A new class of k-rank graph algebras, coming from “k-cube groups”

Sam A. Mutter
with Aura-Cristiana Radu and Alina Vdovina

22nd February 2021

Overview

This talk is based on:

S.A. Mutter, A.-C. Radu, and A. Vdovina

$C^\star$-algebras of higher-rank graphs from groups acting on buildings, and explicit computation of their K-theory

`https://arxiv.org/abs/2012.05561`

Overview

This talk is based on:

S.A. Mutter, A.-C. Radu, and A. Vdovina

$C^\star$-algebras of higher-rank graphs from groups acting on buildings, and explicit computation of their K-theory

https://arxiv.org/abs/2012.05561

It will look like this:

Overview

This talk is based on:

S.A. Mutter, A.-C. Radu, and A. Vdovina

$C^\star$-algebras of higher-rank graphs from groups acting on buildings, and explicit computation of their K-theory

`https://arxiv.org/abs/2012.05561`

It will look like this:

1. Higher-rank graphs

Overview

This talk is based on:

S.A. Mutter, A.-C. Radu, and A. Vdovina

$C^\star$-algebras of higher-rank graphs from groups acting on buildings, and explicit computation of their K-theory

https://arxiv.org/abs/2012.05561

It will look like this:

1. Higher-rank graphs
2. k-cube groups

Overview

This talk is based on:

S.A. Mutter, A.-C. Radu, and A. Vdovina

$C^\star$-algebras of higher-rank graphs from groups acting on buildings, and explicit computation of their K-theory

https://arxiv.org/abs/2012.05561

It will look like this:

1. Higher-rank graphs
2. k-cube groups
3. Why k-cube groups are higher-rank graphs

Overview

This talk is based on:

S.A. Mutter, A.-C. Radu, and A. Vdovina

$C^\star$-algebras of higher-rank graphs from groups acting on buildings, and explicit computation of their K-theory

https://arxiv.org/abs/2012.05561

It will look like this:

1. Higher-rank graphs
2. k-cube groups
3. Why k-cube groups are higher-rank graphs
4. Higher-rank graph $C^\star$-algebras

Overview

This talk is based on:

S.A. Mutter, A.-C. Radu, and A. Vdovina

$C^\star$-algebras of higher-rank graphs from groups acting on buildings, and explicit computation of their K-theory

https://arxiv.org/abs/2012.05561

It will look like this:

1. Higher-rank graphs
2. k-cube groups
3. Why k-cube groups are higher-rank graphs
4. Higher-rank graph $C^\star$-algebras
5. K-theory computations (non-scary version)

Preface

Why do we want more $C^\star$-algebras?

- k-rank graphs are primarily studied for their associated graph algebras, though they themselves are defined categorically.

Preface

Why do we want more $C^\star$-algebras?

- k-rank graphs are primarily studied for their associated graph algebras, though they themselves are defined categorically.
- Our approach is geometrical—certain groups acting on buildings and products of trees have the structure of a k-rank graph.

Preface

Why do we want more $C^\star$-algebras?

- k-rank graphs are primarily studied for their associated graph algebras, though they themselves are defined categorically.
- Our approach is geometrical—certain groups acting on buildings and products of trees have the structure of a k-rank graph.
- The K-theory of these algebras is a great way to distinguish them from each other, showing that our constructions do indeed produce new $C^\star$-algebras.

Preface

Why do we want more $C^\star$-algebras?

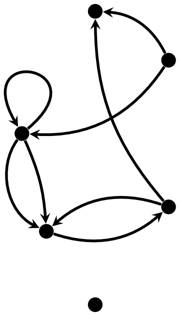
- k-rank graphs are primarily studied for their associated graph algebras, though they themselves are defined categorically.
- Our approach is geometrical—certain groups acting on buildings and products of trees have the structure of a k-rank graph.
- The K-theory of these algebras is a great way to distinguish them from each other, showing that our constructions do indeed produce new $C^\star$-algebras.
- Thus we have new bridges between operator algebras, geometric group theory, and category theory.

Higher-rank graphs

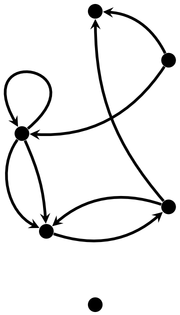
Higher-rank graphs

- We might describe a directed graph G by its vertex set Λ^0 and edge set Λ^1.

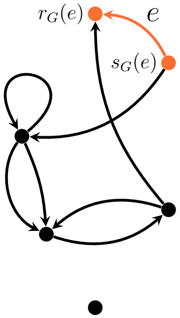
- We might describe a directed graph G by its vertex set Λ^0 and edge set Λ^1.



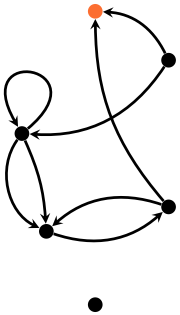
- We might describe a directed graph G by its vertex set Λ^0 and edge set Λ^1.
- An edge $e \in \Lambda^1$ is an arrow connecting two vertices. We can denote this by an ordered pair of vertices $(s_G(e), r_G(e))$, telling us the domain s_G and range r_G of e.



- We might describe a directed graph G by its vertex set Λ^0 and edge set Λ^1.
- An edge $e \in \Lambda^1$ is an arrow connecting two vertices. We can denote this by an ordered pair of vertices $(s_G(e), r_G(e))$, telling us the domain s_G and range r_G of e.

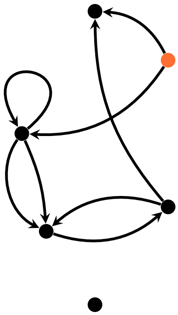


- We might describe a directed graph G by its vertex set Λ^0 and edge set Λ^1.
- An edge $e \in \Lambda^1$ is an arrow connecting two vertices. We can denote this by an ordered pair of vertices $(s_G(e), r_G(e))$, telling us the domain s_G and range r_G of e.
- A sink is a vertex which is the domain of no edges.

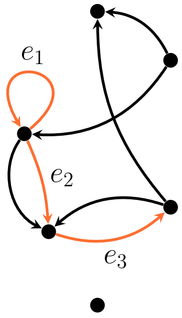


Higher-rank graphs

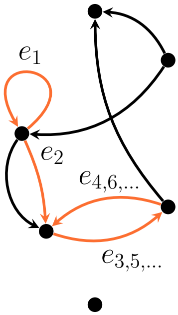
- We might describe a directed graph G by its vertex set Λ^0 and edge set Λ^1.
- An edge $e \in \Lambda^1$ is an arrow connecting two vertices. We can denote this by an ordered pair of vertices $(s_G(e), r_G(e))$, telling us the domain s_G and range r_G of e.
- A sink is a vertex which is the domain of no edges.
- A source is a vertex which is the range of no edges.



- We might describe a directed graph G by its vertex set Λ^0 and edge set Λ^1.
- An edge $e \in \Lambda^1$ is an arrow connecting two vertices. We can denote this by an ordered pair of vertices $(s_G(e), r_G(e))$, telling us the domain s_G and range r_G of e.
- A sink is a vertex which is the domain of no edges.
- A source is a vertex which is the range of no edges.
- A path in a graph is a finite sequence of edges (e_n) such that $r_G(e_i) = s_G(e_{i+1})$.

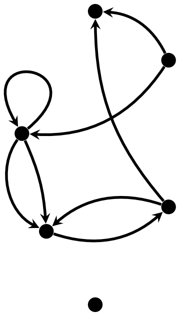


- We might describe a directed graph G by its vertex set Λ^0 and edge set Λ^1.
- An edge $e \in \Lambda^1$ is an arrow connecting two vertices. We can denote this by an ordered pair of vertices $(s_G(e), r_G(e))$, telling us the domain s_G and range r_G of e.
- A sink is a vertex which is the domain of no edges.
- A source is a vertex which is the range of no edges.
- A path in a graph is a finite sequence of edges (e_n) such that $r_G(e_i) = s_G(e_{i+1})$.



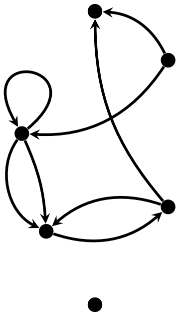
Higher-rank graphs

- We write Λ to denote the set of all paths in G.

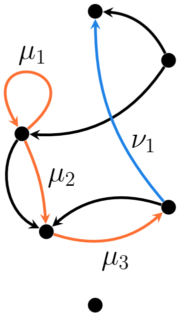


Higher-rank graphs

- We write Λ to denote the set of all paths in G.
- Two paths $\mu = (\mu_i)_{i=1}^m$ and $\nu = (\nu_i)_{i=1}^n$ in Λ are concatenatable iff $r_G(\mu_m) = s_G(\nu_1)$.

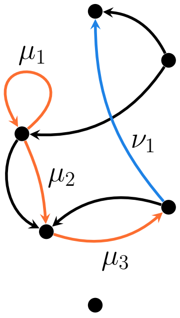


- We write Λ to denote the set of all paths in G.
- Two paths $\mu = (\mu_i)_{i=1}^m$ and $\nu = (\nu_i)_{i=1}^n$ in Λ are concatenatable iff $r_G(\mu_m) = s_G(\nu_1)$.



$\mu\nu \in \Lambda$

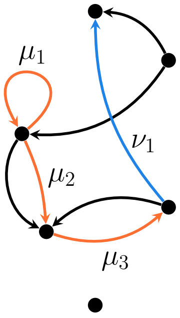
- We write Λ to denote the set of all paths in G.
- Two paths $\mu = (\mu_i)_{i=1}^{m}$ and $\nu = (\nu_i)_{i=1}^{n}$ in Λ are concatenatable iff $r_G(\mu_m) = s_G(\nu_1)$.
- There is a function $d : \Lambda \to \mathbb{N}$ which returns the length of a path.



$$\mu\nu \in \Lambda$$

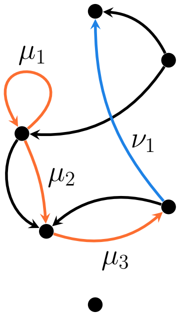
Higher-rank graphs

- We write Λ to denote the set of all paths in G.
- Two paths $\mu = (\mu_i)_{i=1}^m$ and $\nu = (\nu_i)_{i=1}^n$ in Λ are concatenatable iff $r_G(\mu_m) = s_G(\nu_1)$.
- There is a function $d : \Lambda \to \mathbb{N}$ which returns the length of a path.
- Clearly $d(\mu\nu) = d(\mu) + d(\nu)$.



