded free module.

It follows from the displayed formula that there is a non-zero morphism

$${}_R(\Sigma^i k) \xrightarrow{\kappa_0} M\langle 0 \rangle$$

for some i .

Now let $q \geq 1$ and suppose that

$${}_R(\Sigma^i k) \xrightarrow{\kappa_{q-1}} M\langle q-1 \rangle \xrightarrow{\mu_{q-1}} M\langle q-2 \rangle \xrightarrow{\mu_{q-2}} \dots \xrightarrow{\mu_1} M\langle 0 \rangle$$

has already been found with the desired properties. Let $\tau M\langle q-1 \rangle \rightarrow X\langle q \rangle \xrightarrow{\mu'_q} M\langle q-1 \rangle \rightarrow$ be an AR triangle in $D^c(R)$. By Proposition 3.2 it is also an AR triangle in $D^f(R)$. Since ${}_R k$ is not in $D^c(R)$, see Setup 4.1, it is clear that κ_{q-1} is not a split epimorphism, so it factors through μ'_q . Now I can get the situation claimed in the lemma by letting $M\langle q \rangle$ be a suitable indecomposable summand of $X\langle q \rangle$ and μ_q the restriction of μ'_q to $M\langle q \rangle$. \square

Lemma 4.8. *The function φ is unbounded on C .*

Proof. If φ were bounded on C then Lemma 4.6 would apply to sufficiently long sequences of morphisms between indecomposable objects with vertices in C , but this would make impossible the situation established in Lemma 4.7. \square

Recall that the Cartan matrix c of the labelled directed tree T is a matrix with rows and columns indexed by the vertices of T . If s and t are vertices, then

$$c_{st} = \begin{cases} 2 & \text{if } s = t, \\ -a_\mu & \text{if there is an arrow } s \xrightarrow{\mu} t, \\ -b_\nu & \text{if there is an arrow } t \xrightarrow{\nu} s, \\ 0 & \text{if } s \neq t \text{ and } s \text{ and } t \text{ are not connected by an arrow;} \end{cases}$$

cp. [3, sec. 4.5]. The function f on the vertices of T is called additive if it satisfies $\sum_s c_{st} f(s) = 0$ for each t , that is,

$$2f(t) - \sum_{\mu: s \rightarrow t} a_\mu f(s) - \sum_{\nu: t \rightarrow u} b_\nu f(u) = 0 \quad (4.3)$$

for each t , where the sums are over all arrows in T into t and out of t . Indeed:

Proposition 4.9. *The function f is additive and unbounded on T .*

Proof. Using Definition 4.4, the left hand side of Equation (4.3) can be rewritten

$$2\varphi(\Pi(0, t)) - \sum_{\mu: s \rightarrow t} \alpha_{\Pi(0, s) \rightarrow \Pi(0, t)} \varphi(\Pi(0, s)) - \sum_{\nu: t \rightarrow u} \beta_{\Pi(0, t) \rightarrow \Pi(0, u)} \varphi(\Pi(0, u)).$$

The translation of $\mathbb{Z}T/\Pi$ is given by $\tau(\Pi(p, t)) = \Pi(p + 1, t)$. To each arrow $\Pi(0, t) \rightarrow \Pi(0, u)$ corresponds an arrow $\tau(\Pi(0, u)) \rightarrow \Pi(0, t)$, that is, $\Pi(1, u) \rightarrow \Pi(0, t)$. Lemma 4.5(4) gives $\beta_{\Pi(0,t) \rightarrow \Pi(0,u)} = \alpha_{\Pi(1,u) \rightarrow \Pi(0,t)}$. Lemma 4.5(2) gives $\varphi(\Pi(0, u)) = \varphi(\tau(\Pi(0, u))) = \varphi(\Pi(1, u))$, and also implies that $2\varphi(\Pi(0, t)) = \varphi(\Pi(0, t)) + \varphi(\tau(\Pi(0, t)))$.

Substituting all this into the previous expression gives

$$\begin{aligned} & \varphi(\Pi(0, t)) + \varphi(\tau(\Pi(0, t))) \\ & - \sum_{\mu: s \rightarrow t} \alpha_{\Pi(0,s) \rightarrow \Pi(0,t)} \varphi(\Pi(0, s)) - \sum_{\nu: t \rightarrow u} \alpha_{\Pi(1,u) \rightarrow \Pi(0,t)} \varphi(\Pi(1, u)). \end{aligned}$$

Recall that the sums are over all the arrows in T into t and out of t . From the construction of the repetitive quiver $\mathbb{Z}T$, this means that between them, the sums can be viewed as being over all the arrows into $(0, t)$ in $\mathbb{Z}T$. However, the projection $\mathbb{Z}T \rightarrow \mathbb{Z}T/\Pi$ is a covering so induces a bijection between the arrows in $\mathbb{Z}T$ into $(0, t)$ and the arrows in $\mathbb{Z}T/\Pi$ into $\Pi(0, t)$. So in fact, the previous expression can be rewritten

$$\varphi(\Pi(0, t)) + \varphi(\tau(\Pi(0, t))) - \sum_{m \rightarrow \Pi(0,t)} \alpha_{m \rightarrow \Pi(0,t)} \varphi(m)$$

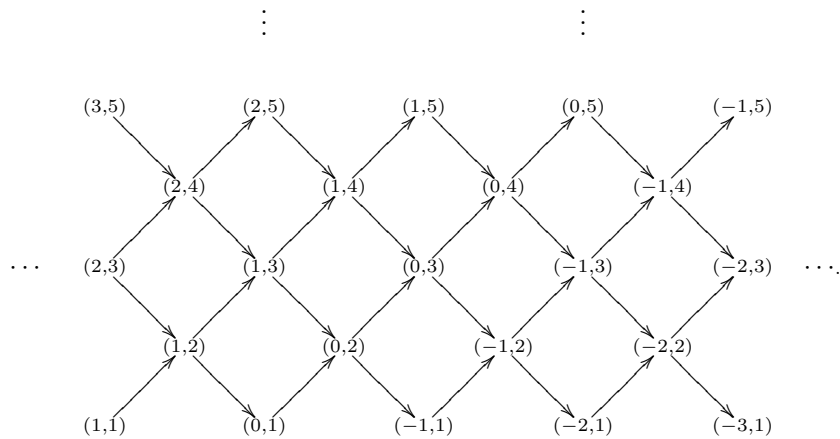
where the sum is over all arrows in $\mathbb{Z}T/\Pi$ into $\Pi(0, t)$. But identifying $\mathbb{Z}T/\Pi$ and C , the displayed expression is zero by Lemma 4.5(6), so f is additive.

Since $f(t) = \varphi(\Pi(0, t))$ by Definition 4.4 and $\varphi(\Pi(p, t)) = \varphi(\tau^p \Pi(0, t)) = \varphi(\Pi(0, t))$ by Lemma 4.5(2), if f were bounded on T then φ would be bounded on C . But this is false by Lemma 4.8. \square

Recall that the graph A_∞ is

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } 4 \text{ --- } 5 \text{ --- } \dots,$$

where a convenient numbering of the vertices has been chosen. A quiver of type A_∞ is an orientation of this graph. The repetitive quiver $\mathbb{Z}A_\infty$ does not depend on the orientation; with a standard numbering of the vertices it is



The translation acts by $\tau(p, t) = (p + 1, t)$.

