DETERMINING THE AMENABILITY OF DISCRETE AND LOCALLY COMPACT GROUPS

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Abstract

The concept of an *amenable group* was introduced by John von Neumann in 1929 [44] as a means to justify the *Banach–Tarski Paradox*. Since then, there have arisen a number of equivalent formulations of amenability in fields as far-reaching as combinatorics, geometric group theory, and functional analysis.

In this article, we develop three of these formulations – each from a different starting point and with a different motivation – and we demonstrate their equivalence.

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1 Amenable Groups

1.1 Measurability

Let a and b be real numbers with $a < b$, and let I denote one of the intervals (a, b), $(a, b]$, $[a, b)$, or $[a, b]$ in $\mathbb{R}$. It is natural to define the *length* of I as $\ell(I) := b - a$, and in 1901, Henri Lebesgue generalised this notion to measure the length of arbitrary subsets in $\mathbb{R}$ [20].

Definition 1.1 (Lebesgue measure). Let $\mathcal{I} \subset \mathbb{R}$ be the union of a finite number of intervals. We can then write $\mathcal{I}$ as the union of a finite number of *mutually disjoint* intervals; $\mathcal{I} = I_1 \cup \cdots \cup I_n$. We define the *Lebesgue measure* of $\mathcal{I}$ to be:

$$\lambda(\mathcal{I}) := \ell(I_1) + \cdots + \ell(I_n).$$

In order to further generalise this function to measure the length of unions of countably-many intervals, we recall the following definitions: $\mathbb{KLM}$

Let X be an arbitrary set, and let Σ be a collection of subsets of X. We say that Σ is a *σ-algebra* if it satisfies the following conditions:

(a) $X \in \Sigma$,

(b) Σ is *closed under complements*: If $A \in \Sigma$, then $A^c := X \setminus A \in \Sigma$,

(c) Σ is *closed under countable unions*: If (A_n) is a sequence of sets in Σ, then $\bigcup_{n=1}^{\infty} A_n \in \Sigma$ [39, p512].

Note that, by *de Morgan's Laws* (see [5, p374]), it follows from (b) and (c) that a σ-algebra is also closed under countable intersections.

The *Borel algebra* on a topological space X, denoted by $\mathcal{B}(X)$, is the smallest σ-algebra which contains all open subsets of X. Here, "smallest" is to say that, if Σ is a σ-algebra which contains all open subsets of X, then $\mathcal{B}(X) \subseteq \Sigma$ [38, p23].

By the fact that a σ-algebra is closed under complements, it can be shown that the Borel algebra on X can alternatively be generated by the *closed* subsets of X. We call the elements of $\mathcal{B}(X)$ *Borel sets*. In the case where $X = \mathbb{R}$, observe that each interval I is a Borel set [5, p4] [39, p529].

Let $A \subseteq X$ be an arbitrary subset. We define an *open covering* of A to be a collection $\mathcal{U}$ of open sets such that:

$$A \subseteq \bigcup_{U \in \mathcal{U}} U.$$

A subset $K \subseteq X$ is said to be *compact* if every open covering of K contains a finite sub-collection which also covers K [23, p164].

In the case where $X = \mathbb{R}$, we can define a countable open covering of a subset $A \subseteq \mathbb{R}$ as a sequence of open intervals (I_n) such that:

$$A \subseteq \bigcup_{n=1}^{\infty} I_n.$$

Now we are able to extend the measure λ to a countably-additive function $\lambda^* : \mathcal{M}(\mathbb{R}) \to \mathbb{R}$, where $\mathcal{M}(\mathbb{R})$ is defined below as a σ-algebra which contains $\mathcal{B}(\mathbb{R})$. The process is as follows.

Let $A \subseteq \mathbb{R}$ be an arbitrary subset. We define the *Lebesgue outer measure* of A to be:

$$\lambda^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) \,\middle|\, (I_n) \text{ is a countable open covering of } A \right\}$$

[35, p27].

Definition 1.2 (Lebesgue σ-algebra). We say that a subset $E \subseteq \mathbb{R}$ is *Lebesgue measurable* if it satisfies *Carathéodory's criterion*, that

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c),$$

for all subsets $A \subseteq \mathbb{R}$. We define the *Lebesgue σ-algebra* on $\mathbb{R}$, denoted by $\mathcal{M}(\mathbb{R})$, to be the collection of all Lebesgue measurable sets [35, pp27-8].

Observe that for any subsets $A, E \subseteq \mathbb{R}$, the equality

$$A = (A \cap E) \cup (A \cap E^c)$$

holds. Since λ^* is countably-subadditive (Prop A.1), it follows that

$$\lambda^*(A) \leq \lambda^*(A \cap E) + \lambda^*(A \cap E^c), \tag{1}$$

and hence to prove that a set E is Lebesgue measurable, we need only show that the reverse inequality holds for all $A \subseteq \mathbb{R}$ [5, pp15-6].

We verify now that the Lebesgue σ-algebra is justly named:

Proposition 1.3. *The collection $\mathcal{M}(\mathbb{R})$ defines a σ-algebra, and for all intervals $I \in \mathcal{M}(\mathbb{R})$, we have $\lambda^*(I) = \lambda(I)$.*

Proof. We split the proof into a number of smaller steps, beginning by showing that $\mathcal{M}(\mathbb{R})$ defines a field of sets (that is, that $\mathcal{M}(\mathbb{R})$ is closed under finite unions, finite intersections, and complements).

Firstly, notice that $\mathbb{R} \in \mathcal{M}(\mathbb{R})$, since $\lambda^*(\mathbb{R}^c) = \lambda^*(\emptyset) = 0$, and so for each subset $A \subseteq \mathbb{R}$ we have:

$$\lambda^*(A \cap \mathbb{R}) + \lambda^*(A \cap \emptyset) = \lambda^*(A) + 0.$$

Now, let $E, F \in \mathcal{M}(\mathbb{R})$. Then for each subset $A \subseteq \mathbb{R}$ we have:

$$\begin{aligned}
\lambda^*(A) &= \lambda^*(A \cap E) + \lambda^*(A \cap E^c) \\
&= \lambda^*(A \cap E) + \lambda^*((A \cap E^c) \cap F) + \lambda^*((A \cap E^c) \cap F^c) \\
&= \lambda^*(A \cap E) + \lambda^*((A \cap E^c) \cap F) + \lambda^*((A \cap (E \cup F)^c) \\
&= \lambda^*(A \cap E \cap (E \cup F)) + \lambda^*(A \cap E^c \cap (E \cup F)) + \lambda^*(A \cap (E \cup F)^c) \\
&= \lambda^*(A \cap (E \cup F)) + 0 + \lambda^*(A \cap (E \cup F)^c),
\end{aligned}$$

and so $E \cup F \in \mathcal{M}(\mathbb{R})$.

Finally, it suffices to check that $E \setminus F \in \mathcal{M}(\mathbb{R})$. Indeed, by the symmetry of *Carathéodory's criterion*, we have that $E^c \in \mathcal{M}(\mathbb{R})$, and since $E \setminus F = (E^c \cup F)^c$, the result follows immediately from the above argument.

Hence $\mathcal{M}(\mathbb{R})$ defines a field of sets.

In order to conclude that $\mathcal{M}(\mathbb{R})$ is a σ-algebra, we must show that it is also closed under countable unions. Let $(E_n) \in \mathcal{M}(\mathbb{R})$ be a sequence of pairwise-disjoint Lebesgue measurable sets. Firstly, we have to show that

$$\lambda^*(A) = \sum_{n=1}^{N} \left(\lambda^*(A \cap E_n) + \lambda^*\left(A \cap \left(\bigcap_{n=1}^{N} E_n^c\right)\right)\right), \tag{2}$$

for each subset $A \subseteq \mathbb{R}$ and each $N \in \mathbb{N}$. To do this, we use induction on N.