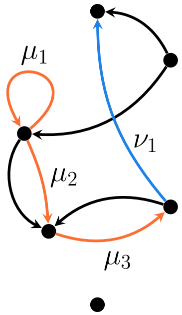
$\mu\nu \in \Lambda$

- We write Λ to denote the set of all paths in G.
- Two paths $\mu = (\mu_i)_{i=1}^{m}$ and $\nu = (\nu_i)_{i=1}^{n}$ in Λ are concatenatable iff $r_G(\mu_m) = s_G(\nu_1)$.
- There is a function $d : \Lambda \to \mathbb{N}$ which returns the length of a path.
- Clearly $d(\mu\nu) = d(\mu) + d(\nu)$.
- We write Λ^n for the set of paths λ with $d(\lambda) = n$.



$\mu\nu \in \Lambda$

- We write Λ to denote the set of all paths in G.
- Two paths $\mu = (\mu_i)_{i=1}^m$ and $\nu = (\nu_i)_{i=1}^n$ in Λ are concatenatable iff $r_G(\mu_m) = s_G(\nu_1)$.
- There is a function $d : \Lambda \to \mathbb{N}$ which returns the length of a path.
- Clearly $d(\mu\nu) = d(\mu) + d(\nu)$.
- We write Λ^n for the set of paths λ with $d(\lambda) = n$.
- Λ and d encode all the necessary information to replicate G (in fact we only need Λ^0 and Λ^1).



$\mu\nu \in \Lambda$

Definition (1-rank graph)

Definition (1-rank graph)

Let Λ be a countable category.

Definition (1-rank graph)

Let Λ be a countable category. We are interested in the morphisms of this category, so we identify $\mathrm{Ob}(\Lambda)$ *with the identity morphisms in* $\mathrm{Hom}(\Lambda)$.

Definition (1-rank graph)

Let Λ *be a countable category. We are interested in the morphisms of this category, so we identify* $\mathrm{Ob}(\Lambda)$ *with the identity morphisms in* $\mathrm{Hom}(\Lambda)$.

For a morphism $\lambda \in \mathrm{Hom}_\Lambda(A, B)$*, define domain and range maps* $s(\lambda) := A$*, and* $r(\lambda) := B$.

DEFINITION (1-RANK GRAPH)

Let Λ be a countable category. We are interested in the morphisms of this category, so we identify $\mathrm{Ob}(\Lambda)$ with the identity morphisms in $\mathrm{Hom}(\Lambda)$.

For a morphism $\lambda \in \mathrm{Hom}_\Lambda(A, B)$, define domain and range maps $s(\lambda) := A$, and $r(\lambda) := B$.

Now, two morphisms ν, μ are composable iff $r(\nu) = s(\mu)$; then $\mu\nu \in \Lambda$.

Definition (1-rank graph)

Let Λ be a countable category. We are interested in the morphisms of this category, so we identify $\mathrm{Ob}(\Lambda)$ with the identity morphisms in $\mathrm{Hom}(\Lambda)$.

For a morphism $\lambda \in \mathrm{Hom}_\Lambda(A, B)$, define domain and range maps $s(\lambda) := A$, and $r(\lambda) := B$.

Now, two morphisms ν, μ are composable iff $r(\nu) = s(\mu)$; then $\mu\nu \in \Lambda$.

Let $d : \Lambda \to \mathbb{N}$ be a functor with the following factorisation property:

Higher-rank graphs

Definition (1-rank graph)

Let Λ be a countable category. We are interested in the morphisms of this category, so we identify $\mathrm{Ob}(\Lambda)$ with the identity morphisms in $\mathrm{Hom}(\Lambda)$.

For a morphism $\lambda \in \mathrm{Hom}_\Lambda(A, B)$, define domain and range maps $s(\lambda) := A$, and $r(\lambda) := B$.

Now, two morphisms ν, μ are composable iff $r(\nu) = s(\mu)$; then $\mu\nu \in \Lambda$.

Let $d : \Lambda \to \mathbb{N}$ be a functor with the following factorisation property:

For every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}$ with $d(\lambda) = m + n$, there are ***unique*** *elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = m$ and $d(\nu) = n$.*

Definition (1-rank graph)

Let Λ be a countable category. We are interested in the morphisms of this category, so we identify $\mathrm{Ob}(\Lambda)$ with the identity morphisms in $\mathrm{Hom}(\Lambda)$.

For a morphism $\lambda \in \mathrm{Hom}_\Lambda(A, B)$, define domain and range maps $s(\lambda) := A$, and $r(\lambda) := B$.

Now, two morphisms ν, μ are composable iff $r(\nu) = s(\mu)$; then $\mu\nu \in \Lambda$.

Let $d : \Lambda \to \mathbb{N}$ be a functor with the following factorisation property:

*For every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}$ with $d(\lambda) = m + n$, there are **unique** elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = m$ and $d(\nu) = n$.*

We call the pair (Λ, d) a 1-rank graph.

(1-rank graph)

- Λ *countable category.*
- $\nu, \mu \in \Lambda$ *composable iff* $r(\nu) = s(\mu)$.
- $d : \Lambda \to \mathbb{N}$ *s.t.* $d(\lambda) = m + n \Rightarrow d(\mu) = m$, $d(\nu) = n$, *and* $\lambda = \mu\nu$ *for some unique* $\mu, \nu \in \Lambda$.

Higher-rank graphs

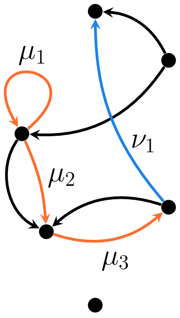
(1-rank graph)

- *Λ countable category.*
- *$\nu, \mu \in \Lambda$ composable iff $r(\nu) = s(\mu)$.*
- *$d : \Lambda \to \mathbb{N}$ s.t. $d(\lambda) = m + n \Rightarrow d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$ for some unique $\mu, \nu \in \Lambda$.*

- How do we build a graph G from (Λ, d)?

(1-RANK GRAPH)

- Λ *countable category.*
- $\nu, \mu \in \Lambda$ *composable iff* $r(\nu) = s(\mu)$.
- $d : \Lambda \to \mathbb{N}$ *s.t.* $d(\lambda) = m + n \Rightarrow d(\mu) = m$, $d(\nu) = n$, *and* $\lambda = \mu\nu$ *for some unique* $\mu, \nu \in \Lambda$.

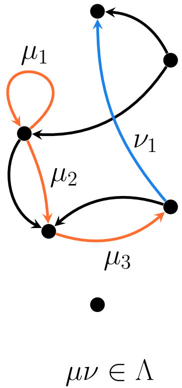


$\mu\nu \in \Lambda$

- How do we build a graph G from (Λ, d)?
- Write $\Lambda^n := \{\lambda \in \Lambda \mid d(\lambda) = n\}$.

(1-rank graph)

- *Λ countable category.*
- *$\nu, \mu \in \Lambda$ composable iff $r(\nu) = s(\mu)$.*
- *$d : \Lambda \to \mathbb{N}$ s.t. $d(\lambda) = m + n \Rightarrow d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$ for some unique $\mu, \nu \in \Lambda$.*

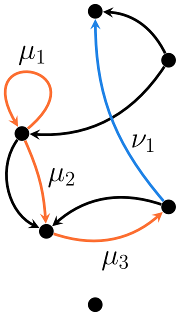


- How do we build a graph G from (Λ, d)?
- Write $\Lambda^n := \{\lambda \in \Lambda \mid d(\lambda) = n\}$.
- The vertex and edge sets of G are Λ^0, Λ^1 resp.

Higher-rank graphs

(1-rank graph)

- *Λ countable category.*
- *$\nu, \mu \in \Lambda$ composable iff $r(\nu) = s(\mu)$.*
- *$d : \Lambda \to \mathbb{N}$ s.t. $d(\lambda) = m + n \Rightarrow d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$ for some unique $\mu, \nu \in \Lambda$.*

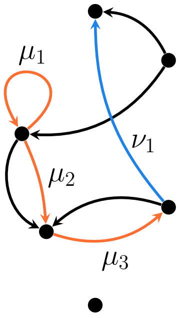


$\mu\nu \in \Lambda$

- How do we build a graph G from (Λ, d)?
- Write $\Lambda^n := \{\lambda \in \Lambda \mid d(\lambda) = n\}$.
- The vertex and edge sets of G are Λ^0, Λ^1 resp.
- $r_G(\lambda) = s(\lambda)$ and $s_G(\lambda) = r(\lambda)$.

(1-rank graph)

- *Λ countable category.*
- *$\nu, \mu \in \Lambda$ composable iff $r(\nu) = s(\mu)$.*
- *$d : \Lambda \to \mathbb{N}$ s.t. $d(\lambda) = m + n \Rightarrow d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$ for some unique $\mu, \nu \in \Lambda$.*



$\mu\nu \in \Lambda$

- How do we build a graph G from (Λ, d)?
- Write $\Lambda^n := \{\lambda \in \Lambda \mid d(\lambda) = n\}$.
- The vertex and edge sets of G are Λ^0, Λ^1 resp.
- $r_G(\lambda) = s(\lambda)$ and $s_G(\lambda) = r(\lambda)$.
- The converse construction also works.

(k-rank graph)

- Λ *countable category, and* $k \geq 1$.
- $\nu, \mu \in \Lambda$ *composable iff* $r(\nu) = s(\mu)$.

(k-RANK GRAPH)

- Λ *countable category, and* $k \geq 1$.
- $\nu, \mu \in \Lambda$ *composable iff* $r(\nu) = s(\mu)$.
- $d : \Lambda \to \mathbb{N}^k$ *s.t.*

(k-rank graph)

- Λ *countable category, and* $k \geq 1$.
- $\nu, \mu \in \Lambda$ *composable iff* $r(\nu) = s(\mu)$.
- $d : \Lambda \to \mathbb{N}^k$ *s.t. if* $d(\lambda) = \mathbf{m} + \mathbf{n}$ *for some* $\mathbf{m}, \mathbf{n} \in \mathbb{N}^k$*, then we can find* ***unique*** *elements* $\mu, \nu \in \Lambda$ *such that* $d(\mu) = \mathbf{m}$*,* $d(\nu) = \mathbf{n}$*, and* $\lambda = \mu\nu$.