Theorem 4.10. (1) *The component C of the AR quiver Γ of $D^c(R)$ is isomorphic to $\mathbb{Z}A_\infty$ as a stable translation quiver.*

(2) *Each label (α_μ, β_μ) on Γ is equal to $(1, 1)$.*

(3) *If the function φ has value φ_1 on the edge of $C \cong \mathbb{Z}A_\infty$, then it has value $n\varphi_1$ on the n 'th horizontal row of vertices in C .*

Proof. By Proposition 4.9, there is an additive unbounded function f on the labelled tree T . Hence T is of type A_∞ with all labels equal to $(1, 1)$ by [3, thm. 4.5.8(iv)]. This proves (2), and it also means that to prove (1), it is sufficient to show that Π acts trivially on $\mathbb{Z}A_\infty$.

But if it did not, then there would exist a vertex m on the edge of $\mathbb{Z}A_\infty$ and a g in Π such that $gm \neq m$. The vertex gm would again be on the edge, and so it would have the form $\tau^p m$ for some $p \neq 0$. But then m and $\tau^p m$ would get identified in $\mathbb{Z}A_\infty/\Pi$, and hence Πm would be a fixed point in $\mathbb{Z}A_\infty/\Pi$ of τ^p , that is, a fixed point in C of τ^p . But this is impossible by Lemma 4.2(1).

Finally, it is a standard consequence of additivity that if the function f has value $f(1) = f_1$ at the first vertex of A_∞ , then it has value $f(n) = nf_1$ at the n th vertex. Since $\varphi(\Pi(p, n)) = \varphi(\tau^p(\Pi(0, n))) = \varphi(\Pi(0, n)) = f(n)$, the claim (3) on φ follows. \square

5. Report on work by Karsten Schmidt

In this section, the study of the AR quiver Γ of $D^c(R)$ is continued, and some aspects of the global structure are revealed. If $\dim_k HR = 2$ then Γ has precisely $d - 1$ components. On the other hand, for Gorenstein algebras with $\dim_k HR \geq 3$, there are infinitely many components. Often, these even form families which are indexed by projective manifolds, and these manifolds can be of arbitrarily high dimension.

With the exception of Theorem 5.1 which is essentially in [11], the results of this section are due to Karsten Schmidt; see [20, thm. 4.1].

Only a sketch is given of the proof of the next theorem; for more information, see [11, sec. 8].

Theorem 5.1. *If $\dim_k HR = 2$ then $D^c(R)$ has AR triangles, and the AR quiver of $D^c(R)$ has $d - 1$ components, each isomorphic to $\mathbb{Z}A_\infty$.*

Proof. The cohomology of R in low degrees is $H^0 R = k$ and $H^1 R = 0$. Since $\dim_k HR = 2$, it follows that the only other non-zero cohomology is $H^d R = k$, and it is easy to check that R therefore satisfies the conditions of Theorem 2.7(2) so R is Gorenstein. Theorem 3.4 says that $D^c(R)$ has AR triangles, and Theorem 4.10(1) says that each component of the AR quiver of $D^c(R)$ is isomorphic to $\mathbb{Z}A_\infty$.

Replacing R with a quasi-isomorphic truncation, it can be supposed that $R^{>d} = 0$, see Lemma A.11(3). Pick a cycle x in R^d with non-zero cohomology class. The

graded algebra $k[X]/(X^2)$ with X in cohomological degree d can be viewed as a DG algebra with zero differential, and the map $k[X]/(X^2) \rightarrow R$ sending X to x is a quasi-isomorphism, so R can be replaced with $k[X]/(X^2)$.

Now consider the algebra $S = k[Y]$ with Y in cohomological degree $-d + 1$, viewed as a DG algebra with zero differential. The DG module k can be viewed as a DG right- R -right- S -module in an obvious way, and it induces adjoint functors

$$\mathrm{D}(S^\circ) \begin{array}{c} \xleftarrow{k \otimes_R^L -} \\ \xrightarrow{\mathrm{RHom}_{S^\circ}(k, -)} \end{array} \mathrm{D}(R).$$

The upper functor clearly sends ${}_R R$ to k_S , and by computing with a semi-free resolution it can be verified that the lower functor sends k_S to ${}_R R$. Hence the functors restrict to quasi-inverse equivalences on the subcategories of objects which are finitely built, respectively, from k_S and ${}_R R$. These subcategories are precisely $\mathrm{D}^f(S^\circ)$ and $\mathrm{D}^c(R)$.

So it is enough to show that the AR quiver of $\mathrm{D}^f(S^\circ)$ has $d - 1$ components. However, S is $k[Y]$ equipped with zero differential, so HS is just $k[Y]$ viewed as a graded algebra. This polynomial algebra in one variable has global dimension 1, and this makes it possible to prove that if M is a DG right- S -module, then M is quasi-isomorphic to HM equipped with zero differential.

This reduces the classification of objects of $\mathrm{D}^f(S^\circ)$ to the classification of graded right- HS -modules. However, using again that $\mathrm{HS} = k[Y]$ is a polynomial algebra in one variable, one shows that its indecomposable finite dimensional graded right-modules are precisely

$$\Sigma^j k[Y]/(Y^{m+1})$$

for j in \mathbb{Z} and $m \geq 0$. Viewing these as DG right- S -modules with zero differential gives the indecomposable objects of $\mathrm{D}^f(S^\circ)$, and knowing the indecomposable objects, it is an exercise in AR theory to compute the AR triangles, find the AR quiver, and verify that it has $d - 1$ components. \square

Setup 5.2. *In the rest of this section, the setup of Section 4 will be kept: R is Gorenstein with $\dim_k \mathrm{HR} \geq 2$.*

The category $\mathrm{D}^c(R)$ has AR triangles by Theorem 3.4, and ${}_R k$ is not in $\mathrm{D}^c(R)$ by Theorem 2.8.

The AR quiver $\Gamma(\mathrm{D}^c(R))$ will abbreviated to Γ .

Since $\mathrm{H}^0 R \cong k$ and $\mathrm{H}^1 R = 0$, by Theorem 2.7(2) it must be the case that $\mathrm{H}^{d-1} R = 0$ and $\mathrm{H}^d R \cong k$. By definition, d is the highest degree in which R has non-zero cohomology; suppose that $e \notin \{0, d\}$ is another degree with $\mathrm{H}^e R \neq 0$ and observe that then

$$2 \leq e \leq d - 2$$

and $d \geq 4$.

Let X be a minimal semi-free DG left- R -module whose semi-free filtration contains only finitely many copies of (de)suspensions of R . In particular, Lemma A.13(4) says that X is in $\mathrm{D}^c(R)$; suppose that it is indecomposable in that category. Let $i \geq 2$ and consider the following cases.