In the case where $N = 1$, the equation above is precisely *Carathéodory's criterion*. For the inductive step, observe that since E_{N+1} is Lebesgue measurable, we have:

$$\begin{aligned}
&\lambda^*\left(A \cap \left(\bigcap_{n=1}^{N} E_n^c\right)\right) \\
&\quad = \lambda^*\left(A \cap \left(\bigcap_{n=1}^{N} E_n^c\right) \cap E_{N+1}\right) + \lambda^*\left(A \cap \left(\bigcap_{n=1}^{N} E_n^c\right) \cap E_{N+1}^c\right),
\end{aligned}$$

and since $E_i \cap E_j = \emptyset$ for all $i \neq j$, it follows that

$$\lambda^*\left(A \cap \left(\bigcap_{n=1}^{N} E_n^c\right)\right) = \lambda^*\left(A \cap E_{N+1}\right) + \lambda^*\left(A \cap \left(\bigcap_{n=1}^{N+1} E_n^c\right)\right),$$

and so (2) is proved. As we let N approach infinity, we obtain from (2) the inequality

$$\lambda^*(A) \geq \sum_{n=1}^{\infty} \left(\lambda^*(A \cap E_n) + \lambda^*\left(A \cap \left(\bigcap_{n=1}^{\infty} E_n^c\right)\right)\right),$$

and hence

$$\lambda^*(A) \geq \sum_{n=1}^{\infty} \Big(\lambda^*(A \cap E_n) + \lambda^*\Big(A \cap \Big(\textstyle\bigcup_{n=1}^{\infty} E_n\Big)^c\Big)\Big). \tag{3}$$

It is not difficult to show that if (A_n) is a sequence of subsets of $\mathbb{R}$, then

$$\lambda^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \lambda^*(A_n) \tag{4}$$

(see Prop A.1 in the appendix). From this it follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \Big(\lambda^*(A \cap E_i) + \lambda^*\Big(A \cap \Big(\textstyle\bigcup_{n=1}^{\infty} E_n\Big)^c\Big)\Big) \\
&\geq \lambda^*\Big(A \cap \Big(\textstyle\bigcup_{n=1}^{\infty} E_n\Big)\Big) + \lambda^*\Big(A \cap \Big(\textstyle\bigcup_{n=1}^{\infty} E_n\Big)^c\Big) \\
&\geq \lambda^*(A),
\end{aligned}$$

which, together with (3), shows that $\bigcup_{n=1}^{\infty} E_n$ is Lebesgue measurable. Since we can write the union of any sequence of sets in $\mathcal{M}(\mathbb{R})$ as the union of a sequence of *disjoint* sets (E_i) [5, p17], we have shown that $\mathcal{M}(\mathbb{R})$ is closed under countable unions, and is therefore a σ-algebra.

Now let $I \subseteq \mathbb{R}$ be an interval. Then for every $\varepsilon > 0$ we can find an open interval $J \supseteq I$ such that $\lambda^*(I) \leq \lambda(J) \leq \lambda(J) + \varepsilon$. Since we can set ε to be arbitrarily small, this shows that $\lambda^*(I) \leq \lambda(I)$, and so it only remains to show the reverse inequality.

Let $\varepsilon > 0$ be an arbitrary constant, and let (I_i) be a covering of I, where each I_i is an open interval in $\mathbb{R}$. Without loss of generality, we may assume that I is bounded, since an unbounded interval has infinite Lebesgue outer measure. Furthermore, we may assume I to be closed, since any bounded interval has length which can be approximated arbitrarily closely by a closed interval containing it [41, p34].

Hence we may suppose that I is compact. Then we can find a finite subcollection of open intervals $I_1, \ldots I_n$ in $\mathbb{R}$ which form an open covering of I [23, p164]. Hence, by induction on n, it follows that

$$\lambda(I) \leq \sum_{i=1}^{n} \lambda(I_i),$$

and hence that

$$\lambda(I) \leq \sum_{i=1}^{\infty} \lambda(I_i).$$

So, by the definition of the Lebesgue outer measure, it follows that $\lambda(I) \leq \lambda^*(I)$ whenever I is compact, and therefore that $\lambda^*(I) = \lambda(I)$ for all intervals $I \subseteq \mathbb{R}$ [5, p14]. $\square$

Proposition 1.4. *Every Borel subset of* $\mathbb{R}$ *is Lebesgue measurable [5, p18].*

Proof. It can be shown that the Borel algebra $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra generated by the collection $\{(-\infty, b] \mid b \in \mathbb{R}\}$ [5, p4]. Hence we need only verify that such an interval $E = (-\infty, b]$ is Lebesgue measurable – using observation (1) on p3, this means checking that

$$\lambda^*(A) \geq \lambda^*(A \cap E) + \lambda^*(A \cap E^c),$$

for all subsets $A \subseteq \mathbb{R}$. Observe that this is certainly true if $\lambda^*(A) = \infty$, so we may assume that A has finite measure.

Let $\varepsilon > 0$ be an arbitrary constant, and let (I_n) be a covering of A, where each I_n is an open interval in $\mathbb{R}$, such that

$$\sum_{n=1}^{\infty} \ell(I_n) < \lambda^*(A) + \varepsilon.$$

For each $n \in \mathbb{N}$, the sets $I_n \cap E$ and $I_n \cap E^c$ are disjoint intervals whose union is I_n, and so

$$\ell(I_n) = \lambda^*(I_n) = \lambda^*(I_n \cap E) + \lambda^*(I_n \cap E^c),$$

by Prop 1.3. But the collections $\{I_n \cap E\}$ and $\{I_n \cap E^c\}$ are open coverings of $A \cap E$ and $A \cap E^c$ respectively, and since λ^* is countably-subadditive (Prop A.1), it follows that

$$\begin{aligned}
\lambda^*(A \cap E) + \lambda^*(A \cap E^c) &\leq \sum_{n=1}^{\infty} \left(\lambda^*(I_n \cap E) + \lambda^*(I_n \cap E^c)\right) \\
&= \sum_{n=1}^{\infty} \ell(I_n) \\
&< \lambda^*(A) + \varepsilon.
\end{aligned}$$

Since we can set ε to be arbitrarily small, this shows that each set $E = (\infty, b]$ is in the Lebesgue σ-algebra $\mathcal{M}(\mathbb{R})$. But, since $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra containing all such sets E, it follows that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}(\mathbb{R})$. $\square$

The Lebesgue measure possesses some interesting properties, including:

Lemmata 1.5 (Interesting properties).

1. Completeness. *Let* $E \subseteq \mathbb{R}$ *be a subset such that* $\lambda^*(E) = 0$. *Then every subset* $F \subseteq E$ *is Lebesgue measurable. In such instances, the measure of* F *is zero [35, p29].*

2. Invariance under translation. *Let $E \subseteq \mathbb{R}$ be a measurable subset. Then $\lambda^*(E+t) = \lambda^*(E)$ for all $t \in \mathbb{R}$ [35, p28].*

Proof.