(k-rank graph)

- Λ *countable category, and* $k \geq 1$.
- $\nu, \mu \in \Lambda$ *composable iff* $r(\nu) = s(\mu)$.
- $d : \Lambda \to \mathbb{N}^k$ *s.t. if* $d(\lambda) = \mathbf{m} + \mathbf{n}$ *for some* $\mathbf{m}, \mathbf{n} \in \mathbb{N}^k$*, then we can find* ***unique*** *elements* $\mu, \nu \in \Lambda$ *such that* $d(\mu) = \mathbf{m}$, $d(\nu) = \mathbf{n}$*, and* $\lambda = \mu\nu$.

- We call d the degree map.

(k-rank graph)

- *Λ countable category, and $k \geq 1$.*
- *$\nu, \mu \in \Lambda$ composable iff $r(\nu) = s(\mu)$.*
- *$d : \Lambda \to \mathbb{N}^k$ s.t. if $d(\lambda) = \mathbf{m} + \mathbf{n}$ for some $\mathbf{m}, \mathbf{n} \in \mathbb{N}^k$, then we can find **unique** elements $\mu, \nu \in \Lambda$ such that $d(\mu) = \mathbf{m}$, $d(\nu) = \mathbf{n}$, and $\lambda = \mu\nu$.*

- We call d the degree map.
- Write $\Lambda^{\mathbf{n}} := \{\lambda \in \Lambda \mid d(\lambda) = \mathbf{n}\}$. We call elements of $\Lambda^{\mathbf{0}}$ vertices.

(k-rank graph)

- *Λ countable category, and $k \geq 1$.*
- *$\nu, \mu \in \Lambda$ composable iff $r(\nu) = s(\mu)$.*
- *$d : \Lambda \to \mathbb{N}^k$ s.t. if $d(\lambda) = \mathbf{m} + \mathbf{n}$ for some $\mathbf{m}, \mathbf{n} \in \mathbb{N}^k$, then we can find **unique** elements $\mu, \nu \in \Lambda$ such that $d(\mu) = \mathbf{m}$, $d(\nu) = \mathbf{n}$, and $\lambda = \mu\nu$.*

- We call d the degree map.
- Write $\Lambda^{\mathbf{n}} := \{\lambda \in \Lambda \mid d(\lambda) = \mathbf{n}\}$. We call elements of $\Lambda^{\mathbf{0}}$ vertices.
- Define the set $\Lambda^{\mathbf{n}}(v) := \{\lambda \in \Lambda^{\mathbf{n}} \mid r(\lambda) = v\}$.

(k-rank graph)

- *Λ countable category, and $k \geq 1$.*
- *$\nu, \mu \in \Lambda$ composable iff $r(\nu) = s(\mu)$.*
- *$d : \Lambda \to \mathbb{N}^k$ s.t. if $d(\lambda) = \mathbf{m} + \mathbf{n}$ for some $\mathbf{m}, \mathbf{n} \in \mathbb{N}^k$, then we can find **unique** elements $\mu, \nu \in \Lambda$ such that $d(\mu) = \mathbf{m}$, $d(\nu) = \mathbf{n}$, and $\lambda = \mu\nu$.*

- We call d the degree map.
- Write $\Lambda^{\mathbf{n}} := \{\lambda \in \Lambda \mid d(\lambda) = \mathbf{n}\}$. We call elements of $\Lambda^{\mathbf{0}}$ vertices.
- Define the set $\Lambda^{\mathbf{n}}(v) := \{\lambda \in \Lambda^{\mathbf{n}} \mid r(\lambda) = v\}$.
- We say Λ is row-finite if $\Lambda^{\mathbf{n}}(v)$ is finite, and that Λ has no sources if $\Lambda^{\mathbf{n}}(v) \neq \emptyset$, for all $v \in \Lambda^{\mathbf{0}}$ and $\mathbf{n} \in \mathbb{N}^k$.

- We can draw any 1-graph as a directed graph.

Higher-rank graphs

Examples

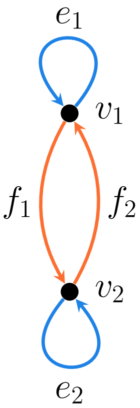
- We can draw any 1-graph as a directed graph.
- When $k \geq 2$, we represent a k-graph as a collection of k coloured graphs which share the same vertex set.

Higher-rank graphs

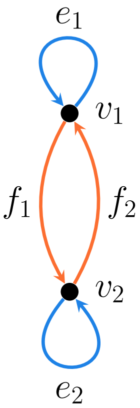
Examples

- We can draw any 1-graph as a directed graph.
- When $k \geq 2$, we represent a k-graph as a collection of k coloured graphs which share the same vertex set.
- In general a k-coloured graph will not have the required factorisation property, so we shoehorn it in.

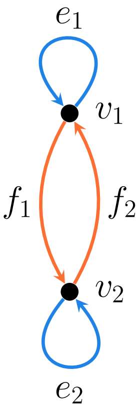
- We can draw any 1-graph as a directed graph.
- When $k \geq 2$, we represent a k-graph as a collection of k coloured graphs which share the same vertex set.
- In general a k-coloured graph will not have the required factorisation property, so we shoehorn it in.
- Consider this example where $k = 2$.



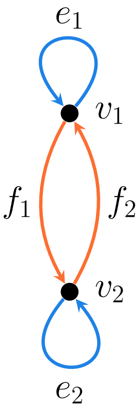
- We can draw any 1-graph as a directed graph.
- When $k \geq 2$, we represent a k-graph as a collection of k coloured graphs which share the same vertex set.
- In general a k-coloured graph will not have the required factorisation property, so we shoehorn it in.
- Consider this example where $k = 2$. Give blue edges degree $(1, 0)$, and orange edges degree $(0, 1)$.



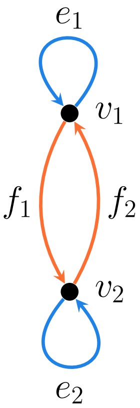
- We can draw any 1-graph as a directed graph.
- When $k \geq 2$, we represent a k-graph as a collection of k coloured graphs which share the same vertex set.
- In general a k-coloured graph will not have the required factorisation property, so we shoehorn it in.
- Consider this example where $k = 2$. Give blue edges degree $(1, 0)$, and orange edges degree $(0, 1)$.
- $d(e_1) = (1, 0)$, $d(f_1) = (0, 1)$, and $d(f_1 e_1) = (1, 1)$.



- We can draw any 1-graph as a directed graph.
- When $k \geq 2$, we represent a k-graph as a collection of k coloured graphs which share the same vertex set.
- In general a k-coloured graph will not have the required factorisation property, so we shoehorn it in.
- Consider this example where $k = 2$. Give blue edges degree $(1, 0)$, and orange edges degree $(0, 1)$.
- $d(e_1) = (1, 0)$, $d(f_1) = (0, 1)$, and $d(f_1e_1) = (1, 1)$. But $d(e_2f_1) = (1, 1)$, and f_1e_1 and e_2f_1 both go from v_1 to v_2.



- We can draw any 1-graph as a directed graph.
- When $k \geq 2$, we represent a k-graph as a collection of k coloured graphs which share the same vertex set.
- In general a k-coloured graph will not have the required factorisation property, so we shoehorn it in.
- Consider this example where $k = 2$. Give blue edges degree $(1, 0)$, and orange edges degree $(0, 1)$.
- $d(e_1) = (1, 0)$, $d(f_1) = (0, 1)$, and $d(f_1 e_1) = (1, 1)$. But $d(e_2 f_1) = (1, 1)$, and $f_1 e_1$ and $e_2 f_1$ both go from v_1 to v_2.
- So, in the 2-graph, we must define $f_1 e_1 = e_2 f_1$. Likewise, we identify $f_2 e_2 = e_1 f_2$.

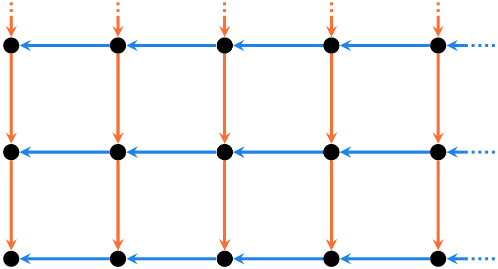


- Let Ω_k be the countable category defined by object set $\mathrm{Ob}(\Omega_k) = \Omega_k^{\mathbf{0}} := \mathbb{N}^k$, and morphism set $\mathrm{Hom}(\Omega_k) := \{(\mathbf{m}, \mathbf{n}) \in \mathbb{N}^k \times \mathbb{N}^k \mid \mathbf{m} \leq \mathbf{n} \text{ (entrywise)}\}$.

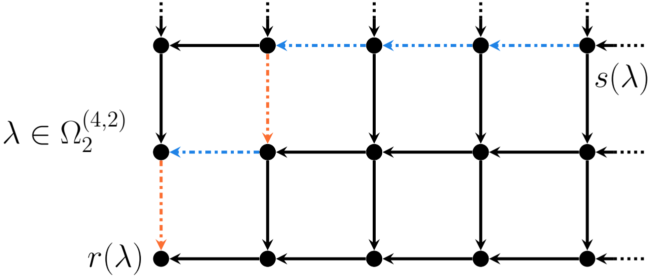
- Let Ω_k be the countable category defined by object set $\mathrm{Ob}(\Omega_k) = \Omega_k^{\mathbf{0}} := \mathbb{N}^k$, and morphism set $\mathrm{Hom}(\Omega_k) := \{(\mathbf{m}, \mathbf{n}) \in \mathbb{N}^k \times \mathbb{N}^k \mid \mathbf{m} \leq \mathbf{n} \text{ (entrywise)}\}$.
- Define range and domain maps $r(\mathbf{m}, \mathbf{n}) := \mathbf{m}$, $s(\mathbf{m}, \mathbf{n}) := \mathbf{n}$, degree map $d(\mathbf{m}, \mathbf{n}) := \mathbf{n} - \mathbf{m}$, and composition $(\mathbf{l}, \mathbf{m})(\mathbf{m}, \mathbf{n}) := (\mathbf{l}, \mathbf{n})$.