Case (1). Suppose that

$$\inf X = 0 \quad \text{and} \quad \sup X = i.$$

A non-zero cohomology class in $H^i X$ permits a non-zero morphism $\Sigma^{-i} R \xrightarrow{g} X$; denoting the mapping cone by $X(1)$, there is a distinguished triangle

$$\Sigma^{-i} R \xrightarrow{g} X \rightarrow X(1) \rightarrow . \quad (5.1)$$

Case (2). Suppose that

$$\inf X = 0, \quad \sup X = i, \quad \text{and} \quad H^{i-d+e} X \neq 0.$$

A non-zero cohomology class in $H^{i-d+e} X$ permits a non-zero morphism $\Sigma^{-i+d-e} R \xrightarrow{h} X$; denoting the mapping cone by $X(2)$, there is a distinguished triangle

$$\Sigma^{-i+d-e} R \xrightarrow{h} X \rightarrow X(2) \rightarrow . \quad (5.2)$$

Case (2 _{α}). In Case (2), suppose moreover that $H^i X \cong k$ and that scalar multiplication induces a non-degenerate bilinear form

$$H^{d-e}(R) \times H^{i-d+e}(X) \rightarrow H^i(X) \cong k. \quad (5.3)$$

The morphism $\Sigma^{-i+d-e} R \xrightarrow{h} X$ corresponds to an element α in $H^{i-d+e} X$; denote h by h_α and $X(2)$ by $X(2_\alpha)$.

It follows from the mapping cone construction that $X(1)$, $X(2)$, and $X(2_\alpha)$ are again minimal semi-free DG left- R -modules whose semi-free filtrations contain only finitely many copies of (de)suspensions of R .

Lemma 5.3. (1) *In Case (1) of the above construction, the DG module $X(1)$ is indecomposable in $D^c(R)$. It has*

$$\inf X(1) = 0, \quad \sup X(1) = i + d - 1$$

and

$$H^{i+e-1}(X(1)) \cong H^e(R) \neq 0, \quad H^{i+d-1}(X(1)) \cong H^d(R) \cong k.$$

It satisfies $\text{amp}(X(1)) = \text{amp}(X) + d - 1$ and $\varphi(X(1)) = \varphi(X) + 1$. Moreover, if the construction is applied to X and X' then $X(1) \cong X'(1)$ implies $X \cong X'$ in $D^c(R)$. Finally, scalar multiplication induces a non-degenerate bilinear form

$$H^{d-e}(R) \times H^{i+e-1}(X(1)) \rightarrow H^{i+d-1}(X(1)) \cong k.$$

(2) *In Case (2), the DG module $X(2)$ is indecomposable in $D^c(R)$. It has*

$$\inf X(2) = 0 \quad \text{and} \quad \sup X(2) = i + e - 1.$$

It satisfies $\text{amp}(X(2)) = \text{amp}(X) + e - 1$ and $\varphi(X(2)) = \varphi(X) + 1$. Moreover, if the construction is applied to X and X' then $X(2) \cong X'(2)$ implies $X \cong X'$ in $D^c(R)$.

(3) In Case (2_α) , if α and α' are elements of $H^{i-d+e}X$ then

$$X(2_\alpha) \cong X(2_{\alpha'}) \text{ in } D^c(R) \iff \alpha = \kappa\alpha' \text{ for a } \kappa \text{ in } k.$$

Proof. (1). Indecomposability will follow from [10, lem. 6.5] if I can show in $D^c(R)$ that g is non-zero (clear), non-invertible (clear since $\inf \Sigma^{-i}R = i \geq 2$ but $\inf X = 0$), and that $\text{Hom}_{D^c(R)}(X, \Sigma\Sigma^{-i}R) = 0$. However,

$$\begin{aligned} \text{Hom}_{D^c(R)}(X, \Sigma\Sigma^{-i}R) &\cong \text{Hom}_{D^c(R^o)}(D\Sigma\Sigma^{-i}R, DX) \\ &\cong \text{Hom}_{D^c(R^o)}(\Sigma^{i-1}DR, DX) \\ &\stackrel{(a)}{\cong} \text{Hom}_{D^c(R^o)}(\Sigma^{i-1+d}R, DX) \\ &\cong H^{-i+1-d}(DX) \\ &\cong \text{DH}^{i-1+d}(X) \\ &\stackrel{(b)}{=} 0 \end{aligned}$$

where (a) is by Theorem 2.7(3) and (b) is because $\sup X = i$.

The statements $\inf X(1) = 0$, $\sup X(1) = i+d-1$, $H^{i+e-1}(X(1)) \cong H^e(R) \neq 0$, and $H^{i+d-1}(X(1)) \cong H^d(R) \cong k$ follow from the long exact cohomology sequence of the distinguished triangle (5.1). The statement about the amplitude is a consequence, and $\varphi(X(1)) = \varphi(X) + 1$ because $X(1)$ is minimal semi-free with one more copy of a desuspension of R in its semi-free filtration than X ; cf. Lemma 4.5(1).

To get the statement on isomorphisms, first observe that by a computation like the one above,

$$\text{Hom}_{D^c(R)}(X(1), \Sigma\Sigma^{-i}R) \cong \text{DH}^{i+d-1}(X(1)) \cong D(k) \cong k.$$

Now suppose that there is an isomorphism $X(1) \xrightarrow{\sim} X'(1)$ in $D^c(R)$. This gives a diagram between the distinguished triangles defining $X(1)$ and $X'(1)$,

$$\begin{array}{ccccccc} \Sigma^{-i}R & \longrightarrow & X & \longrightarrow & X(1) & \longrightarrow & \Sigma^{-i+1}R \\ & & & & \downarrow & & \\ \Sigma^{-i}R & \longrightarrow & X' & \longrightarrow & X'(1) & \longrightarrow & \Sigma^{-i+1}R. \end{array}$$

The last morphism in the upper distinguished triangle is non-zero, for otherwise the triangle would be split contradicting that X is indecomposable. Since $\text{Hom}_{D^c(R)}(X(1), \Sigma\Sigma^{-i}R)$ is one-dimensional, there exists a morphism $\Sigma^{-i+1}R \rightarrow \Sigma^{-i+1}R$ to give a commutative square. Adding this morphism and its desuspension to the diagram gives

$$\begin{array}{ccccccc} \Sigma^{-i}R & \longrightarrow & X & \longrightarrow & X(1) & \longrightarrow & \Sigma^{-i+1}R \\ \downarrow & & & & \downarrow & & \downarrow \\ \Sigma^{-i}R & \longrightarrow & X' & \longrightarrow & X'(1) & \longrightarrow & \Sigma^{-i+1}R, \end{array}$$

and the two new vertical arrows are also isomorphisms since they are non-zero and since $\mathrm{Hom}_{\mathcal{D}^c(R)}(R, R) \cong k$. By the axioms of triangulated categories, there is a vertical morphism $X \rightarrow X'$ which completes to a commutative diagram, and this morphism is an isomorphism by the triangulated five lemma.

Finally, to get the non-degenerate bilinear form, observe that R is Gorenstein so by Theorem 2.7(2) scalar multiplication gives a non-degenerate bilinear form

$$\mathrm{H}^{d-e}(R) \times \mathrm{H}^{i+e-1}(\Sigma^{-i+1}R) \rightarrow \mathrm{H}^{i+d-1}(\Sigma^{-i+1}R) \cong k.$$

But $X(1)$ is a mapping cone which in degrees $\geq i+1$ is equal to $\Sigma^{-i+1}R$, so this gives a non-degenerate bilinear form

$$\mathrm{H}^{d-e}(R) \times \mathrm{H}^{i+e-1}(X(1)) \rightarrow \mathrm{H}^{i+d-1}(X(1)) \cong k$$

as claimed.

(2) follows by similar arguments.