1. Let $\lambda^*(E) = 0$. Then $\lambda^*(A \cap E) = 0$ for all $A \subseteq \mathbb{R}$, since $(A \cap E) \subseteq E$. Hence
$$\lambda^*(A \cap E) + \lambda^*(A \cap E^c) = \lambda^*(A \cap E^c) \leq \lambda^*(A),$$
for all $A \subseteq \mathbb{R}$, and so E is Lebesgue measurable by observation (1). Thus any subset F of E is Lebesgue measurable with $\lambda^*(F) = 0$, by the subadditivity of λ^* [32, §2.3].

2. Let (I_n) be an covering of E, where each I_n is an open interval in $\mathbb{R}$. Then $(I_n + t) = (\{i_n + t \,|\, i_n \in I_n\})$ is an open covering of $E + t$, and
$$\lambda^*(E+t) \leq \sum_{n=1}^{\infty} \ell(I_n + t) = \sum_{n=1}^{\infty} \ell(I_n).$$
Since this is true for any open covering (I_n) of E, it follows that $\lambda^*(E+t) \leq \lambda^*(E)$.

   Similarly, by considering an open covering (I'_n) of $E + t$, such that $(I'_n - t)$ covers E, we can show that the reverse inequality also holds, and hence that $\lambda^*(E+t) = \lambda^*(E)$ [32, §2.2] [39, p526]. □

From now on, we are mainly going to be interested in the way the Lebesgue measure λ behaves on the Borel algebra on $\mathbb{R}$. In fact, λ is practically unique when we restrict our vision to $\mathcal{B}(\mathbb{R})$.

Theorem 1.6. *The Lebesgue measure λ is the* unique *complete translation-invariant measure on $\mathcal{B}(\mathbb{R})$ such that $\lambda[0,1] = 1$ [31, p623].*

Proof. We defer a proof of this result until §1.4, whereupon it becomes a corollary of Thm 1.22. If the reader is eager to see a direct proof, we direct them to [5, p26]. □

So far, we have observed that the Lebesgue measure λ is the unique non-negative, real-valued function on $\mathcal{B}(\mathbb{R})$ which satisfies the conditions:

(a) λ is complete,

(b) λ is translation-invariant,

(c) $\lambda[0,1] = 1$,

(d) λ is countably-additive.

At the turn of the twentieth century, Stanisław Ruziewicz asked the natural question as to whether λ remains unique when the *countable* additivity of (d) is replaced by *finite* additivity. In 1923, Stefan Banach confirmed that uniqueness is not preserved in $\mathbb{R}$, and in fact that there exists a complete translation-invariant measure μ on $\mathbb{R}$ which is finitely- but not countably-additive, and which is defined on *all* subsets of $\mathbb{R}$ [31, p623]. Such a measure μ is called an *invariant mean* on $\mathbb{R}$.

1.2 Invariant Means on Discrete Groups

Before considering the definition of an invariant mean on an arbitrary group, we first restrict our observations to discrete groups (that is, topological groups with the largest possible topology, in which all subsets are open [23, p77]. In this way, we can use the tools already at our disposal from the previous section and from elementary measure theory, before adding some structure which will motivate the more general definition.

Definition 1.7 (Means on discrete groups). Let Γ be a discrete group, and write $\mathcal{P}(\Gamma)$ for the collection of all subsets of Γ. We call a function $\mu : \mathcal{P}(\Gamma) \to [0, 1]$ a *mean* on Γ if it has the following properties:

(a) μ is *finitely-additive*: if $A, B \subseteq \Gamma$ are disjoint subsets, then $\mu(A \cup B) = \mu(A) + \mu(B)$,

(b) μ is a *probability measure*: $\mu(\emptyset) = 0$, and $\mu(\Gamma) = 1$.

If, in addition, the function μ has the property that

(c) μ is *left-invariant*: $\mu(xA) = \mu(A)$ for all $x \in \Gamma$ and $A \subseteq \Gamma$,

then we call μ a *left-invariant mean*, or *measure*, on Γ. Likewise, we call μ a *right-invariant mean* if $\mu(Ax) = \mu(A)$ [10, p4] [18, §4.1].

Example 1.8. Let Γ be a discrete group, and let $\nu : l^\infty(\Gamma) \to \mathbb{R}$ be the *counting measure*, defined by:

$$\nu(A) := \begin{cases} |A| & \text{if } |A| < \infty, \\ \infty & \text{if } A \text{ is infinite}, \end{cases}$$

for all subsets $A \subseteq \Gamma$. If $|\Gamma| < \infty$, we can use the counting measure to construct the function $\nu' : l^\infty(\Gamma) \to [0, 1]$ given by:

$$\nu'(A) := \frac{\nu(A)}{|\Gamma|},$$

for all subsets $A \subseteq \Gamma$. It is not difficult to see that ν' defines an invariant mean on Γ [35, p27].

Recall that a group Γ is called a *topological group* when equipped with a topology with respect to which the binary and inverse functions of the group are continuous. A topological space X is said to be *locally compact* if each point $x \in X$ has an open neighbourhood U_x which itself is contained in some compact set K. One might be forgiven for assuming, then, that a *locally compact group* Γ is a topological group which is locally compact. This is almost the case, but we permit ourselves to make one extra assumption about Γ [8, p35] [22, §28].

We define a *locally compact group* to be a topological group which is both locally compact and Hausdorff as a topological space [5, p279] [22, p112]. It turns out that this restriction is almost negligible [8, p37].

It can be shown that all discrete groups are also locally compact. [1] If our group Γ is locally compact (but not necessarily discrete), we can define a mean μ on Γ in a similar way, but we will first need some new definitions.

One of the most elementary examples of a locally compact group is the real line $\mathbb{R}$. We have already found an invariant mean for $\mathcal{B}(\mathbb{R})$, the Lebesgue measure λ, and this was derived from the function λ^*, the Lebesgue outer measure on $\mathbb{R}$. Then, in order to find a suitable mean for a general locally compact group, we will begin in the same manner.

Definition 1.9 (Outer measure). Let X be an arbitrary set, and let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be a function. We call μ^* an *outer measure* on X if it has the following properties:

(a) μ^* is *monotonic*: if $A \subseteq B \subseteq X$, then $\mu^*(A) \leq \mu(B)$,

(b) μ^* is *countably-subadditive*: If (A_n) is a sequence of subsets of X (not necessarily disjoint), then

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

(c) $\mu^*(\emptyset) = 0$.

As a generalisation of Lebesgue measurability (Def 1.2), we say that a subset $E \subseteq X$ is *μ^*-measurable* if it satisfies the condition:

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c),$$

[1] To see this, observe that a singleton $\{x\}$ in a discrete group Γ is open by the definition of the discrete topology. But $\{x\}$ is also finite, and hence compact. Thus $\{x\}$ is a compact set containing an open neighbourhood of x, and so Γ is locally compact at each $x \in \Gamma$ (see §1.3).

for all subsets $A \subseteq X$. Hence a μ^*-measurable subset $E \subseteq X$ splits each subset $A \subseteq X$ in such a way that the measures of the pieces of A which fall inside and outside of E add up to the total measure of A [34, §2.1].

Definition 1.10 (Borel regular measure). Let X be a locally compact Hausdorff space. An outer measure μ^* whose domain is $\mathcal{B}(X)$ is called a *Borel outer measure* on X if $\mu^*(K) < \infty$ for all compact subsets $K \subseteq X$ [14, p223].

We say that a Borel outer measure μ^* is *regular* if each Borel set $E \in \mathcal{B}(X)$ satisfies the conditions:

(a) $\inf\{\mu^*(U) \mid E \subseteq U, \text{ and } U \subseteq X \text{ is open}\} = \mu^*(E)$,

(b) $\sup\{\mu^*(K) \mid K \subseteq E, \text{ and } K \text{ is compact}\} = \mu^*(E)$.