- Let Ω_k be the countable category defined by object set $\mathrm{Ob}(\Omega_k) = \Omega_k^{\mathbf{0}} := \mathbb{N}^k$, and morphism set $\mathrm{Hom}(\Omega_k) := \{(\mathbf{m}, \mathbf{n}) \in \mathbb{N}^k \times \mathbb{N}^k \mid \mathbf{m} \leq \mathbf{n} \text{ (entrywise)}\}$.
- Define range and domain maps $r(\mathbf{m}, \mathbf{n}) := \mathbf{m}$, $s(\mathbf{m}, \mathbf{n}) := \mathbf{n}$, degree map $d(\mathbf{m}, \mathbf{n}) := \mathbf{n} - \mathbf{m}$, and composition $(\mathbf{l}, \mathbf{m})(\mathbf{m}, \mathbf{n}) := (\mathbf{l}, \mathbf{n})$.
- This is a k-rank graph, which we can draw as a non-negative integer lattice.

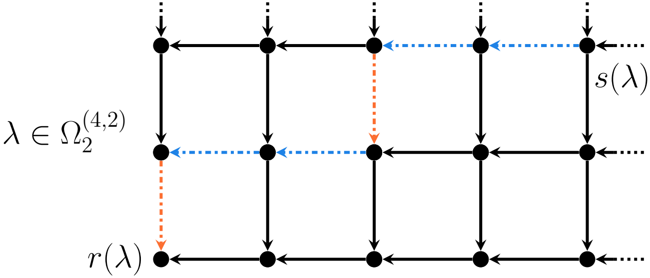
- Let Ω_k be the countable category defined by object set $\mathrm{Ob}(\Omega_k) = \Omega_k^{\mathbf{0}} := \mathbb{N}^k$, and morphism set $\mathrm{Hom}(\Omega_k) := \{(\mathbf{m}, \mathbf{n}) \in \mathbb{N}^k \times \mathbb{N}^k \mid \mathbf{m} \leq \mathbf{n} \text{ (entrywise)}\}$.
- Define range and domain maps $r(\mathbf{m}, \mathbf{n}) := \mathbf{m}$, $s(\mathbf{m}, \mathbf{n}) := \mathbf{n}$, degree map $d(\mathbf{m}, \mathbf{n}) := \mathbf{n} - \mathbf{m}$, and composition $(\mathbf{l}, \mathbf{m})(\mathbf{m}, \mathbf{n}) := (\mathbf{l}, \mathbf{n})$.
- This is a k-rank graph, which we can draw as a non-negative integer lattice.



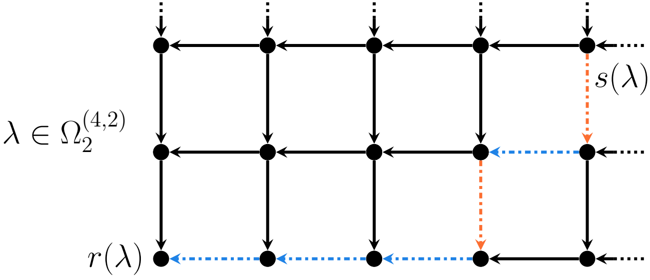
- Let Ω_k be the countable category defined by object set $\mathrm{Ob}(\Omega_k) = \Omega_k^{\mathbf{0}} := \mathbb{N}^k$, and morphism set $\mathrm{Hom}(\Omega_k) := \{(\mathbf{m}, \mathbf{n}) \in \mathbb{N}^k \times \mathbb{N}^k \mid \mathbf{m} \leq \mathbf{n} \text{ (entrywise)}\}$.
- Define range and domain maps $r(\mathbf{m}, \mathbf{n}) := \mathbf{m}$, $s(\mathbf{m}, \mathbf{n}) := \mathbf{n}$, degree map $d(\mathbf{m}, \mathbf{n}) := \mathbf{n} - \mathbf{m}$, and composition $(\mathbf{l}, \mathbf{m})(\mathbf{m}, \mathbf{n}) := (\mathbf{l}, \mathbf{n})$.
- This is a k-rank graph, which we can draw as a non-negative integer lattice.



- Let Ω_k be the countable category defined by object set $\mathrm{Ob}(\Omega_k) = \Omega_k^{\mathbf{0}} := \mathbb{N}^k$, and morphism set $\mathrm{Hom}(\Omega_k) := \{(\mathbf{m}, \mathbf{n}) \in \mathbb{N}^k \times \mathbb{N}^k \mid \mathbf{m} \leq \mathbf{n} \text{ (entrywise)}\}$.
- Define range and domain maps $r(\mathbf{m}, \mathbf{n}) := \mathbf{m}$, $s(\mathbf{m}, \mathbf{n}) := \mathbf{n}$, degree map $d(\mathbf{m}, \mathbf{n}) := \mathbf{n} - \mathbf{m}$, and composition $(\mathbf{l}, \mathbf{m})(\mathbf{m}, \mathbf{n}) := (\mathbf{l}, \mathbf{n})$.
- This is a k-rank graph, which we can draw as a non-negative integer lattice.

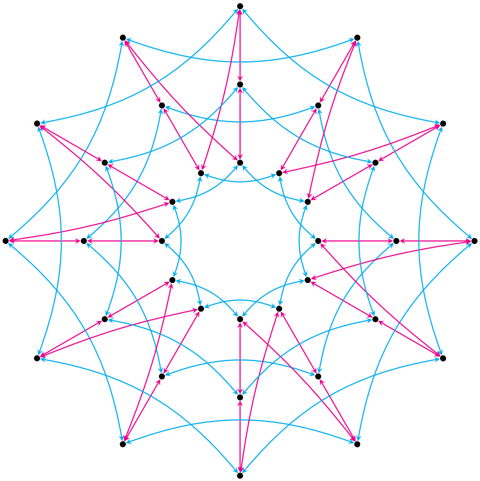


- Let Ω_k be the countable category defined by object set $\mathrm{Ob}(\Omega_k) = \Omega_k^{\mathbf{0}} := \mathbb{N}^k$, and morphism set $\mathrm{Hom}(\Omega_k) := \{(\mathbf{m}, \mathbf{n}) \in \mathbb{N}^k \times \mathbb{N}^k \mid \mathbf{m} \leq \mathbf{n} \text{ (entrywise)}\}$.
- Define range and domain maps $r(\mathbf{m}, \mathbf{n}) := \mathbf{m}$, $s(\mathbf{m}, \mathbf{n}) := \mathbf{n}$, degree map $d(\mathbf{m}, \mathbf{n}) := \mathbf{n} - \mathbf{m}$, and composition $(\mathbf{l}, \mathbf{m})(\mathbf{m}, \mathbf{n}) := (\mathbf{l}, \mathbf{n})$.
- This is a k-rank graph, which we can draw as a non-negative integer lattice.



Higher-rank graphs

Examples

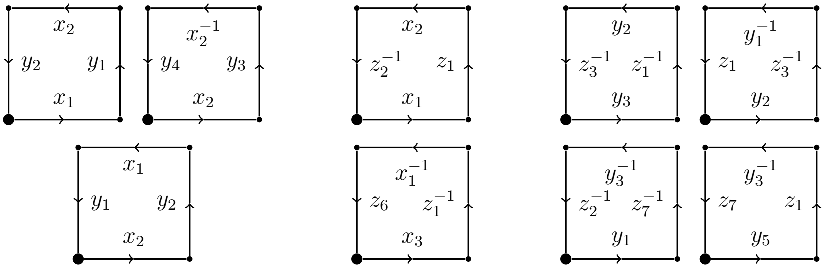


k-CUBE GROUPS

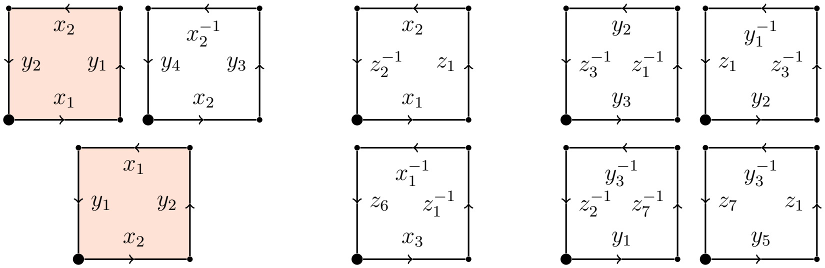
- Let $E_1, \ldots, E_k$ be alphabets, sets of even size ≥ 4, each equipped with a fixed-point-free involution $e \mapsto e^{-1}$. E.g. $E_1 = \{x_1, x_2, x_1^{-1}, x_2^{-1}\}$.

- Let $E_1, \ldots, E_k$ be alphabets, sets of even size ≥ 4, each equipped with a fixed-point-free involution $e \mapsto e^{-1}$. E.g. $E_1 = \{x_1, x_2, x_1^{-1}, x_2^{-1}\}$.
- We are going to construct $\binom{k}{2}$ sets of squares, with each pair of parallel sides labelled from one alphabet.

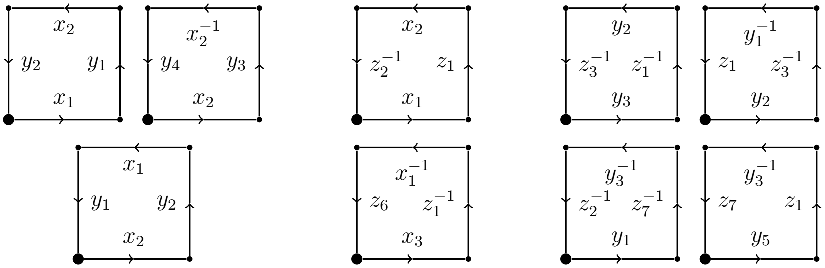
- Let $E_1, \ldots, E_k$ be alphabets, sets of even size ≥ 4, each equipped with a fixed-point-free involution $e \mapsto e^{-1}$. E.g. $E_1 = \{x_1, x_2, x_1^{-1}, x_2^{-1}\}$.
- We are going to construct $\binom{k}{2}$ sets of squares, with each pair of parallel sides labelled from one alphabet.
- The squares will be pointed and oriented, and the sides will be directed arrows, with e^{-1} being the reverse arrow to e.