(3). \Leftarrow is elementary. \Rightarrow : Given the isomorphism $X(2_\alpha) \rightarrow X(2_{\alpha'})$, the method applied in the proof of (1) produces a diagram between the distinguished triangles defining $X(2_\alpha)$ and $X(2_{\alpha'})$,

$$\begin{array}{ccccccc} \Sigma^{-i+d-e}R & \xrightarrow{h_\alpha} & X & \longrightarrow & X(2_\alpha) & \longrightarrow & \Sigma^{-i+d-e+1}R \\ \downarrow & & \downarrow \gamma & & \downarrow & & \downarrow \\ \Sigma^{-i+d-e}R & \xrightarrow{h_{\alpha'}} & X & \longrightarrow & X(2_{\alpha'}) & \longrightarrow & \Sigma^{-i+d-e+1}R, \end{array}$$

where the vertical maps are isomorphisms. Commutativity of the first square implies $(\mathrm{H}^{i-d+e}(\gamma))(\alpha) = \alpha'$.

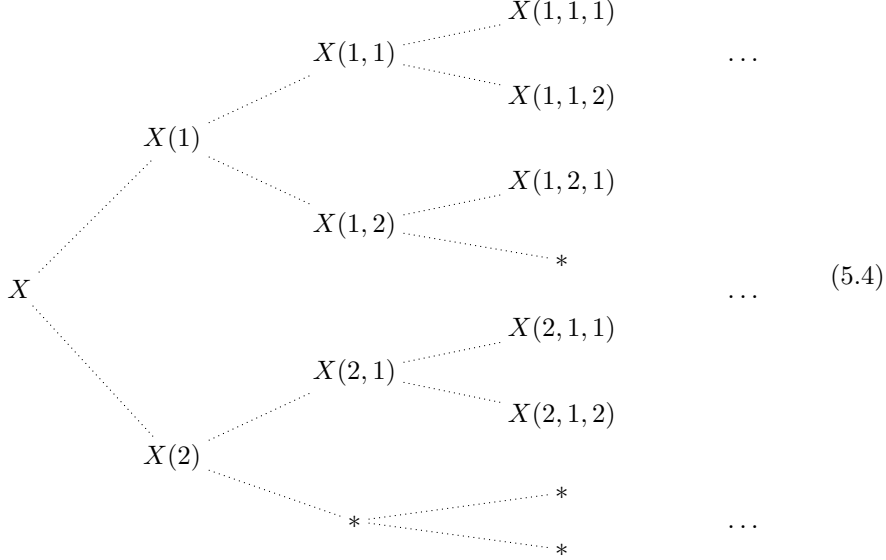
Consider x in $\mathrm{H}^{d-e}R$. Then

$$x\alpha' = 0 \Leftrightarrow x(\mathrm{H}^{i-d+e}(\gamma))(\alpha) = 0 \Leftrightarrow (\mathrm{H}^{i-d+e}(\gamma))(x\alpha) = 0 \Leftrightarrow x\alpha = 0,$$

the last \Leftrightarrow because γ is an isomorphism in $\mathcal{D}^c(R)$ whence $\mathrm{H}^{i-d+e}(\gamma)$ is bijective. Seeing that the bilinear form (5.3) is non-degenerate, this means that $\alpha = \kappa\alpha'$ for a κ in k . \square

Observe that it makes sense to insert $X(1)$ into either of Cases (1), (2), and (2_α) . Likewise, it makes sense to insert $X(2)$ and $X(2_\alpha)$ into Case (1). Iterating

Cases (1) and (2), the following tree can be constructed.



The notation is straightforward; for instance, by $X(1, 2)$ is denoted the DG module obtained by first performing the construction of Case (1), then the construction of Case (2). The rule for omitting nodes of the tree is that no $X(\dots)$ must contain two neighbouring digits 2.

Theorem 5.4. *Suppose that $\dim_k HR \geq 3$. Then the AR quiver Γ of $D^c(R)$ has infinitely many components.*

Proof. It is a standing assumption in this section that R is Gorenstein, so each component C of Γ is isomorphic to $\mathbb{Z}A_\infty$ as a stable translation quiver by Theorem 4.10(1).

Since $\dim_k HR \geq 3$, there exists an $e \notin \{0, d\}$ such that R has non-zero cohomology in degree e , so the above constructions make sense. Start with $X = R$ and consider the tree (5.4). It follows from Lemma 5.3, (1) and (2), that the function φ is constant with value r on the r 'th column of the tree. On the other hand, by Theorem 4.10(3), the value of φ on the n 'th horizontal row of a component $C \cong \mathbb{Z}A_\infty$ of Γ is $n\varphi_1$. Hence, if the vertices corresponding to two modules in the r 'th column of the tree (5.4) both belong to C , then they sit in the same horizontal row of vertices in C .

Equation (3.4) implies that $\text{amp } \tau Y = \text{amp } Y$ for each Y in $D^c(R)$. However, on C , the action of τ is to move a vertex one step to the left. It follows that the amplitude is constant on each horizontal row of C .

Combining these arguments, if the vertices corresponding to two modules in the r 'th column of the tree (5.4) both belong to C , then the modules have the same amplitude.

On the other hand, in the construction above, Case (1) makes the amplitude grow by $d - 1$ and Case (2) makes the amplitude grow by $e - 1$. Let a_1, \dots, a_r be

a sequence of the digits 1 and 2 which does not contain two neighbouring digits 2. Suppose that the sequence contains s digits 1 and $r - s$ digits 2. Then since $\text{amp } X = \text{amp } R = d$ it holds that $\text{amp } X(a_1, \dots, a_r) = d + s(d - 1) + (r - s)(e - 1)$, and since $e < d$ it is clear that this value changes when s changes. So by choosing r sufficiently large, a column of the tree (5.4) can be achieved with an arbitrarily large number of DG modules with pairwise different amplitudes.

By the first part of the proof, this results in an arbitrarily large number of different components of Γ , so Γ has infinitely many components. \square

Theorem 5.5. *Suppose that there is an e with $\dim_k H^e R \geq 2$. Then the AR quiver Γ of $D^c(R)$ has families of distinct components which are indexed by projective manifolds over k , and these manifolds can be of arbitrarily high dimension.*

Proof. Again, it is a standing assumption in this section that R is Gorenstein, so each component C of Γ is isomorphic to $\mathbb{Z}A_\infty$ as a stable translation quiver by Theorem 4.10(1).

Set $X = R$. With an obvious notation, consider $X(2_\alpha, 1, 2_\beta)$. Then an isomorphism $X(2_\alpha, 1, 2_\beta) \cong X(2_{\alpha'}, 1, 2_{\beta'})$ implies $X(2_\alpha, 1) \cong X(2_{\alpha'}, 1)$ by Lemma 5.3(2), and then $\beta = \lambda\beta'$ for a λ in k by Lemma 5.3(3). And $X(2_\alpha, 1) \cong X(2_{\alpha'}, 1)$ implies $X(2_\alpha) \cong X(2_{\alpha'})$ by Lemma 5.3(1), and then $\alpha = \kappa\alpha'$ for a κ in k by Lemma 5.3(3).

The $X(2_\alpha, 1, 2_\beta)$ hence give a family of pairwise non-isomorphic objects of $D^c(R)$ parametrized by the Cartesian product $\{\text{rays of } \alpha\text{'s}\} \times \{\text{rays of } \beta\text{'s}\}$.