An outer measure which satisfies only one of (a) or (b) can be called *outer regular* or *inner regular* respectively.

It can be shown that a regular outer measure becomes a regular measure if restricted to the Borel sets, since Borel sets are formed from countable unions and intersections [34, §3.1] [35, p158].

Clearly the Lebesgue measure λ restricted to $\mathcal{B}(\mathbb{R})$ is a Borel measure, and we claim that λ is also regular [5, p23]. Indeed, since $\lambda(A) \leq \lambda(B)$ whenever $A \subseteq B \in \mathcal{B}(\mathbb{R})$, we need only to show that:

(a) $\inf\{\lambda(U) \mid E \subseteq U, \text{ and } U \subseteq \mathbb{R} \text{ is open}\} \leq \lambda(E)$,

(b) $\sup\{\lambda(K) \mid K \subseteq E, \text{ and } K \text{ is compact}\} \geq \lambda(E)$,

for each Borel set $E \in \mathcal{B}(\mathbb{R})$.

Proof. To show part (a), we first set an arbitrary $\varepsilon > 0$, and let $\lambda(E) < \infty$ (the result automatically holds when $\lambda(E) = \infty$). Then there is an open covering (I_n) of E such that

$$\sum_{n=1}^{\infty} \lambda(I_n) < \lambda(E) + \varepsilon.$$

Define $U := \bigcup_{n=1}^{\infty} I_n$. Clearly U is open, and $E \subseteq U$. Moreover, since λ is an outer measure, it is countably-subadditive. Hence

$$\lambda(U) \leq \sum_{n=1}^{\infty} \lambda(I_n) = \sum_{n=1}^{\infty} \ell(I_n),$$

by Prop 1.3. So $\lambda(U) < \lambda(E) + \varepsilon$, and since we can set ε to be arbitrarily small, this proves part (a).

Now, to show part (b), we consider two distinct cases: where E is either bounded or unbounded. Firstly, suppose that E is bounded, and let $E \subseteq K' \subseteq \mathbb{R}$, where K' is compact (and hence Lebesgue measurable [42, chp2]). Write $F = K' \setminus E$. As above, we can find an open set U containing F and such that

$$\lambda(U) < \lambda(F) + \varepsilon,$$

where $\varepsilon > 0$ is an arbitrary constant. Now $K := K' \setminus U$ is a compact subset of E such that $K' \subseteq K \cup U$, and hence

$$\lambda(K) \geq \lambda(K') - \lambda(U).$$

But

$$\lambda(U) < \lambda(F) + \varepsilon = \lambda(K') - \lambda(E) + \varepsilon,$$

and so $\lambda(K) > \lambda(E) - \varepsilon$. Since we can set ε to be arbitrarily small, this proves part (b) when E is bounded.

For the case where E is unbounded, let (E_n) be an increasing sequence of bounded (measurable) subsets of E, such that $E = \bigcup_{n=1}^{\infty} E_n$. Then $\lambda(E) = \lim_{n\to\infty} \lambda(E_n)$ [14, pp37-8].

Let d be an arbitrary real number such that $\lambda(E) > d$, and choose a natural number N such that $\lambda(E_N) > d$. Then, since E_N is bounded, we can use the arguments in the above case to generate a compact subset $K \subseteq E_N \subseteq E$. Since our choice of $d < \lambda(E)$ was arbitrary, this proves part (b) when E is unbounded [34, p61]. $\square$

Hence the Lebesgue measure λ, when restricted to $\mathcal{B}(\mathbb{R})$, is a Borel regular measure on $\mathbb{R}$. Our more general notion of measure is therefore compatible with the idea of length on the real line, and so we are ready to introduce the idea of "size" in an arbitrary locally compact group. Firstly, we make a quick digression to learn some facts about locally compact groups which will be of continued use.

1.3 Locally Compact Groups

We turn our attention to a handful of lemmata which we will require in order to define measure on a locally compact group.

Theorem 1.11 (Tychonoff). *Let $\mathcal{Y}$ be an arbitrary collection of compact spaces. Then the space X defined by*

$$X := \prod_{Y \in \mathcal{Y}} Y$$

is compact with respect to the product topology. (Refer to [23, p234] for a proof.) $\square$

Lemma 1.12 (Separation). *Let Γ be a locally compact group, and let $K, L \subseteq \Gamma$ be disjoint compact subsets. Then we can find disjoint open subsets $U, V \subseteq \Gamma$ such that $K \subseteq U$ and $L \subseteq V$. The sets K and L are said to be* separated *by U and V [5, p182].*

Proof. Firstly, consider the case where K is the empty set. Then we can set $U = \emptyset$ and $V = \Gamma$, and we are done. The same is true if L is empty, so we assume neither K nor L is the empty set.

Let $x \in K$. Since Γ is Hausdorff, for each $y \in L$, we can find a pair of disjoint open subsets $U_y^x, V_y^x \subseteq \Gamma$ such that $x \in U_y^x$ and $y \in V_y^x$. Then, as L is compact, there is a finite list of elements $y_1, \ldots, y_n$ such that the sets $V_{y_1}^x, \ldots, V_{y_n}^x$ form an open covering of L. For each $x \in K$, we can then define

$$U^x := \bigcap_{i=1}^{n} U_{y_i}, \qquad \text{and} \qquad V^x := \bigcup_{i=1}^{n} V_{y_i},$$

such that $U^x \ni x$ and $V^x \subseteq L$ are disjoint and open. Since K is compact, we can find a finite list of elements $x_1, \ldots, x_m$ such that $U^{x_1}, \ldots, U^{x_m}$ form an open covering of K. Define the subsets

$$U := \bigcup_{i=1}^{m} U^{x_i}, \qquad \text{and} \qquad V := \bigcap_{i=1}^{m} V^{x_i}.$$

Then U and V are disjoint and open, with $K \subseteq U$ and $L \subseteq V$ [5, p182]. $\square$

Lemma 1.13. *Let Γ be a locally compact group, let $K \subseteq \Gamma$ be a closed subset, and let $U \subseteq \Gamma$ be an open subset containing K. Then we can find an open subset $V \subseteq \Gamma$ with compact closure and such that $K \subseteq V \subseteq \overline{V} \subseteq U$ [5, p183].*

Proof. Let $x \in K$. Since Γ is locally compact, there is an open neighbourhood W_x of x with compact closure [23, p185]. Write $V_x := W_x \cap U$, such that $V_x \subseteq U$, and use Lem 1.12 to find disjoint open sets Y and Z which separate the compact sets $\{x\}$ and $\overline{V_x} \setminus V_x$. The closure of $V_x \cap Y$ is therefore compact and contained in V_x, and hence in U. Thus $V_x \cap Y$ is an open neighbourhood of x whose closure is compact and contained in U.