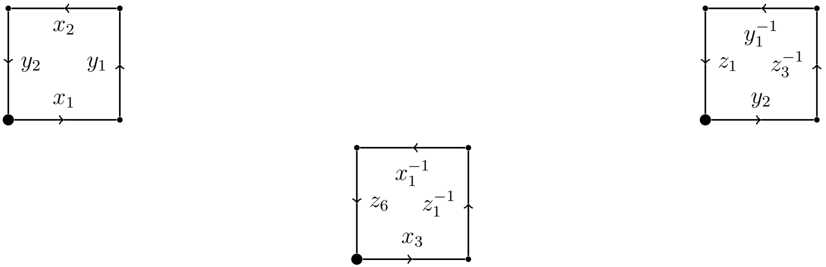
- Let $E_1, \ldots, E_k$ be alphabets, sets of even size ≥ 4, each equipped with a fixed-point-free involution $e \mapsto e^{-1}$. E.g. $E_1 = \{x_1, x_2, x_1^{-1}, x_2^{-1}\}$.
- We are going to construct $\binom{k}{2}$ sets of squares, with each pair of parallel sides labelled from one alphabet.
- The squares will be pointed and oriented, and the sides will be directed arrows, with e^{-1} being the reverse arrow to e.



- Let $E_1, \ldots, E_k$ be alphabets, sets of even size ≥ 4, each equipped with a fixed-point-free involution $e \mapsto e^{-1}$. E.g. $E_1 = \{x_1, x_2, x_1^{-1}, x_2^{-1}\}$.
- We are going to construct $\binom{k}{2}$ sets of squares, with each pair of parallel sides labelled from one alphabet.
- The squares will be pointed and oriented, and the sides will be directed arrows, with e^{-1} being the reverse arrow to e.

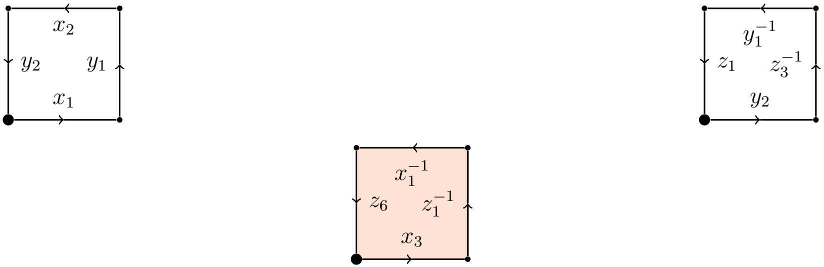


- We want to glue these squares together to make k-dimensional cubes.



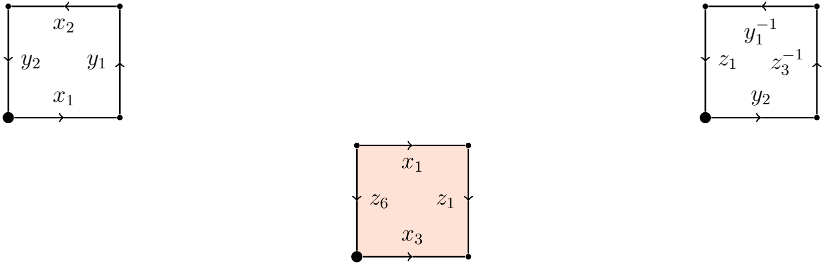
- We want to glue these squares together to make k-dimensional cubes.

k-cube groups

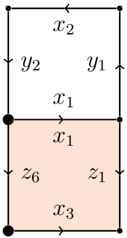


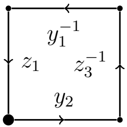
- We want to glue these squares together to make k-dimensional cubes.

k-CUBE GROUPS

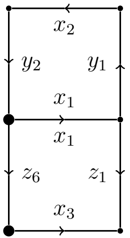


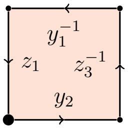
- We want to glue these squares together to make k-dimensional cubes.



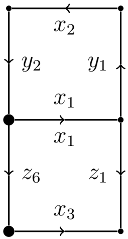


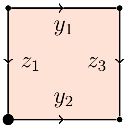
- We want to glue these squares together to make k-dimensional cubes.



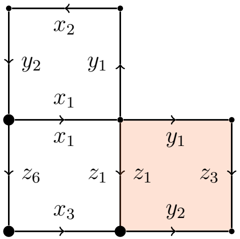


- We want to glue these squares together to make k-dimensional cubes.

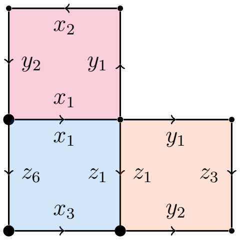


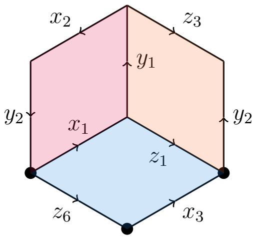


- We want to glue these squares together to make k-dimensional cubes.

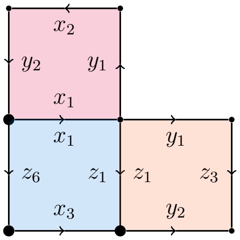


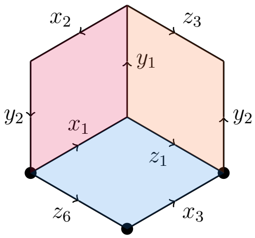
- We want to glue these squares together to make k-dimensional cubes.



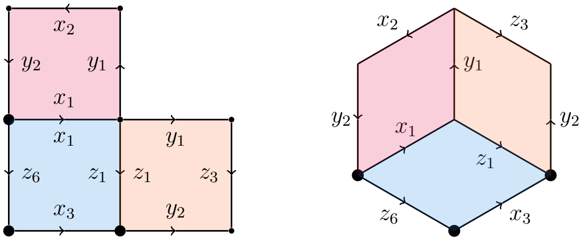


- We want to glue these squares together to make k-dimensional cubes.

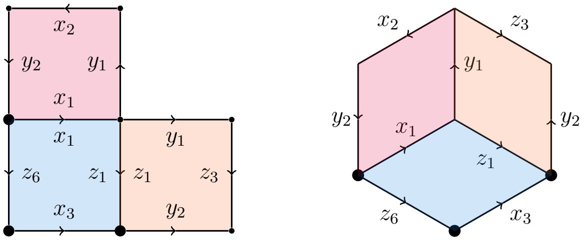




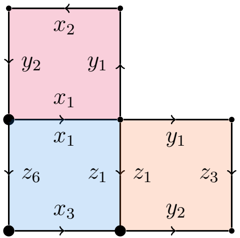
- We want to glue these squares together to make k-dimensional cubes.
- We design our squares in such a way as to make this possible.

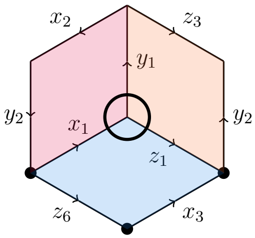


- We want to glue these squares together to make k-dimensional cubes.
- We design our squares in such a way as to make this possible.
- We also want the property that each possible "corner" of a cube occurs exactly once.

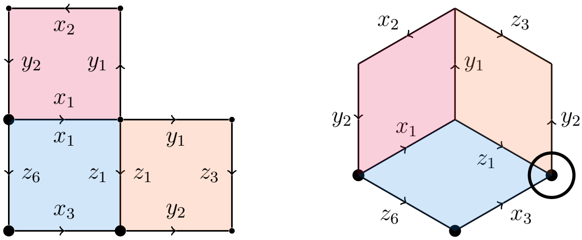


- We want to glue these squares together to make k-dimensional cubes.
- We design our squares in such a way as to make this possible.
- We also want the property that each possible "corner" of a cube occurs exactly once.
- Here, for example, we have





- We want to glue these squares together to make k-dimensional cubes.
- We design our squares in such a way as to make this possible.
- We also want the property that each possible “corner” of a cube occurs exactly once.
- Here, for example, we have $\{x_1, y_1^{-1}, z_1^{-1}\}$,



- We want to glue these squares together to make k-dimensional cubes.
- We design our squares in such a way as to make this possible.
- We also want the property that each possible "corner" of a cube occurs exactly once.
- Here, for example, we have $\{x_1, y_1^{-1}, z_1^{-1}\}$, $\{x_3, y_2^{-1}, z_1\}$, and so on.

DEFINITION (k-CUBE GROUP)

Definition (k-cube group)

Let $E_1, \ldots, E_k$ be a collection of alphabets, and let R be the set of pointed squares labelled by (x, y, x', y') with $x, x' \in E_i$ and $y, y' \in E_j$ for $i < j$.

DEFINITION (k-CUBE GROUP)

Let $E_1, \ldots, E_k$ be a collection of alphabets, and let R be the set of pointed squares labelled by (x, y, x', y') with $x, x' \in E_i$ and $y, y' \in E_j$ for $i < j$.

Suppose that R has the following properties:

Definition (k-cube group)

Let $E_1, \ldots, E_k$ be a collection of alphabets, and let R be the set of pointed squares labelled by (x, y, x', y') with $x, x' \in E_i$ and $y, y' \in E_j$ for $i < j$.

Suppose that R has the following properties:

1. *If $(x, y, x', y') \in R$, then $(x^{-1}, (y')^{-1}, (x')^{-1}, y^{-1})$, (x', y', x, y), and $((x')^{-1}, y^{-1}, x^{-1}, (y')^{-1})$ are all distinct, and in R.*

Definition (k-cube group)

Let $E_1, \ldots, E_k$ be a collection of alphabets, and let R be the set of pointed squares labelled by (x, y, x', y') with $x, x' \in E_i$ and $y, y' \in E_j$ for $i < j$.

Suppose that R has the following properties:

1. *If $(x,y,x',y') \in R$, then $(x^{-1}, (y')^{-1}, (x')^{-1}, y^{-1})$, (x', y', x, y), and $((x')^{-1}, y^{-1}, x^{-1}, (y')^{-1})$ are all distinct, and in R.*
2. *The squares in R can be glued together to make pointed k-cubes.*

DEFINITION (k-CUBE GROUP)

Let $E_1, \ldots, E_k$ be a collection of alphabets, and let R be the set of pointed squares labelled by (x, y, x', y') with $x, x' \in E_i$ and $y, y' \in E_j$ for $i < j$.

Suppose that R has the following properties:

1. *If $(x, y, x', y') \in R$, then $(x^{-1}, (y')^{-1}, (x')^{-1}, y^{-1})$, (x', y', x, y), and $((x')^{-1}, y^{-1}, x^{-1}, (y')^{-1})$ are all distinct, and in R.*
2. *The squares in R can be glued together to make pointed k-cubes.*
3. *Each combination $\{e_1, \ldots, e_k \mid e_i \in E_i\}$ occurs at the corner of precisely one k-cube, up to symmetry.*

DEFINITION (k-CUBE GROUP)

Let $E_1, \ldots, E_k$ be a collection of alphabets, and let R be the set of pointed squares labelled by (x, y, x', y') with $x, x' \in E_i$ and $y, y' \in E_j$ for $i < j$.