Now, $\text{sup } X = d$ so the class α is in $H^{d-d+e}(X)$, cf. the construction in Case (2). However,

$$H^{d-d+e}(X) = H^e(X) = H^e(R).$$

Hence $\{\text{rays of } \alpha\text{'s}\} = \mathbb{P}(H^e R)$ where \mathbb{P} denotes the projective space of rays in a vector space. Moreover, $\text{sup } X(2_\alpha, 1) = d + (e - 1) + (d - 1) = 2d + e - 2$ by Lemma 5.3, (1) and (2), so the class β is in $H^{(2d+e-2)-d+e}(X(2_\alpha, 1))$. However,

$$\begin{aligned} H^{(2d+e-2)-d+e}(X(2_\alpha, 1)) &= H^{d+2e-2}(X(2_\alpha, 1)) \\ &= H^{(d+e-1)+e-1}(X(2_\alpha, 1)) \\ &\cong H^e(R), \end{aligned}$$

where \cong is by Lemma 5.3(1) because $\text{sup } X(2_\alpha) = d + e - 1$. Hence it is also the case that $\{\text{rays of } \beta\text{'s}\} = \mathbb{P}(H^e R)$.

This shows that the $X(2_\alpha, 1, 2_\beta)$ give a family of pairwise non-isomorphic objects of $D^c(R)$ indexed by $\mathbb{P}(H^e R) \times \mathbb{P}(H^e R)$. Note that the projective space $\mathbb{P}(H^e R)$ is non-trivial since $\dim_k H^e R \geq 2$.

By Lemma 5.3, (1) and (2), all the $X(2_\alpha, 1, 2_\beta)$ have the same value of φ (it is 4), so if the vertices of two non-isomorphic ones belonged to the same component C of Γ , then they would be different vertices in the same horizontal row of $C \cong \mathbb{Z}A_\infty$ because the value of φ on the n 'th row of C is $n\varphi_1$ by Theorem 4.10(3). However, it follows from Equation (3.4) that $\text{inf}(\tau Y) = \text{inf}(Y) - d + 1$, so different vertices in the n 'th row of C correspond to DG modules with different inf , but the $X(2_\alpha, 1, 2_\beta)$

all have the same inf by Lemma 5.3, (1) and (2) (it is 0). Hence the vertices of two non-isomorphic $X(2_\alpha, 1, 2_\beta)$'s must belong to different components of Γ , so a family has been found of distinct components of Γ parametrized by the projective manifold $\mathbb{P}(\mathbb{H}^e R) \times \mathbb{P}(\mathbb{H}^e R)$ over k .

An analogous argument with objects of the form $X(2_\alpha, 1, 2_\beta, 1, \dots, 1, 2_\gamma)$ produces families of distinct components of the AR quiver indexed by projective manifolds of arbitrarily high dimension, as claimed. \square

6. Poincaré duality spaces

This section makes explicit the highlights of the previous sections for DG algebras of the form $C^*(X; k)$. The results first appeared in [11], [12], and [20].

Setup 6.1. *In this section, the field k will have characteristic 0. By X will be denoted a simply connected topological space with $\dim_k H^*(X; k) < \infty$. Write*

$$n = \sup\{i \mid H^i(X; k) \neq 0\}.$$

When the singular cochain complex $C^(X; k)$ and singular cohomology $H^*(X; k)$ appear below, it is always with coefficients in k , so I will use the shorthands $C^*(X)$ and $H^*(X)$.*

Remark 6.2. The singular chain complex $C^*(X)$ is a DG algebra under cup product, and by [6, exa. 6, p. 146], it is quasi-isomorphic to a commutative DG algebra A satisfying the conditions of Setup 2.1.

Remark 6.3. For X to be simply connected means that it is path connected and that each closed path in X can be shrunk continuously to a point. Equivalently, X is path connected and its fundamental group $\pi_1(X)$ is trivial.

The space X is said to have Poincaré duality over k if there is an isomorphism

$$DH^*(X) \cong \Sigma^n H^*(X)$$

of graded left- $H^*(X)$ -modules. It is a classical theorem that any compact n -dimensional manifold has Poincaré duality; indeed, this is one of the oldest results of algebraic topology.

A consequence of Poincaré duality over k is that there are isomorphisms of vector spaces

$$DH^i(X) \cong H^{n-i}(X)$$

for each i , and hence that the singular cohomology $H^*(X)$ with coefficients in k is concentrated between dimensions 0 and n and has the same vector space dimension in degrees i and $n - i$. Geometrically, this is in a sense the statement that the number of holes with i -dimensional boundary enclosed by X is equal to the number of holes with $(n - i)$ -dimensional boundary enclosed by X .

Algebraically, spaces with Poincaré duality emulate Gorenstein algebras; see [5].

For the definition of n -Calabi-Yau categories, see Definitions B.9 and B.11.

Theorem 6.4. *The following conditions are equivalent.*

- (1) $D^c(C^*(X))$ is an n -Calabi-Yau category.
- (2) $D^c(C^*(X)^o)$ is an n -Calabi-Yau category.
- (3) X has Poincaré duality over k .

Proof. This will involve showing that the conditions of the theorem are also equivalent to the following two conditions.

- (4) $D^c(C^*(X))$ has AR triangles.
- (5) $D^c(C^*(X)^o)$ has AR triangles.

For the proof, $C^*(X)$ can be replaced with the commutative DG algebra A by Remark 6.2. So it is clear that (1) \Leftrightarrow (2) and that (4) \Leftrightarrow (5).

Condition (3), that X has Poincaré duality, means ${}_{HA}(DHA) \cong {}_{HA}(\Sigma^n HA)$; since A is commutative, Theorem 2.7(2) implies that this is equivalent to A being Gorenstein. Condition (4) is also equivalent to A being Gorenstein by Theorem 3.4. It follows that (3) \Leftrightarrow (4).

(1) \Rightarrow (4) holds since a Calabi-Yau category has a Serre functor and hence AR triangles, see Definition B.9, Theorem B.10, and Definition B.11.

(3) \Rightarrow (1). The DG algebra A is commutative, so Theorem 2.7(3) implies that condition (3) is equivalent to

$$DA \cong \Sigma^n A$$

in the derived category of DG bi- A -modules. Inserting this into Equation (3.2) shows that the Serre functor of $D^c(A)$ is Σ^n so (1) holds, cf. Definition B.11. \square

Theorem 6.5. *Suppose that X has Poincaré duality over k and that it satisfies $\dim_k H^*(X) \geq 2$. Then each component of the AR quiver Γ of $D^c(C^*(X))$ is isomorphic to $\mathbb{Z}A_\infty$.*

If $\dim_k H^(X) = 2$, then Γ has $n - 1$ components.*

If $\dim_k H^(X) \geq 3$, then Γ has infinitely many components.*

If $\dim_k H^e(X) \geq 2$ for some e , then Γ has families of distinct components which are indexed by projective manifolds over k , and these manifolds can be of arbitrarily high dimension.

Proof. Since $C^*(X)$ is quasi-isomorphic to A , the theory of the previous sections applies to $C^*(X)$. As in the proof of Theorem 6.4, since X has Poincaré duality, $C^*(X)$ is Gorenstein. The present theorem hence follows from Theorems 4.10, 5.1, 5.4, and 5.5. \square

Theorem 6.4 and its proof imply that if X has Poincaré duality over k , then the AR quiver of $D^c(C^*(X))$ is a stable translation quiver.