Since K is compact, some finite collection of these neighbourhoods covers K. Write V to denote the union of these sets. Then V satisfies $K \subseteq V \subseteq \overline{V} \subseteq U$ [5, pp182-3]. $\square$

Lemma 1.14. *Let Γ be a locally compact group, let $K \subseteq \Gamma$ be a compact subset, and let $U_1, U_2 \subseteq \Gamma$ be open subsets such that $K \subseteq U_1 \cup U_2$. Then* *we can find compact subsets $K_1, K_2 \subseteq \Gamma$ such that $K_1 \subseteq U_1$, $K_2 \subseteq U_2$, and $K = K_1 \cup K_2$ [5, p184].*

Proof. Observe that the sets $K \setminus U_1$ and $K \setminus U_2$ are disjoint and compact, so we can use the *Separation Lemma* (Lem 1.12) to find disjoint open sets $V_1 \supseteq (K \setminus U_1)$ and $V_2 \supseteq (K \setminus U_2)$. Now define $K_1 := K \setminus V_1$ and $K_2 := K \setminus V_2$. Then K_1 and K_2 are compact sets such that $K_1 \subseteq U_1$, $K_2 \subseteq U_2$, and $K = K_1 \cup K_2$ [5, p184]. $\square$

Lemma 1.15. *Let Γ be a topological group, let $K \subseteq \Gamma$ be a compact subset, and let $U \subseteq \Gamma$ be an open subset containing K. Then we can find open neighbourhoods V_R and V_L of the group identity e such that*

$$KV_R := \{xv \,|\, v \in V_R, x \in K\} \subseteq U,$$

and

$$V_L K := \{vx \,|\, v \in V_L, x \in K\} \subseteq U.$$

[5, p281]

Proof. Recall that a *neighbourhood base* for an element $x \in \Gamma$ is a family $\mathcal{W}$ of subsets $W \subseteq \Gamma$ such that:

(a) Each $W \in \mathcal{W}$ is open, and

(b) For each open neighbourhood U of x, there is some $W \in \mathcal{W}$ such that $U \subseteq W$ [5, p280].

Let $x \in K$, and let $U \subseteq \Gamma$ be an open neighbourhood of x. We claim that, for each x, we can choose an open neighbourhood W_x of e such that $xW_x \subseteq U$. To see why, consider the map $l_x : \Gamma \to \Gamma$ defined by $l_x(t) := xt$, for all $t \in \Gamma$. The map l_x is clearly continuous, and has continuous inverse $(l_x)^{-1}(t) := x^{-1}t$, so l_x is a homeomorphism. Hence if $\mathcal{W}_e$ forms a neighbourhood base for e, then $l_x[\mathcal{W}_e] := \{xW \,|\, W \in \mathcal{W}_e\}$ is a neighbourhood base for x [5, p281] [16, §4.2].

Now, let U be an open set which contains K. Then in particular, U is an open neighbourhood of x, and so we can find a $W_x \in \mathcal{W}_e$ such that $xW_x \subseteq U$.

We further claim that, for each such set W_x, we can find an open neighbourhood V_x of e such that

$$V_x V_x := \{ab \,|\, a, b \in V_x\} \subseteq W_x.$$

Indeed, since the function $f : \Gamma \times \Gamma \to \Gamma$ defined by $f(x, y) := xy$ is continuous, the preimage

$$Y := \{(x, y) \,|\, xy \in W_x, \text{ and } W_x \in \mathcal{U}\} \subseteq \Gamma \times \Gamma$$

forms an open neighbourhood of (e, e). Hence we can find open neighbourhoods V_1, V_2 of e such that $V_1 \times V_2 \subseteq Y$, and we define $V_x := V_1 \cap V_2$ such that $V_x V_x \subseteq W_x$.

Now, the collection $\{xV_x\}_{x\in K}$ forms an open covering of K, and since K is compact, we can find a finite sub-collection of points $x_1, \ldots x_n$ in K such that

$$K \subseteq \bigcup_{i=1}^{n} x_i V_{x_i}.$$

Define the set V_R by:

$$V_R := \bigcap_{i=1}^{n} V_{x_i}.$$

Then, if $x \in K$, there is an element x_i such that $x \in x_i V_{x_i}$, and so

$$xV_R \subseteq x_i V_{x_i} V_{x_i} \subseteq x_i W_{x_i} \subseteq U.$$

Since our choice of x was arbitrary, it follows that $KV_R \subseteq U$. The construction of V_L follows the same arguments [5, pp280-1]. $\square$

1.4 Haar's Theorem

A theorem of Haar (Thm 1.17) shows that every locally compact group can be furnished with a particularly nice measure, a *Haar measure*, which possesses some desirable properties:

Definition 1.16 (Haar measure). Let Γ be a locally compact group equipped with a (non-zero) regular Borel measure μ. We call μ a *left Haar measure* if it left-invariant, that is, if:

$$\mu(xA) = \mu(A),$$

for all $x \in \Gamma$ and $A \in \mathcal{B}(\Gamma)$. Similarly, we call μ a *right Haar measure* if:

$$\mu(Ax) = \mu(A),$$

for all $x \in \Gamma$ and $A \in \mathcal{B}(\Gamma)$. Observe that, by the fact that the translation map $l_x : \Gamma \to \Gamma$ defined by $l_x(t) := xt$ is a homeomorphism (see p13), xA is a Borel subset of Γ whenever $A \in \mathcal{B}(\Gamma)$, for all $x \in \Gamma$. Likewise, Ax is a Borel subset, and so the measures of xA and Ax are well-defined [5, p285].

Theorem 1.17 (Haar). *Let Γ be a locally compact group. Then Γ can be equipped with a left Haar measure.*

Proof. The proof we present here closely follows the structure of that in [5, §9.2], which in turn is based on the classical proof of Weil [46]. Let Γ be a locally compact group, let $K \subseteq \Gamma$ be compact, and let $V \subseteq \Gamma$ be a subset with non-empty interior V^o. Then the collection

$$\mathcal{V} := \{xV^o \,|\, x \in \Gamma\} = \{xv \,|\, x \in \Gamma, v \in V^o\}$$

forms an open covering of K, and since K is compact, there is a finite subcovering $\mathcal{V}' \subseteq \mathcal{V}$ of K. This is to say that we can find a finite list of elements $x_1, \ldots, x_n \in \Gamma$ such that

$$\mathcal{V}' = \{x_i V^o \,|\, i = 1, \ldots n\} \tag{5}$$

is an open covering of K. Let $(K : V)$ denote the smallest non-negative integer n for which such a covering $\mathcal{V}'$ exists. Note that $(K : V) = 0$ if and only if $K = \emptyset$.

Write $\mathcal{K}$ to denote the collection of all compact subsets of Γ, and write $\mathcal{U}$ to denote the collection of all open subsets of Γ which contain the identity e. Let $K_0 \subseteq \Gamma$ be a fixed compact subset with non-empty interior: such a set exists since Γ is locally compact. For each $U \in \mathcal{U}$, we define a function $\varphi_U : \mathcal{K} \to \mathbb{R}$ by:

$$\varphi_U(K) := \frac{(K : U)}{(K_0 : U)},$$

for all $K \in \mathcal{K}$. This function will be an indicator of the "size" of a compact subset K relative to some open neighbourhood U of e.

Now, write $m := (K : K_0)$ and $n := (K_0 : U)$, for some $K \in \mathcal{K}$ and $U \in \mathcal{U}$. Let $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ be finite sequences in Γ such that $\{x_i K_0^o\}_{i=1}^m$ and $\{y_j U\}_{j=1}^n$ are open coverings of K and K_0 respectively. Then

$$K \subseteq \bigcup_{i=1}^{m} \left(\bigcup_{j=1}^{n} x_i y_j U \right),$$

which is to say that $(K : U) \leq mn = (K : K_0)(K_0 : U)$. Since $(K : U) \geq 0$, it follows that $\varphi_U \geq 0$, and so

$$0 \leq \varphi_U(K) \leq (K : K_0), \tag{6}$$

for all $K \in \mathcal{K}$ and $U \in \mathcal{U}$.