Suppose that R has the following properties:

1. *If $(x, y, x', y') \in R$, then $(x^{-1}, (y')^{-1}, (x')^{-1}, y^{-1})$, (x', y', x, y), and $((x')^{-1}, y^{-1}, x^{-1}, (y')^{-1})$ are all distinct, and in R.*
2. *The squares in R can be glued together to make pointed k-cubes.*
3. *Each combination $\{e_1, \ldots, e_k \mid e_i \in E_i\}$ occurs at the corner of precisely one k-cube, up to symmetry.*

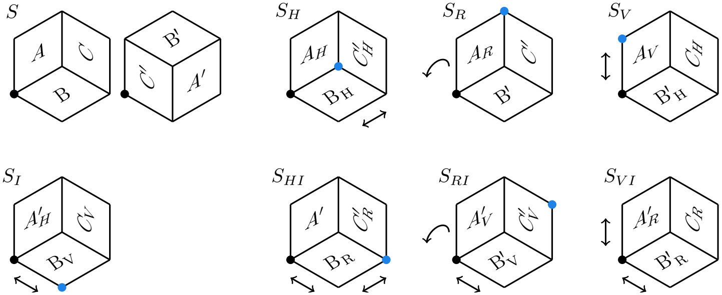
*We call the group $\Gamma := \langle E_1 \sqcup \cdots \sqcup E_k \mid xyx'y' = 1 \text{ whenever } (x, y, x', y') \in R \rangle$ a **k-cube group**.*

- What does it mean "up to symmetry"?

k-CUBE GROUPS

- What does it mean "up to symmetry"?
- Each geometric k-cube produces 2^k pointed k-cubes.

- What does it mean "up to symmetry"?
- Each geometric k-cube produces 2^k pointed k-cubes.



- The symmetries map squares labelled from alphabets E_i, E_j to other squares labelled from E_i, E_j.

Example (Rungtanapirom, Stix, Vdovina, 2019)

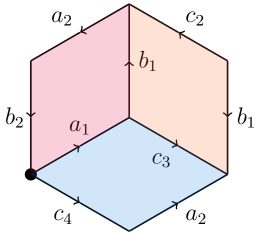
$\Gamma'_{\{3,5,7\}} := \langle a_1, a_2, b_1, b_2, b_3, c_1, c_2, c_3, c_4 \mid R\rangle$, *where*

$$R := \{a_1b_1a_2b_2, a_1b_2a_2b_1^{-1}, a_1b_3a_2^{-1}b_1, a_1b_3^{-1}a_1b_2^{-1}, a_1b_1^{-1}a_2^{-1}b_3, a_2b_3a_2b_2^{-1}, a_1c_1a_2^{-1}c_2^{-1}, a_1c_2a_1^{-1}c_3, a_1c_3a_2^{-1}c_4^{-1}, a_1c_4a_1c_1^{-1}, a_1c_4^{-1}a_2c_2, a_1c_3^{-1}a_2c_1, a_2c_3a_2c_2^{-1}, a_2c_4a_2^{-1}c_1, c_1b_1c_3b_3^{-1}, c_1b_2c_4b_2^{-1}, c_1b_3c_4^{-1}b_2, c_1b_3^{-1}c_4b_3, c_1b_2^{-1}c_2b_1, c_1b_1^{-1}c_4b_1^{-1}, c_2b_2c_3^{-1}b_3^{-1}, c_2b_3c_4b_1, c_2b_3^{-1}c_3b_3, c_2b_2^{-1}c_3b_2, c_2b_1^{-1}c_3b_1^{-1}, c_3b_1c_4b_2\}.$$

EXAMPLE (RUNGTANAPIROM, STIX, VDOVINA, 2019)

$\Gamma'_{\{3,5,7\}} := \langle a_1, a_2, b_1, b_2, b_3, c_1, c_2, c_3, c_4 \mid R \rangle$, *where*

$R := \{a_1b_1a_2b_2, a_1b_2a_2b_1^{-1}, a_1b_3a_2^{-1}b_1, a_1b_3^{-1}a_1b_2^{-1}, a_1b_1^{-1}a_2^{-1}b_3, a_2b_3a_2b_2^{-1}, a_1c_1a_2^{-1}c_2^{-1}, a_1c_2a_1^{-1}c_3, a_1c_3a_2^{-1}c_4^{-1}, a_1c_4a_1c_1^{-1}, a_1c_4^{-1}a_2c_2, a_1c_3^{-1}a_2c_1, a_2c_3a_2c_2^{-1}, a_2c_4a_2^{-1}c_1, c_1b_1c_3b_3^{-1}, c_1b_2c_4b_2^{-1}, c_1b_3c_4^{-1}b_2, c_1b_3^{-1}c_4b_3, c_1b_2^{-1}c_2b_1, c_1b_1^{-1}c_4b_1^{-1}, c_2b_2c_3^{-1}b_3^{-1}, c_2b_3c_4b_1, c_2b_3^{-1}c_3b_3, c_2b_2^{-1}c_3b_2, c_2b_1^{-1}c_3b_1^{-1}, c_3b_1c_4b_2\}$.



EXAMPLE ($k = 4$, FREE GROUP)

Consider the product of four free groups, each with two generators, defined as follows:

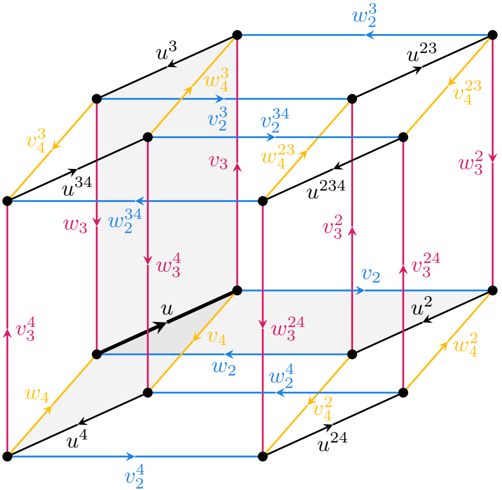
$$\mathbb{F}_2^4 := \langle a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \mid [a_i, b_j], [a_i, c_j], [a_i, d_j], [b_i, c_j], [b_i, d_j], [c_i, d_j] \text{ for all } i, j \in \{1, 2\} \rangle.$$

This is a 4-cube group.

EXAMPLE ($k = 4$, MRV)

$\Gamma_{\{1,2,3,4\}} := \langle a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3 \mid R\rangle$, *where*

$R := \{a_1b_1a_3^{-1}b_1, a_1b_1^{-1}a_2^{-1}b_3, a_1b_2a_2b_2, a_1b_2^{-1}a_3b_3^{-1}, a_1b_3a_2^{-1}b_1^{-1}, a_1b_3^{-1}a_3b_2^{-1},$
$a_2b_3a_3b_3, a_3b_1a_2^{-1}b_2, a_3b_2a_2^{-1}b_1, a_1c_1a_2^{-1}c_1, a_1c_1^{-1}a_1c_3^{-1}, a_1c_2a_1^{-1}c_2^{-1}, a_1c_3a_3c_3,$
$a_2c_1a_2c_2^{-1}, a_2c_2a_3^{-1}c_2, a_2c_3a_2^{-1}c_3^{-1}, a_3c_1^{-1}a_3^{-1}c_1, a_3c_2a_3c_3^{-1}, a_1d_1a_3^{-1}d_3, a_1d_1^{-1}a_2d_2,$
$a_1d_2a_2d_1^{-1}, a_1d_2^{-1}a_1d_3^{-1}, a_1d_3a_3^{-1}d_1, a_2d_1a_2d_3^{-1}, a_2d_2^{-1}a_3d_3, a_2d_3a_3d_2^{-1}, a_3d_1a_3d_2,$
$b_1c_1b_3^{-1}c_1, b_1c_1^{-1}b_2^{-1}c_3, b_1c_2b_2c_2, b_1c_2^{-1}b_3c_3^{-1},$
$b_1c_3b_2^{-1}c_1^{-1}, b_1c_3^{-1}b_3c_2^{-1}, b_2c_3b_3c_3, b_3c_1b_2^{-1}c_2, b_3c_2b_2^{-1}c_1,$
$b_1d_1b_2^{-1}d_1, b_1d_1^{-1}b_1d_3^{-1}, b_1d_2b_1^{-1}d_2^{-1}, b_1d_3b_3d_3,$
$b_2d_1b_2d_2^{-1}, b_2d_2b_3^{-1}d_2, b_2d_3b_2^{-1}d_3^{-1}, b_3d_1^{-1}b_3^{-1}d_1, b_3d_2b_3d_3^{-1},$
$c_1d_1c_3^{-1}d_1, c_1d_1^{-1}c_2^{-1}d_3, c_1d_2c_2d_2, c_1d_2^{-1}c_3d_3^{-1},$
$c_1d_3c_2^{-1}d_1^{-1}, c_1d_3^{-1}c_3d_2^{-1}, c_2d_3c_3d_3, c_3d_1c_2^{-1}d_2, c_3d_2c_2^{-1}d_1\}.$



w_2^3
u^3
u^{23}
w_4^3
v_4^{23}
v_2^3
v_2^{34}
v_4^3
v_3
w_4^{23}
w_3^2
u^{34}
u^{234}
w_3
w_2^{34}
v_3^2
w_3^4
v_2
v_3^{24}
u
u^2
v_3^4
v_4
w_3^{24}
w_4^2
w_2
w_2^4
w_4
v_4^2
u^4
u^{24}
v_2^4

k-CUBE GROUPS

THEOREM

A group Γ is a k-cube group iff it acts freely and transitively on the set of vertices of the product of k trees.

THEOREM

A group Γ is a k-cube group iff it acts freely and transitively on the set of vertices of the product of k trees.

Given a k-cube group $\Gamma = \langle E_1, \dots, E_k \mid \mathcal{R} \rangle$, the subgroup generated by alphabets $E_1, \dots, E_{k-1}$ is a $(k-1)$-cube group.

THEOREM

A group Γ is a k-cube group iff it acts freely and transitively on the set of vertices of the product of k trees.

Given a k-cube group $\Gamma = \langle E_1, \ldots, E_k \mid \mathcal{R} \rangle$, the subgroup generated by alphabets $E_1, \ldots, E_{k-1}$ is a $(k-1)$-cube group.