Theorem 6.6. *The AR quiver of $D^c(C^*(X))$ is a weak homotopy invariant of X .*

If X is restricted to spaces with Poincaré duality over k , then the AR quiver of $D^c(C^(X))$, viewed as a stable translation quiver, is a weak homotopy invariant of X .*

Proof. If X and X' have the same weak homotopy type, then by [6, thm. 4.15] there exists a series of quasi-isomorphisms of DG algebras linking $C^*(X)$ and $C^*(X')$. Hence $D^c(C^*(X))$ and $D^c(C^*(X'))$ are equivalent triangulated categories, and this implies both parts of the theorem. \square

7. Open problems

Let me close the paper by proposing the following open problems. The first one is due to Karsten Schmidt, see [20, sec. 6].

Problem 7.1. Develop a theory of representation type of simply connected cochain DG algebras.

What is known so far is the following.

- (1) By Theorem 5.1, if $\dim_k HR = 2$, then the AR quiver Γ of $D^c(R)$ has a finite number of components.

Suppose that R is Gorenstein.

- (2) By Theorem 5.4, if $\dim_k HR \geq 3$, then Γ has infinitely many components.
- (3) By Theorem 5.5, if $\dim_k H^e R \geq 2$ for some e , then Γ has families of distinct components which are indexed by projective manifolds, and these manifolds can be of arbitrarily high dimension.

It is tempting to interpret the DG algebras of (1) as having finite representation type, and the ones of (3) as having wild representation type.

If $\dim_k HR \geq 3$ but $\dim_k H^i R \leq 1$ for each i , then it is not clear whether the infinitely many components of Γ form discrete or continuous families, or indeed, what these words precisely mean in the context.

Note that some previous work does exist on the representation type of derived categories, see [8], but it does not apply to the categories considered in this paper.

Problem 7.2. What is the structure of the AR quiver of $D^c(R)$ if R is not Gorenstein?

Do components of a different shape than $\mathbb{Z}A_\infty$ become possible?

Problem 7.3. Generalize the theory to cochain DG algebras which are not simply connected.

Presently, not even the structure of $D^c(C^*(S^1; \mathbb{Q}))$ is known because S^1 and hence $C^*(S^1; \mathbb{Q})$ is not simply connected.

A generalization to the non-simply connected case may impact on non-commutative geometry for which more general cochain DG algebras are being considered as vehicles.

Problem 7.4. Is there a link between the categories $D^c(R)$ which have AR quivers consisting of $\mathbb{Z}A_\infty$ -components, and the appearance of $\mathbb{Z}A_\infty$ -components in representation theory?

See for instance [3, thm. 4.17.4].

Problem 7.5. If a simply connected topological space X has $\dim_{\mathbb{Q}} H^*(X; \mathbb{Q}) = 2$, then it has the same rational homotopy type as a sphere of dimension ≥ 2 . Theorem 6.5 implies that these are the only simply connected spaces with Poincaré duality for which the AR quiver of $D^c(C^*(X; \mathbb{Q}))$ has only finitely many components.

Is this linked to any topological property which is special to these spaces?

Problem 7.6. Let X and T be topological spaces. Suppose that X is simply connected with $\dim_k H^*(X; k) < \infty$, that T has $\dim_k H^i(T; k) < \infty$ for each i , and let

$$F \rightarrow T \rightarrow X$$

be a fibration. The induced morphism $C^*(X; k) \rightarrow C^*(T; k)$ turns $C^*(T; k)$ into a DG left- $C^*(X; k)$ -module. By [6, thm. 7.5] there is a quasi-isomorphism $k \otimes_{C^*(X; k)}^L C^*(T; k) \simeq C^*(F; k)$, and this implies that if $\dim_k H^*(F; k) < \infty$ then $C^*(T; k)$ is an object of $D^c(C^*(X; k))$.

Hence $C^*(T; k)$ corresponds to a collection of vertices with multiplicities of the AR quiver Γ of $D^c(C^*(X; k))$. If X has Poincaré duality over k , then the theory of this paper gives information about the structure of Γ , both locally and globally.

Does this have applications to the topological theory of fibrations?

Do the structural results on Γ correspond to structural results on topological fibrations?

Problem 7.7. By considering the fibration $F \rightarrow T \rightarrow X$, looking at $C^*(T; k)$ as a DG left- $C^*(X; k)$ -module, and using the theory of this paper, one is in effect doing “AR theory with topological spaces”.

Is there a way to do so directly with the spaces themselves?

Problem 7.8. If X is a topological space with $\dim_k H^*(X; k) < \infty$ and Poincaré duality over the field k of characteristic 0, then $D^c(C^*(X; k))$ is an n -Calabi-Yau category for some n by Theorem 6.4. More generally, if R is the DG algebra from setup 2.1 and R is commutative and Gorenstein, then $D^c(R)$ is a d -Calabi-Yau category.

These categories appear to behave quite differently from higher cluster categories which are standard examples of Calabi-Yau categories. For instance, an m -cluster category contains an m -cluster tilting object in terms of which every other object can be built in a single step; this seems to be far from true for $D^c(C^*(X; k))$ and $D^c(R)$.

Which role do $D^c(C^*(X; k))$ and $D^c(R)$ play in the taxonomy of Calabi-Yau categories?

In the context of Calabi-Yau categories, there is a “Morita” theorem for higher cluster categories, see [15, thm. 4.2]. Is there also a Morita theorem for the categories $D^c(R)$?

A. Differential Graded homological algebra

This appendix is an introduction to Differential Graded (DG) homological algebra, written for a reader who is already familiar with the formalism of derived categories of rings. Some useful references are [2], [5, appendix], [6, chps. 3, 6, 18, 19, 20], [13], and [14].

Let k be a commutative ring.

Definition A.1 (DG algebras and modules). A Differential Graded (DG) algebra R over k is a complex of k -modules equipped with a product which

- turns R into a \mathbb{Z} -graded k -algebra, and
- satisfies the Leibniz rule $\partial^R(rs) = \partial^R(r)s + (-1)^i r\partial^R(s)$ when r is in R^i .

A DG left- R -module M is a complex of k -modules equipped with an R -scalar multiplication which

- turns it into a graded module over the underlying graded algebra of R , and
- satisfies the Leibniz rule $\partial^M(rm) = \partial^R(r)m + (-1)^i r\partial^M(m)$ when r is in R^i .

DG right- R -modules and DG bi-modules are defined analogously. Note that R itself is an important DG bi- R -module. Sometimes the notations ${}_R M$ and N_R are used to emphasize that M is a DG left- R -module, N a DG right- R -module.

The opposite DG algebra of R is denoted by R° . Its product \cdot is given by $r \cdot s = (-1)^{ij} sr$ in terms of the product of R , when r and s are elements of R^i and R^j . DG right- R -modules can be viewed as DG left- R° -modules.

Remark A.2 (DG homological algebra). It is possible to do homological algebra with DG modules. A test case is when the DG algebra R is concentrated in degree zero, that is, when $R^i = 0$ for $i \neq 0$. Then the zeroth component, R^0 , is an ordinary k -algebra, DG left- R -modules are just complexes of left- R^0 -modules, and DG homological algebra over R specializes to ordinary homological algebra over R^0 .