For each $K \in \mathcal{K}$, consider the interval $[0, (K : K_0)] \subset \mathbb{R}$. Define the space X to be the product of all such intervals:

$$X := \prod_{K \in \mathcal{K}} [0, (K : K_0)],$$

together with the product topology. Since each interval is compact, it follows by *Tychonoff's Theorem* (Thm 1.11) that the space X is compact [22, p11]. It follows from (6) that each function φ_U can be viewed as a point in X. Then, for each $V \in \mathcal{U}$, we define the set

$$C(V) := \overline{\{\varphi_U \,|\, U \in \mathcal{U} \text{ and } U \subseteq V\}}.$$

Write V as the intersection of a finite sequence of sets $V_1, \dots V_n$ in $\mathcal{U}$. Then $\varphi_V \in \bigcap_{i=1}^n C(V_i)$, and so $\bigcap_{i=1}^n C(V_i)$ is non-empty. Hence the space of closed sets $\{C(V) \,|\, V \in \mathcal{U}\}$ satisfies the *finite intersection property* (see Prop A.3).

But since X is compact, it follows that the intersection of the sets in the collection $\{C(V) \,|\, V \in \mathcal{U}\}$ is non-empty, and so we are able to pick an element $\varphi \in \bigcap_{V \in \mathcal{U}} C(V)$.

⋆ ⋆ ⋆

We show that the function φ possesses the properties of a finitely-additive measure on $\mathcal{K}$, namely:

(a) If $K_1 \subseteq K_2$, then $\varphi(K_1) \leq \varphi(K_2)$,

(b) $\varphi(K_1 \cup K_2) \leq \varphi(K_1) + \varphi(K_2)$,

(c) $\varphi(K_1 \cup K_2) = \varphi(K_1) + \varphi(K_2)$ whenever K_1 and K_2 are disjoint,

(d) $\varphi(K) \geq 0$ for all $K \in \mathcal{K}$, and $\varphi(\emptyset) = 0$.

To show part (a), first let $K_1 \subseteq K_2$ be compact subsets of Γ. Observe that $\varphi_U(K_1) \leq \varphi_U(K_2)$ for each $U \in \mathcal{U}$, since a covering of K_2 is automatically a covering of K_1. Recall that X has the product topology, so that for each $K \in \mathcal{K}$ and $U \in \mathcal{U}$, the projection $f \mapsto f(K)$ from X to $\mathbb{R}$ is continuous [5, p392].

Let $f \in X$, and consider the map $\Phi : X \to \mathbb{R}$ defined by $\Phi(f) := f(K_2) - f(K_1)$. Clearly Φ is continuous by virtue of being the composition of continuous maps, and also non-negative on each $C(V)$, since $\varphi_U(K_1) \leq \varphi_U(K_2)$ for each $U \in \mathcal{U}$. Hence Φ is non-negative at each point in each closure $C(V)$, and in particular, non-negative at φ. So $\varphi(K_2) - \varphi(K_1) \geq 0$, and this proves part (a).

We recycle this argument to prove part (b): let K_1, K_2 be compact subsets of Γ which are not necessarily disjoint. We claim that $\varphi_U(K_1 \cup K_2) \leq \varphi_U(K_1) + \varphi_U(K_2)$ for each $U \in \mathcal{U}$.

Indeed, if $\mathcal{V}_1$ is a covering of K_1 (as in (5)) consisting of $m = (K_1 : U)$ cosets of U, and $\mathcal{V}_2$ is a covering of K_2 with $n = (K_2 : U)$ cosets, then $\mathcal{V}_1 \cup \mathcal{V}_2$ is a covering of $K_1 \cup K_2$ with $m+n$ elements, and hence $(K_1 \cup K_2 : U) \leq m+n$.

Now, in a similar manner to above, the map $\Psi : X \to \mathbb{R}$ defined by $\Psi(f) := f(K_1) + f(K_2) - f(K_1 \cup K_2)$ is continuous and non-negative on each $C(V)$, and in particular on φ. So $\varphi(K_1 \cup K_2) \leq \varphi(K_1) + \varphi(K_2)$, and we have proved (b).

To show part (c), we will require the fact that, for all $U \in \mathcal{U}$,

$$\varphi_U(K_1 \cup K_2) = \varphi_U(K_1) + \varphi_U(K_2), \tag{7}$$

whenever K_1U^{-1} and K_2U^{-1} are disjoint. Since we have already proven one inequality in part (b) above, we can show this by checking that

$$\varphi_U(K_1 \cup K_2) \geq \varphi_U(K_1) + \varphi_U(K_2), \tag{8}$$

whenever K_1U^{-1} and K_2U^{-1} are disjoint. So let $n = (K_1 \cup K_2 : U)$, and let $x_1, \ldots, x_n$ be a finite sequence of elements of Γ such that $\{x_iU\}_{i=1}^n$ is an open covering of $K_1 \cup K_2$. Each set x_iU intersects at most one of K_1 or K_2, since otherwise $K_1U^{-1} \cap K_2U^{-1}$ would be non-empty. Thus we can re-index the x_i to form two shorter finite sequences $\{x_i\}_{i=1}^k$ and $\{x_i\}_{i=k+1}^n$ such that

$$K_1 \subseteq \bigcup_{i=1}^{k} x_iU, \qquad \text{and} \qquad K_2 \subseteq \bigcup_{i=k+1}^{n} x_iU.$$

From this, we obtain inequality (8), and together with part (b), this proves the equality of (7).

Now, let $K_1, K_2 \in \mathcal{K}$ be disjoint, and use the *Separation Lemma* (Lem 1.12) to find disjoint open subsets U_1 and U_2 of Γ which contain K_1 and K_2 respectively. By Lem 1.15, there exist $V_1, V_2 \in \mathcal{U}$ such that $K_1V_1 \subseteq U_1$ and $K_2V_2 \subseteq U_2$. Write $V := V_1 \cap V_2$, so that K_1V and K_2V are disjoint. Then for each $U \in \mathcal{U}$ such that $U \subseteq V^{-1}$,

$$\varphi_U(K_1 \cup K_2) = \varphi_U(K_1) + \varphi_U(K_2),$$

by property (7). Hence the map $\Psi(f) := f(K_1) + f(K_2) - f(K_1 \cup K_2)$ takes value zero at each element of $C(V^{-1})$. In particular $\varphi \in C(V^{-1})$, and so $\varphi(K_1 \cup K_2) = \varphi(K_1) + \varphi(K_2)$ whenever K_1 and K_2 are disjoint.

Part (d) can be shown by a similar argument to those of (a) and (b), using the simple projection $f \mapsto f(K)$ for $f \in X$, and the fact that $\varphi_U \geq 0$.

⋆ ⋆ ⋆

Our candidate measure φ still does not fit all the criteria of the theorem. We must now extend φ so it is defined on all subsets of Γ. For all open subsets $E \subseteq \Gamma$, define the function μ^* by:

$$\mu^*(E) = \sup\{\varphi(K) \mid K \subseteq E \text{ and } K \in \mathcal{K}\},$$

and as an extension of this to arbitrary subsets $A \subseteq \Gamma$, define $\mu^*(A)$ by:

$$\mu^*(A) := \inf\{\mu^*(E) \mid A \subseteq U \text{ and } E \text{ is open}\}.$$

We claim that this extended function μ^* is an outer measure on Γ.[2] It is easy to see that $\mu^*(A) \geq 0$ for all $A \subseteq \Gamma$: we need only verify that μ^* is non-negative on $\mathcal{K}$, which becomes clear once we consider again the projection $f \mapsto f(K)$ from X to $\mathbb{R}$.