We can find k different $(k-1)$-cube subgroups in this way. The group Γ is the product of these groups, amalgamated over the free groups generated by their pairwise intersections.

THEOREM

A group Γ is a k-cube group iff it acts freely and transitively on the set of vertices of the product of k trees.

Given a k-cube group $\Gamma = \langle E_1, \ldots, E_k \mid \mathcal{R} \rangle$, the subgroup generated by alphabets $E_1, \ldots, E_{k-1}$ is a $(k-1)$-cube group.

We can find k different $(k-1)$-cube subgroups in this way. The group Γ is the product of these groups, amalgamated over the free groups generated by their pairwise intersections.

- In general the converse is not true—it is difficult to find a family of k-cube groups whose amalgamated product forms a $(k+1)$-cube group.

- Let Δ be the rank k affine building which is the k-dimensional cube complex $T_{|E_1|} \times \cdots \times T_{|E_k|}$.

- Let Δ be the rank k affine building which is the k-dimensional cube complex $T_{|E_1|} \times \cdots \times T_{|E_k|}$. We identify elements of Γ with edges of Δ, so the set of k-cubes $\mathcal{S}_k$ can be identified with the set of pointed, oriented chambers of Δ.

- Let Δ be the rank k affine building which is the k-dimensional cube complex $T_{|E_1|} \times \cdots \times T_{|E_k|}$. We identify elements of Γ with edges of Δ, so the set of k-cubes $\mathcal{S}_k$ can be identified with the set of pointed, oriented chambers of Δ.

DEFINITION (ADJACENCY FUNCTIONS)

- Let Δ be the rank k affine building which is the k-dimensional cube complex $T_{|E_1|} \times \cdots \times T_{|E_k|}$. We identify elements of Γ with edges of Δ, so the set of k-cubes $\mathcal{S}_k$ can be identified with the set of pointed, oriented chambers of Δ.

Definition (Adjacency functions)

Let $A, B \in \mathcal{S}_k$. We define adjacency functions $M_1, \ldots, M_k : \mathcal{S}_k \times \mathcal{S}_k \to \{0,1\}$, where $M_i(A,B) = 1$ iff:

- Let Δ be the rank k affine building which is the k-dimensional cube complex $T_{|E_1|} \times \cdots \times T_{|E_k|}$. We identify elements of Γ with edges of Δ, so the set of k-cubes $\mathcal{S}_k$ can be identified with the set of pointed, oriented chambers of Δ.

DEFINITION (ADJACENCY FUNCTIONS)

Let $A, B \in \mathcal{S}_k$. We define adjacency functions $M_1, \ldots, M_k : \mathcal{S}_k \times \mathcal{S}_k \to \{0, 1\}$, where $M_i(A, B) = 1$ iff:

- *The first $(k-1)$-dimensional face of A labelled by alphabets $E_1, \ldots, \hat{E}_i, \ldots, E_k$ coincide with the second such $(k-1)$-face of B, and*

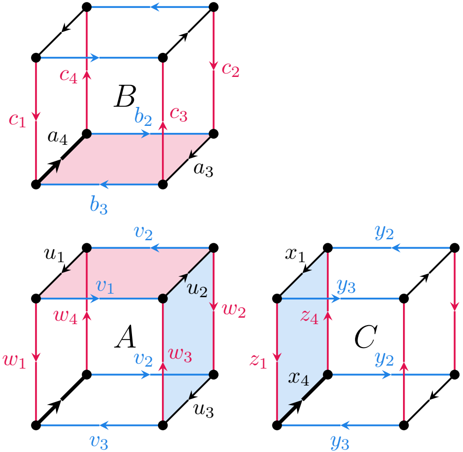
- Let Δ be the rank k affine building which is the k-dimensional cube complex $T_{|E_1|} \times \cdots \times T_{|E_k|}$. We identify elements of Γ with edges of Δ, so the set of k-cubes $\mathcal{S}_k$ can be identified with the set of pointed, oriented chambers of Δ.

Definition (Adjacency functions)

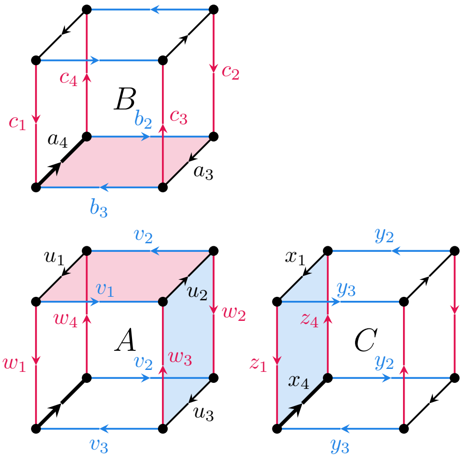
Let $A, B \in \mathcal{S}_k$. We define adjacency functions $M_1, \ldots, M_k : \mathcal{S}_k \times \mathcal{S}_k \to \{0, 1\}$, where $M_i(A, B) = 1$ iff:

- *The first $(k-1)$-dimensional face of A labelled by alphabets $E_1, \ldots, \hat{E}_i, \ldots, E_k$ coincide with the second such $(k-1)$-face of B, and*
- *Whenever $M_i(A, B) = 1$, if we stack A and B together so that their common $(k-1)$-faces overlap, we never have e and e^{-1} pointing to the same vertex.*

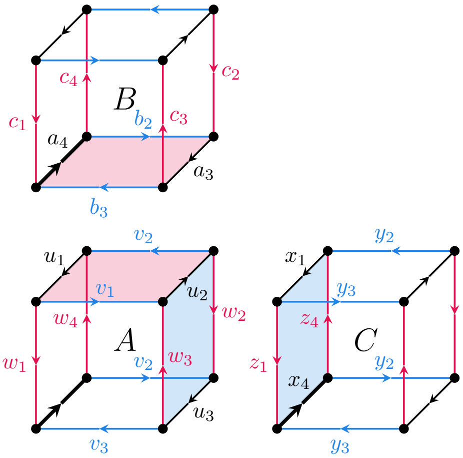
- Consider this example for $k = 3$.



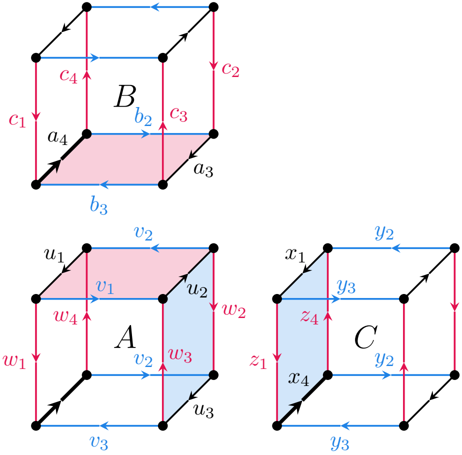
- Consider this example for $k = 3$.
- Suppose all black edges are labelled from E_1, magenta from E_2, and blue from E_3.



- Consider this example for $k = 3$.
- Suppose all black edges are labelled from E_1, magenta from E_2, and blue from E_3.
- $M_2(A, B) = 1$ iff $(u_1, v_1, u_2, v_2) = (a_4^{-1}, b_3^{-1}, a_3^{-1}, b_2^{-1})$ and $c_1 \neq w_1^{-1}$, $c_3 \neq w_3^{-1}$, etc.



- Consider this example for $k = 3$.
- Suppose all black edges are labelled from E_1, magenta from E_2, and blue from E_3.
- $M_2(A,B) = 1$ iff $(u_1, v_1, u_2, v_2) = (a_4^{-1}, b_3^{-1}, a_3^{-1}, b_2^{-1})$ and $c_1 \neq w_1^{-1}$, $c_3 \neq w_3^{-1}$, etc.
- $M_3(A,C) = 1$ iff $(u_2, w_2, u_3, w_3) = (x_1^{-1}, z_4^{-1}, x_4^{-1}, z_1^{-1})$ and $y_3 \neq v_3^{-1}$, etc.



Theorem (MRV)

Theorem (MRV)

Let Γ be a k-cube group, and write the adjacency functions $M_1, \ldots, M_k$ as matrices. Then each M_i has at least 3 non-zero entries in each row, and $M_i M_j = M_j M_i$ for all i, j.

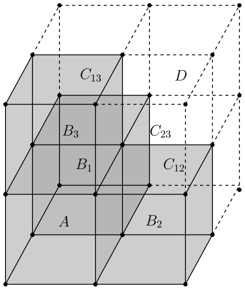
Theorem (MRV)

Let Γ be a k-cube group, and write the adjacency functions $M_1, \ldots, M_k$ as matrices. Then each M_i has at least 3 non-zero entries in each row, and $M_i M_j = M_j M_i$ for all i, j.

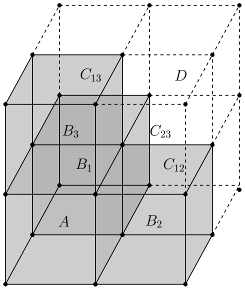
Theorem (MRV)

A k-cube group Γ has the unique common extension property.

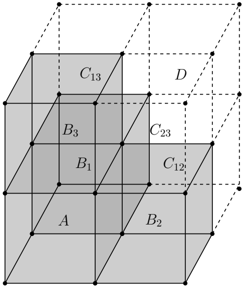
- The UCE property says that in every dimension $2 \leq n \leq k$, if we start with an n-dimensional cube and find n adjacent n-cubes (one in each direction), then we can uniquely extend the arrangement into a 2×2 supercube.

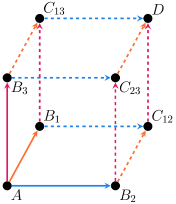


- The UCE property says that in every dimension $2 \leq n \leq k$, if we start with an n-dimensional cube and find n adjacent n-cubes (one in each direction), then we can uniquely extend the arrangement into a 2×2 supercube.
- Assign to each k-cube in Γ a vertex.

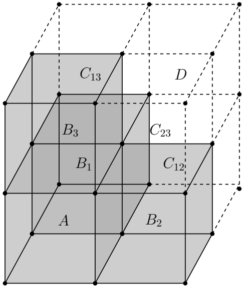


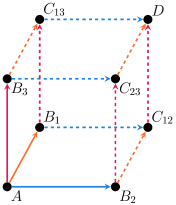
- The UCE property says that in every dimension $2 \leq n \leq k$, if we start with an n-dimensional cube and find n adjacent n-cubes (one in each direction), then we can uniquely extend the arrangement into a 2×2 supercube.
- Assign to each k-cube in Γ a vertex.
- Draw an i-coloured arrow from A to B if B is adjacent to A in the i direction.