Definition A.3 (inf, sup, and amp). The infimum and supremum of a DG module are

$$\inf M = \inf\{i \mid H^i M \neq 0\}, \quad \sup M = \sup\{i \mid H^i M \neq 0\},$$

and the amplitude is

$$\text{amp } M = \sup M - \inf M.$$

Note that $\inf 0 = \infty$, $\sup 0 = -\infty$, and $\text{amp } 0 = -\infty$.

Definition A.4 (Morphisms, suspensions, and mapping cones). The notation $(-)^{\natural}$ is used for the operation of forgetting the differential. It sends DG algebras and DG modules to graded algebras and graded modules.

A morphism $\rho : R \rightarrow S$ of DG algebras is a homomorphism $R^{\natural} \rightarrow S^{\natural}$ of the underlying graded algebras which respects the differentials, $\rho\partial^R = \partial^S\rho$.

A morphism $\mu : M \rightarrow N$ of DG R -modules is a homomorphism $M^{\natural} \rightarrow N^{\natural}$ of the underlying graded R^{\natural} -modules which respects the differentials, $\mu \partial^M = \partial^N \mu$.

The morphism μ is called null homotopic if there exists a homomorphism $\theta : M^{\natural} \rightarrow N^{\natural}$ of degree -1 of graded R^{\natural} -modules such that $\mu = \partial^N \theta + \theta \partial^M$. Morphisms μ and μ' are called homotopic if $\mu - \mu'$ is null homotopic.

Suspension of complexes is denoted by Σ . Suspensions and mapping cone constructions of DG left- R -modules inherit DG left- R -module structures. Some sign issues are involved here as well as in other parts of the theory; I will not go into details but refer the reader to the references given.

Definition A.5 (Cohomology). The product on R and the scalar multiplication of R on M induces a product on the cohomology HR and a scalar multiplication of HR on HM , whereby HR becomes a graded k -algebra and HM becomes a graded HR -module.

A morphism μ of DG modules is called a quasi-isomorphism if the induced homomorphism $H\mu$ of graded HR -modules is an isomorphism.

Definition A.6 (Homotopy and derived categories). The homotopy category $K(R)$ has as objects the DG left- R -modules, and as morphisms the homotopy classes of morphisms of DG modules.

The derived category $D(R)$ is obtained from $K(R)$ by formally inverting the quasi-isomorphisms. Both $K(R)$ and $D(R)$ are triangulated categories with distinguished triangles induced by the mapping cone construction.

The categories $K(R)$ and $D(R)$ have set indexed coproducts which are given by ordinary direct sums.

The categories $K(R^{\circ})$ and $D(R^{\circ})$ can be viewed as being the homotopy and derived categories of DG right- R -modules.

A quasi-isomorphism $R \rightarrow S$ of DG algebras induces an equivalence of triangulated categories $D(S) \rightarrow D(R)$ given by change of scalars.

Denote by $D^f(R)$ the full subcategory of $D(R)$ consisting of DG modules M with HM finitely presented over k .

Denote by $D^c(R)$ the full subcategory of $D(R)$ consisting of DG modules which are finitely built in $D(R)$ from R using distinguished triangles, (de)suspensions, coproducts, and direct summands; these are the so-called compact objects of $D(R)$.

Definition A.7 (Hom and Tensor). If M and N are DG left- R -modules, then there is a graded k -module $\text{Hom}_{R^{\natural}}(M^{\natural}, N^{\natural})$ of graded R^{\natural} -homomorphisms $M^{\natural} \rightarrow N^{\natural}$ of different degrees. This can be turned into a complex $\text{Hom}_R(M, N)$ with the differential induced by the differentials of M and N . Note that $\text{Hom}_R(M, N)^{\natural} = \text{Hom}_{R^{\natural}}(M^{\natural}, N^{\natural})$.

If A is a DG right- R -module and B is a DG left- R -module, then the tensor product $A^{\natural} \otimes_{R^{\natural}} B^{\natural}$ is a graded k -module. It can be turned into a complex $A \otimes_R B$ with the differential induced by the differentials of A and B . Note that $(A \otimes_R B)^{\natural} = A^{\natural} \otimes_{R^{\natural}} B^{\natural}$.

These constructions induce functors between homotopy categories, and there are induced derived functors

$$\text{RHom}_R(-, -) : D(R) \times D(R) \rightarrow D(k)$$

and

$$- \overset{\mathbb{L}}{\otimes}_R - : \mathcal{D}(R^\circ) \times \mathcal{D}(R) \rightarrow \mathcal{D}(k).$$

These are often computed using resolutions. For instance, let M be a DG left- R -module and let $P \rightarrow M$ be a \mathbb{K} -projective resolution of M . This is a quasi-isomorphism of DG modules for which P is \mathbb{K} -projective, that is, $\mathrm{Hom}_R(P, -)$ preserves quasi-isomorphisms. Then $\mathrm{Hom}_R(P, -)$ is a well defined functor $\mathcal{D}(R) \rightarrow \mathcal{D}(k)$, and there is an equivalence of functors $\mathrm{RHom}_R(M, -) \simeq \mathrm{Hom}_R(P, -)$.

The functor RHom_R has the useful property

$$\mathrm{H}^0 \mathrm{RHom}_R(M, N) \cong \mathrm{Hom}_{\mathcal{D}(R)}(M, N);$$

more generally, the notation

$$\mathrm{H}(\mathrm{RHom}_R(M, N)) = \mathrm{Ext}_R(M, N)$$

is used so $\mathrm{H}^i \mathrm{RHom}_R(M, N) = \mathrm{Ext}_R^i(M, N)$.

The functors RHom and $\overset{\mathbb{L}}{\otimes}$ are compatible with DG bi-modules. For instance, if A is a DG bi- R -module then $A \overset{\mathbb{L}}{\otimes}_R B$ inherits a left- R -structure from A , so there is a functor

$$A \overset{\mathbb{L}}{\otimes}_R - : \mathcal{D}(R) \rightarrow \mathcal{D}(R).$$

Setup A.8. *Now consider the special case of this paper: k is a field and R is a DG algebra over k which has the form*

$$\cdots \rightarrow 0 \rightarrow k \rightarrow 0 \rightarrow R^2 \rightarrow R^3 \rightarrow \cdots,$$

that is, $R^{<0} = 0$, $R^0 = k$, and $R^1 = 0$. It will also be assumed that $\dim_k R < \infty$, and d will be defined by

$$d = \sup R.$$

Definition A.9 (Duality). By $\mathcal{D}(-)$ will be denoted the functor $\mathrm{Hom}_k(-, k)$. When applied to graded objects, it is understood to be applied degreewise. It sends DG left- R -modules to DG right- R -modules and vice versa. It is well defined at the level of homotopy and derived categories.

Remark A.10. Over a DG algebra of the present special form, $k \cong R/R^{\geq 1}$ is a DG bi- R -module. Moreover, $\mathcal{D}^f(R)$ is the full subcategory of $\mathcal{D}(R)$ consisting of objects M with $\dim_k \mathrm{HM} < \infty$, and $\mathcal{D}^f(R)$ consists precisely of the objects finitely built from ${}_R k$. This can be shown using the first two parts of the following result on truncations, the proof of which uses only linear algebra over the field k ; see [11, lem. 3.4] and [6, ex. 6, p. 146].