Trivially $\mu^*(\emptyset) = 0$, since $(\emptyset : U) = 0$ for all $U \in \mathcal{U}$. By construction, it is also clear that $\mu^*(A_1) \leq \mu^*(A_2)$ whenever $A_1 \subseteq A_2$, and so it only remains to check countable subadditivity, that is, that each sequence (A_i) of subsets of Γ satisfies

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i).$$

Firstly, let (E_i) be a sequence of open subsets of Γ, and let $K \subseteq \bigcup_i E_i$ be compact. Then (E_i) has a finite subsequence $E_1, \dots, E_k$ which covers K, and by applying Lem 1.14 inductively, we can partition K into compact subsets $K_1, \dots, K_k$ such that

$$\bigcup_{i=1}^{k} K_i = K, \qquad \text{and} \qquad K_i \subseteq E_i,$$

for each $1 \leq i \leq k$. Then, by using property (a) of φ inductively, and from the definition of μ^*, it follows that

$$\begin{aligned} \varphi(K) &\leq \sum_{i=1}^{k} \varphi(K_i) \\ &\leq \sum_{i=1}^{k} \mu^*(E_i) \\ &\leq \sum_{i=1}^{\infty} \mu^*(E_i). \end{aligned}$$

Then

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup\left\{\varphi(K) \,\middle|\, K \subseteq \bigcup_{i=1}^{\infty} E_i \text{ and } K \in \mathcal{K}\right\} \leq \sum_{i=1}^{\infty} \mu^*(E_i),$$

and so μ^* is countably-subadditive on the collection of *open* subsets of Γ. Now, let (A_n) be a sequence of *arbitrary* subsets of Γ, and assume that $\sum_{i=n}^{\infty} A_i < \infty$ (if the sum totalled infinity, the result would trivially hold). Let $\varepsilon > 0$ be an arbitrary constant, and for each $n \in \mathbb{N}$, pick an open set U_n

[2]Recall from Def 1.9 that an outer measure μ^* takes non-negative values, is monotonic, countably-subadditive, and has null empty set, that is, $\mu^*(\emptyset) = 0$.

containing A_n and such that $\mu^*(U_n) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}$. Then

$$\begin{aligned}\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) &\leq \mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \\ &\leq \sum_{n=1}^{\infty} \mu^*(E_n) \\ &\leq \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= \sum_{n=1}^{\infty} \mu^*(A_n) + \frac{\varepsilon}{2}.\end{aligned}$$

Since we can set ε to be arbitrarily small, it follows that μ^* is countably-subadditive on the collection of *all* subsets of Γ, and so μ^* is an outer measure on Γ.

⋆ ⋆ ⋆

We now need to prove that each open subset of Γ is μ^*-measurable. Using observation (1) of *Carathéodory's criterion*, this entails checking that

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c),$$

for all subsets $A \subseteq \Gamma$ and open subsets $E \subseteq \Gamma$. We may assume that $\mu^*(A) < \infty$, since if $\mu^*(A)$ were infinite, then the result would trivially hold. Let $\varepsilon > 0$ be an arbitrary constant, and pick an open subset $F \subseteq \Gamma$ containing A and such that $\mu^*(F) \leq \mu^*(A) + \frac{\varepsilon}{3}$. Pick a compact subset $K \subseteq F \cap E$ such that

$$\varphi(K) > \mu^*(F \cap E) - \frac{\varepsilon}{3}, \tag{9}$$

and pick a compact subset $L \subseteq F \cap K^c$ such that

$$\varphi(L) > \mu^*(F \cap K^c) - \frac{\varepsilon}{3}.$$

Then K and L are disjoint, and by the fact that $K \subseteq E$, it follows that $F \cap E^c \subseteq F \cap K^c$. Hence

$$\varphi(L) > \mu^*(F \cap K^c) - \frac{\varepsilon}{3} \geq \mu^*(F \cap E^c) - \frac{\varepsilon}{3}. \tag{10}$$

Thus, by combining (9), (10), and property (c) of φ, it follows that

$$\begin{aligned}\mu^*(A\cap E)+\mu^*(A\cap E^c)-\frac{2\varepsilon}{3} &\leq \mu^*(F\cap E)+\mu^*(F\cap E^c)-\frac{2\varepsilon}{3}\\ &< \varphi(K)+\varphi(L)\\ &= \varphi(K\cup L)\\ &\leq \varphi(F)\\ &\leq \mu^*(F)\\ &\leq \mu^*(A)+\frac{\varepsilon}{3},\end{aligned}$$

and so

$$\mu^*(A\cap E)+\mu^*(A\cap E^c)<\mu^*(A)+\varepsilon.$$

Since we can set ε to be arbitrarily small, it follows that U is μ^*-measurable, and so $\mathcal{B}(\Gamma)$ is contained in the σ-algebra generated by all μ^*-measurable sets. Then, by the same method as in Prop 1.3, we can restrict μ^* to $\mathcal{B}(\Gamma)$ so that it becomes a full measure μ.

We now check that μ is regular on $\mathcal{B}(\Gamma)$. Let $K\in\mathcal{K}$ and let $E\subseteq\Gamma$ be an open subset containing K. Then $\varphi(K)\leq\mu(E)$, and by the definition of $\mu^*(K)$, it follows that

$$\varphi(K)\leq\mu(K). \tag{11}$$

Furthermore, we can use Lem 1.13 to find an open subset $V\subseteq\Gamma$ with compact closure, such that $K\subseteq V\subseteq\overline{V}\subseteq U$. Then

$$\mu(K)\leq\mu(V)\leq\varphi(\overline{V}),$$

and hence $\mu(K)<\infty$ for all $K\in\mathcal{K}$. Clearly μ is outer regular by the infimum definition of μ^*, and is inner regular by the supremum condition together with (11). Hence μ is a Borel regular measure.

It remains to check that μ is non-zero and translation-invariant. Observe that $\varphi_U(K_0)=1$ for each $U\in\mathcal{U}$, and the projection from $X\to\mathbb{R}$ which maps $f\mapsto f(K_0)$ is constant on each $C(U)$. In particular, $\varphi(K_0)=1$, and so $\mu(K_0)\geq 1$.

Finally, let $K\in\mathcal{K}$, let $U\in\mathcal{U}$, and let $x_1,\ldots,x_n$ be a finite list of elements of Γ such that

$$K\subseteq\bigcup_{i=1}^{n}x_iU.$$

Then

$$yK\subseteq\bigcup_{i=1}^{n}yx_iU,$$

for some fixed element $y\in\Gamma$, and so $(K:U)=(gK:U)$ for each $U\in\mathcal{U}$. Hence $\varphi_U(K)=\varphi_U(yK)$ for each $U\in\mathcal{U}$. Consider the map $\Theta:X\to\mathbb{R}$

defined by $\Theta(f) := f(K) - f(yK)$ for $f \in X$ and $K \in \mathcal{K}$. As above (see p16), we can show that this map is continuous and zero on each $C(U)$, and hence that $\mu(K) = \mu(yK)$.

So μ is a non-zero regular Borel measure which is left-invariant, and therefore is a left Haar measure on Γ [17, §7]. $\square$

Examples 1.18.

1. The Lebesgue measure λ on the additive group $\mathbb{R}$, when restricted to the Borel algebra $\mathcal{B}(\mathbb{R})$, is both a left and right Haar measure.
2. Write $\mathbb{R}^\times$ to denote the multiplicative group of positive real numbers, and define the function $\mu : \mathcal{B}(\mathbb{R}^\times) \to \mathbb{R}$ by:
$$\mu(A) := \int_A \frac{\mathrm{d}x}{x},$$
for all subsets $A \in \mathcal{B}(\mathbb{R}^\times)$. Then μ is a left Haar measure [37].
3. The counting measure ν on a discrete group Γ is clearly both a left and right Haar measure.