- The UCE property says that in every dimension $2 \leq n \leq k$, if we start with an n-dimensional cube and find n adjacent n-cubes (one in each direction), then we can uniquely extend the arrangement into a 2×2 supercube.
- Assign to each k-cube in Γ a vertex.
- Draw an i-coloured arrow from A to B if B is adjacent to A in the i direction.
- The UCE property ensures that this k-coloured graph has all the factorisation properties of a k-rank graph.





Definition (k-rank graph algebra of Kumjian-Pask)

Definition (k-rank graph algebra of Kumjian-Pask)

Let $\Lambda = (\Lambda, d)$ be a row-finite k-rank graph with no sources.

DEFINITION (k-RANK GRAPH ALGEBRA OF KUMJIAN-PASK)

Let $\Lambda = (\Lambda, d)$ be a row-finite k-rank graph with no sources. We define the k-rank graph algebra $\mathcal{A}(\Lambda)$ to be the universal $C^\star$-algebra generated by a family $\{s_\lambda \mid \lambda \in \Lambda\}$ of partial isometries which have the following properties:

Definition (k-rank graph algebra of Kumjian-Pask)

Let $\Lambda = (\Lambda, d)$ be a row-finite k-rank graph with no sources. We define the k-rank graph algebra $\mathcal{A}(\Lambda)$ to be the universal $C^\star$-algebra generated by a family $\{s_\lambda \mid \lambda \in \Lambda\}$ of partial isometries which have the following properties:

1. *The set $\{s_v \mid v \in \Lambda^{\mathbf{0}}\}$ satisfies $(s_v)^2 = s_v = s_v^*$ and $s_u s_v = 0$ for all $u \neq v$.*
2. *If $r(\lambda) = s(\mu)$ for some $\lambda, \mu \in \Lambda$, then $s_{\mu\lambda} = s_\mu s_\lambda$.*
3. *For all $\lambda \in \Lambda$, we have $s_\lambda^* s_\lambda = s_{s(\lambda)}$.*
4. *For all vertices $v \in \Lambda^{\mathbf{0}}$ and $\mathbf{n} \in \mathbb{N}^k$, we have: $s_v = \sum_{\lambda \in \Lambda^{\mathbf{n}}(v)} s_\lambda s_\lambda^*$.*

Definition (k-rank graph algebra of Kumjian-Pask)

Let $\Lambda = (\Lambda, d)$ be a row-finite k-rank graph with no sources. We define the k-rank graph algebra $\mathcal{A}(\Lambda)$ to be the universal $C^\star$-algebra generated by a family $\{s_\lambda \mid \lambda \in \Lambda\}$ of partial isometries which have the following properties:

1. *The set $\{s_v \mid v \in \Lambda^{\mathbf{0}}\}$ satisfies $(s_v)^2 = s_v = s_v^*$ and $s_u s_v = 0$ for all $u \neq v$.*
2. *If $r(\lambda) = s(\mu)$ for some $\lambda, \mu \in \Lambda$, then $s_{\mu\lambda} = s_\mu s_\lambda$.*
3. *For all $\lambda \in \Lambda$, we have $s_\lambda^* s_\lambda = s_{s(\lambda)}$.*
4. *For all vertices $v \in \Lambda^{\mathbf{0}}$ and $\mathbf{n} \in \mathbb{N}^k$, we have: $s_v = \sum_{\lambda \in \Lambda^{\mathbf{n}}(v)} s_\lambda s_\lambda^*$.*

Without the row-finiteness condition, property 4 is not well-defined.

Theorem (Evans, 2008)

Theorem (Evans, 2008)

Let Λ be a row-finite 2-rank graph with no sources. Then the K-theory of $\mathcal{A}(\Lambda)$ is well understood.

Let $k = 3$, and $\partial_1 := [I - M_1^T, I - M_2^T, I - M_3^T]$, $\partial_3 := [I - M_3^T, M_2^T - I, I - M_1^T]^T$,

$$\partial_2 := \begin{bmatrix} M_2^T - I & M_3^T - I & 0 \\ I - M_1^T & 0 & M_3^T - I \\ 0 & I - M_1^T & I - M_2^T \end{bmatrix}.$$

Then $K_1(\mathcal{A}(\Lambda)) \cong \ker(\partial_1)/\operatorname{im}(\partial_2) \oplus G_1$, *and there is a short exact sequence*

$$0 \longrightarrow \operatorname{coker}(\partial_1)/G_0 \longrightarrow K_0(\mathcal{A}(\Lambda)) \longrightarrow \ker(\partial_2)/\operatorname{im}(\partial_3) \longrightarrow 0,$$

where $G_0 \subseteq \operatorname{coker}(\partial_1)$ *and* $G_1 \subseteq \ker(\partial_3)$.

THEOREM (MRV)

There are similar (nastier) short exact sequences and isomorphisms for $k = 4$ *and* $k = 5$.

Theorem (MRV)

There are similar (nastier) short exact sequences and isomorphisms for $k = 4$ and $k = 5$.

Theorem (MRV)

Let Γ be a k-cube group, and Λ be its induced k-rank graph. Then $\mathcal{A}(\Lambda)$ is a Kirchberg algebra— in particular, it is completely classified (up to isomorphism) by its K-theory.

Example (Rungtanapirom, Stix, Vdovina, 2019)

$\Gamma'_{\{3,5,7\}} := \langle a_1, a_2, b_1, b_2, b_3, c_1, c_2, c_3, c_4 \mid R \rangle.$

EXAMPLE (RUNGTANAPIROM, STIX, VDOVINA, 2019)

$\Gamma'_{\{3,5,7\}} := \langle a_1, a_2, b_1, b_2, b_3, c_1, c_2, c_3, c_4 \mid R \rangle$.

We have $K_1(\mathcal{A}(\Gamma)) \cong \mathbb{Z}^{21} \oplus (\mathbb{Z}/2)^6 \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/12)^2 \oplus G_1$ *and*

$$0 \longrightarrow \frac{\mathbb{Z}^7 \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/12)}{G_0} \longrightarrow K_0(\mathcal{A}(\Gamma)) \longrightarrow \mathbb{Z}^{21} \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/12) \longrightarrow 0,$$

for some $G_0 \subseteq \mathbb{Z}^7 \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/12)$ *and* $G_1 \subseteq \mathbb{Z}^7$.

EXAMPLE (RUNGTANAPIROM, STIX, VDOVINA, 2019)

$\Gamma'_{\{3,5,7\}} := \langle a_1, a_2, b_1, b_2, b_3, c_1, c_2, c_3, c_4 \mid R \rangle$.

We have $K_1(\mathcal{A}(\Gamma)) \cong \mathbb{Z}^{21} \oplus (\mathbb{Z}/2)^6 \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/12)^2 \oplus G_1$ *and*

$$0 \longrightarrow \frac{\mathbb{Z}^7 \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/12)}{G_0} \longrightarrow K_0(\mathcal{A}(\Gamma)) \longrightarrow \mathbb{Z}^{21} \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/12) \longrightarrow 0,$$

for some $G_0 \subseteq \mathbb{Z}^7 \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/12)$ *and* $G_1 \subseteq \mathbb{Z}^7$.

We can deduce that the torsion-free part of K_0 *is isomorphic to* $\mathbb{Z}^r$, *and that* $K_1 \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2)^6 \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/12)^2$, *for some* $21 \leq r \leq 28$.

Example ($k = 4$, free group)

$\mathbb{F}_2^4 := \langle a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \mid [a_i, b_j], [a_i, c_j], [a_i, d_j], [b_i, c_j], [b_i, d_j], [c_i, d_j] \textit{ for all } i, j \in \{1, 2\} \rangle$.

Example ($k = 4$, free group)

$\mathbb{F}_2^4 := \langle a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \mid [a_i, b_j], [a_i, c_j], [a_i, d_j],$
$[b_i, c_j], [b_i, d_j], [c_i, d_j]$ *for all* $i, j \in \{1, 2\}\rangle$.

Our short exact sequences give us (something like)
$K_0(\mathcal{A}(\mathbb{F}_2^4)) \cong K_1(\mathcal{A}(\mathbb{F}_2^4)) \cong \mathbb{Z}^{64} \oplus \mathbb{Z}^r$, *where* $0 \leq r \leq 64$.

EXAMPLE ($k = 4$, FREE GROUP)

$\mathbb{F}_2^4 := \langle a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \mid [a_i, b_j], [a_i, c_j], [a_i, d_j],$
$[b_i, c_j], [b_i, d_j], [c_i, d_j]$ *for all* $i, j \in \{1, 2\}\rangle$.

Our short exact sequences give us (something like)
$K_0(\mathcal{A}(\mathbb{F}_2^4)) \cong K_1(\mathcal{A}(\mathbb{F}_2^4)) \cong \mathbb{Z}^{64} \oplus \mathbb{Z}^r$, *where* $0 \leq r \leq 64$.

Compare our sequences with the values obtained for K_0 *and* K_1 *by the Künneth Theorem for tensor products:* $K_0 \cong K_1 \cong \mathbb{Z}^{128}$ *and we see that* r *is maximal.*

Example (MRV)

$\Gamma_{\{1,2,3,4\}} := \langle a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3 \mid R \rangle.$

EXAMPLE (MRV)

$\Gamma_{\{1,2,3,4\}} := \langle a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3 \mid R \rangle$.

We have written a program in Python which determines whether a group is a 4-cube group, and if so, outputs four adjacency matrices. In this example, Γ is a 4-cube group, but the adjacency matrices are very large. We are still awaiting the computations of the K-theory in MAGMA.

Epilogue

- The K-theory of these algebras is a great way to distinguish them from each other, showing that our constructions do indeed produce new $C^\star$-algebras.
- Computing K-theory of k-rank graph algebras is hard (e.g. SAM 2020).
- Next step: make the code work!

S.A. Mutter, A.-C. Radu, and A. Vdovina, *$C^\star$-algebras of higher-rank graphs from groups acting on buildings, and explicit computation of their K-theory*
https://arxiv.org/abs/2012.05561

S.A. Mutter, *The K-theory of the $C^\star$-algebras of 2-rank graphs associated to complete bipartite graphs*
https://arxiv.org/abs/2004.11602