Lemma A.11 (Truncations). (1) *If M is a DG left- R -module with $\inf M$ finite, then there exists a quasi-isomorphism of DG left- R -modules $U \rightarrow M$ with $U^i = 0$ for $i < \inf M$.*

(2) *If N is a DG left- R -module with $\sup N$ finite, then there exists a quasi-isomorphism of DG left- R -modules $N \rightarrow V$ with $V^j = 0$ for $j > \sup N$.*

- (3) *The DG algebra R is quasi-isomorphic to a quotient DG algebra S with $S^{>d} = 0$.*

Definition A.12 (Semi-free DG modules). A DG left- R -module F is called semi-free if it permits a semi-free filtration, that is, a filtration by DG left- R -modules

$$0 = F\langle -1 \rangle \subseteq F\langle 0 \rangle \subseteq F\langle 1 \rangle \subseteq \dots \subseteq F$$

where $F = \bigcup_i F\langle i \rangle$ and where each $F\langle i \rangle / F\langle i-1 \rangle$ is a direct sum of (de)suspensions of ${}_R R$.

If $\partial^F(F) \subseteq R^{\geq 1} \cdot F$, then F is called minimal. If M is in $D(R)$ then a (minimal) semi-free resolution of M is a quasi-isomorphism $F \rightarrow M$ where F is (minimal) semi-free.

The following lemma collects useful facts; for references see [2], [4], [5, appendix], [6, sec. 6], [11, sec. 3], [13, sec. 3], and [21].

Lemma A.13 (Semi-free resolutions). (1) *Each M in $D(R)$ has a semi-free resolution.*

- (2) *A semi-free DG module is K -projective, so if F is a semi-free resolution of M then $\mathrm{RHom}_R(M, -) \simeq \mathrm{Hom}_R(F, -)$ and $-\overset{\mathbb{L}}{\otimes}_R M \simeq - \otimes_R F$.*
- (3) *Each M in $D^f(R)$ has a minimal semi-free resolution F , and for each such resolution there are finite numbers β_i such that*

$$F^{\natural} \cong \bigoplus_{i \leq -\inf M} \Sigma^i (R^{\natural})^{(\beta_i)},$$

where $(R^{\natural})^{(\beta_i)}$ is a direct sum of β_i copies of R^{\natural} .

- (4) *Let M in $D^f(R)$ have minimal semi-free resolution F . Then M is in $D^c(R)$ if and only the numbers β_i from part (3) satisfy $\beta_i = 0$ for $i \ll 0$.*
- (5) *If F is minimal semi-free then $\mathrm{Hom}_R(F, k)$ has zero differential, so*

$$\mathrm{H}(\mathrm{Hom}_R(F, k)) \cong \mathrm{Hom}_R(F, k)^{\natural} = \mathrm{Hom}_{R^{\natural}}(F^{\natural}, k^{\natural})$$

as graded k -vector spaces.

B. Auslander-Reiten theory for triangulated categories

This appendix is a brief introduction to the version of Auslander-Reiten (AR) theory used in the rest of the paper. Some useful references are [1], [3], [9], [16], [17], and [18], with [9] being the source of the theory.

Let \mathbf{T} be a triangulated category. The following definition is taken from [17, def. 2.1]; it generalizes the earlier definition from [9, 3.1].

Definition B.1 (AR triangles). An AR triangle in \mathbb{T} is a distinguished triangle

$$M \xrightarrow{\mu} N \xrightarrow{\nu} P \xrightarrow{\pi} \quad (\text{B.1})$$

for which

- Each morphism $M \rightarrow N'$ which is not a split monomorphism factors through μ .
- Each morphism $N' \rightarrow P$ which is not a split epimorphism factors through ν .
- $\pi \neq 0$.

In an AR triangle, the end terms determine each other up to isomorphism by [9, prop. 3.5(i)], so the following definition makes sense.

Definition B.2 (AR translation). Let P be an object of \mathbb{T} and suppose that there is an AR triangle $M \rightarrow N \rightarrow P \rightarrow$. Then M is denoted by τP , and the operation τ which is defined up to isomorphism is called the AR translation of \mathbb{T} .

In an AR triangle, the end terms have local endomorphism rings by [17, lem. 2.3]; this explains the following terminology.

Definition B.3. The triangulated category \mathbb{T} is said to have right AR triangles if, for each object P with local endomorphism ring, there is an AR triangle (B.1).

The category \mathbb{T} is said to have left AR triangles if, for each object M with local endomorphism ring, there is an AR triangle (B.1).

The category \mathbb{T} is said to have AR triangles if it has right and left AR triangles.

Definition B.4 (The AR quiver). A morphism in \mathbb{T} is called irreducible if it is not an isomorphism, but has the property that when it is factored as $\rho\sigma$, then either ρ is a split epimorphism or σ is a split monomorphism.

The AR quiver $\Gamma(\mathbb{T})$ of \mathbb{T} has one vertex $[M]$ for each isomorphism class of objects with local endomorphism rings, and one arrow $[M] \rightarrow [N]$ when there is an irreducible morphism $M \rightarrow N$.

If \mathbb{T} has right AR triangles, then the AR translation τ induces a map from the set of vertices of $\Gamma(\mathbb{T})$ to itself. By abuse of notation, this map is also referred to as the AR translation and denoted by τ .

Setup B.5. *Now consider the special case of this paper: k is a field and \mathbb{T} is k -linear and has finite dimensional Hom spaces and split idempotents; cf. Proposition 2.4.*

Then \mathbb{T} is a Krull-Schmidt category by [19, p. 52]; that is, each indecomposable object has local endomorphism ring and each object splits into a finite direct sum of indecomposable objects which are unique up to isomorphism.

The following lemma holds by [9, prop. 3.5].

Lemma B.6. *Let $M \rightarrow N \rightarrow P \rightarrow$ be an AR triangle and let $N \cong \coprod_i N_i$ where each N_i is indecomposable. Then the following statements are equivalent for an indecomposable object N' .*

- (1) *There is an irreducible morphism $M \rightarrow N'$.*
- (2) *There is an irreducible morphism $N' \rightarrow P$.*
- (3) *There is an i such that $N' \cong N_i$.*

Hence if \mathbb{T} has AR triangles, knowledge of these triangles implies knowledge of the AR quiver $\Gamma(\mathbb{T})$.

Definition B.7 (Stable translation quivers). A stable translation quiver is a quiver equipped with an injective map τ from the set of vertices to itself such that the number of arrows from $\tau(t)$ to s is equal to the number of arrows from s to t .

The following proposition follows easily from Lemma B.6.

Proposition B.8. *If \mathbb{T} has AR triangles, then the AR translation τ turns the AR quiver $\Gamma(\mathbb{T})$ into a stable translation quiver.*

Definition B.9 (Serre functors). A Serre functor for \mathbb{T} is an autoequivalence S for which there are natural isomorphisms

$$D(\mathrm{Hom}_{\mathbb{T}}(M, N)) \cong \mathrm{Hom}_{\mathbb{T}}(N, SM).$$

The following was proved in [18, thm. I.2.4].

Theorem B.10. *The category \mathbb{T} has AR triangles if and only if it has a Serre functor S . If it does, then $\tau = \Sigma^{-1}S$ on indecomposable objects.*

This implies that if \mathbb{T} has AR triangles, then the AR translation τ can be extended to the autoequivalence $\Sigma^{-1}S$.

Definition B.11 (Calabi-Yau categories). The category \mathbb{T} is called n -Calabi-Yau if n is the smallest non-negative integer for which Σ^n is a Serre functor.

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