Given a locally compact group Γ, there may not necessarily be a canonical choice of Haar measure with which to endow it. It turns out, however, that this decision is unimportant, as any two Haar measures on Γ differ only by a constant – hence Haar measure is, in a sense, unique.

Definition 1.19 (Compact support). Let X be a locally compact Hausdorff space, and let $f : X \to \mathbb{R}$ be a map. Recall that the *support* of f is defined as the set
$$\mathrm{supp}(f) := \overline{\{x \in X \mid f(x) \neq 0\}}.$$
We say that f has *compact support* if $\mathrm{supp}(f)$ is compact as a subset of X. We write $\mathcal{K}_0(X)$ to denote the set of all continuous functions $f : X \to \mathbb{R}$ with compact support [23, p225].

Lemma 1.20 (Left and right uniform continuity). *Let Γ be a locally compact group, let $f \in \mathcal{K}_0(\Gamma)$, and let $\varepsilon > 0$ be an arbitrary constant. Then there exists an open neighbourhood W^L of the identity e in Γ such that $|f(x) - f(y)| < \varepsilon$ whenever $y \in xW^L$, for all $x \in \Gamma$. This is to say that f is* left uniformly continuous.

Likewise, there exists an open neighbourhood W^R of e such that $|f(x) - f(y)| < \varepsilon$ whenever $y \in W^R x$, for all $x \in \Gamma$. This is to say that f is right uniformly continuous *[5, p282].*

Proof. Let $f \in \mathcal{K}_0(\Gamma)$, and write $K := \text{supp}(f)$. Since f is continuous, for each $x \in K$ we can find an open neighbourhood U_x of the identity e such that $|f(x) - f(y)| < \frac{\varepsilon}{2}$ whenever $y \in xU_x$ (as in the proof of Lem 1.15). Then, again by Lem 1.15, we can find an open neighbourhood V_x of e such that $V_xV_x \subseteq U_x$. The collection $\{xV_x\}_{x\in K}$ forms an open covering of K, and since K is compact, there is a finite sub-collection of points $x_1, \ldots, x_n$ in K such that

$$K \subseteq \bigcup_{i=1}^{n} x_i V_{x_i}.$$

Define the set V by:

$$V := \bigcap_{i=1}^{n} V_{x_i},$$

and then define $W := V \cap V^{-1}$, such that W is a symmetric open neighbourhood of e.

Let $y \in xW$, and suppose firstly that $x, y \notin K$. Then $|f(x) - f(y)| = 0$, and the result immediately follows. Assume, then, that $x \in K$. Then there exists some natural number k with $1 \leq k \leq n$ such that $x \in x_k V_{x_k}$, and hence that $x \in x_k U_{x_k}$. But also

$$y \in xW \subseteq x_k V_{x_k} V_{x_k} \subseteq x_k U_{x_k},$$

and so

$$|f(x) - f(y)| \leq |f(x) - f(x_k)| + |f(x_k) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence f is continuous in the case where $x \in K$, and it remains to consider the case where $y \in K$. Since $y \in xW$, we can write $y = xw$ for some $w \in W$, and hence $x = yw^{-1}$. But W is symmetric, so $w^{-1} \in W$, and hence $x \in yW$. We can therefore use the same argument as above, interchanging x and y, to show that f is again continuous. Taking $W^L := W$, we have therefore shown that f is left uniformly continuous.

We can use an almost identical argument to show that f is also right uniformly continuous [5, p282]. $\square$

Lemma 1.21. *Let Γ be a locally compact group equipped with a Haar measure μ, and let $f \in \mathcal{K}_0(\Gamma)$. Then the functionals $L, R \in (\mathcal{K}(\Gamma))^*$ defined by:*

$$L(f)(x) := \int_\Gamma f(tx)\, d\mu(t), \quad \text{and} \quad R(f)(x) := \int_\Gamma f(xt)\, d\mu(t)$$

are continuous [5, p282].

Proof. Let $x_0 \in \Gamma$, and use Lem 1.13 to find an open neighbourhood V of x_0 with compact closure. Write $K := \operatorname{supp}(g)$, and note that since K is compact, it follows by *Tychonoff's Theorem* (Thm 1.11) that the set

$$K \times \overline{V}^{-1} := \left\{ (x, v) \,\middle|\, x \in K \text{ and } v^{-1} \in \overline{V} \right\}$$

is compact. Define the function $p(x, v) := xv$ for all $x \in K$ and $v \in \overline{V}^{-1}$, and observe that p is continuous. Hence

$$K\overline{V}^{-1} := \left\{ xv \,\middle|\, x \in K \text{ and } v \in \overline{V}^{-1} \right\} = p\left[K \times \overline{V}^{-1} \right],$$

and so $K\overline{V}^{-1} \subseteq \Gamma$ is a compact subset. Now, let $\varepsilon > 0$ be an arbitrary constant. Since $K\overline{V}^{-1}$, it follows that $\mu(K\overline{V}^{-1}) < \infty$, and so we can find a constant $\delta > 0$ such that

$$\delta\mu\left(K\overline{V}^{-1}\right) < \varepsilon.$$

Since $g \in \mathcal{K}_0$, it is left uniformly continuous by Lem 1.20. We can therefore find an open neighbourhood W of the identity e such that $|g(x) - g(y)| < \delta$ whenever $y \in xW$, for all $x \in \Gamma$. Then for each $x \in (V \cap x_0 W)$ and each $t \in \Gamma$, we have that $tx \in tx_0 W$. It follows that

$$\begin{aligned} |L(f)(x) - L(f)(x_0)| &\leq \left| \int_\Gamma f(tx)\, \mathrm{d}\mu(t) - \int_\Gamma f(tx_0)\, \mathrm{d}\mu(t) \right| \\ &\leq \int_\Gamma |f(tx) - f(tx_0)|\, \mathrm{d}\mu(t) \\ &\leq \delta\mu\left(K\overline{V}^{-1}\right) < \varepsilon, \end{aligned}$$

since the continuous map which takes $t \mapsto f(tx)$ vanishes whenever $t \notin K\overline{V}^{-1}$. Since we can set ε to be arbitrarily small, and our choice of $x_0 \in \Gamma$ was also arbitrary, this proves that $L(f)(x)$ is continuous. The proof that $R(f)(x)$ is continuous follows an almost identical argument [5, p282]. $\square$

Theorem 1.22 (Haar-von Neumann). *Let Γ be a locally compact group, and let μ and μ' be left Haar measures on Γ. Then there exists a positive constant k such that $\mu' = k\mu$ [5, p290] [25, p65].*

Proof. Firstly, we show that each non-empty open subset $U \subseteq \Gamma$ has non-zero Haar measure. Since μ is not identically zero, we can find some subset of Γ with non-zero measure, and hence by the inner regularity of μ, we can find a compact set K with $\mu(K) > 0$. Let $U \subseteq \Gamma$ be a non-empty open subset, such that $\{xU\}_{x \in \Gamma}$ is an open covering of K. Then, since K is compact, there is a finite sequence of elements $x_1, \dots x_n$ of Γ such that

$\{x_i U\}_{i=1}^n$ covers K. Since

$$\mu(K) \leq \sum_{i=1}^{n} \mu(x_i U),$$

it follows by the fact that μ is translation-invariant that

$$\mu(K) \leq \underbrace{\mu(U) + \cdots + \mu(U)}_{n \text{ times}} = n\mu(U),$$

and hence that $\mu(U) > 0$.

Fix a non-negative function $g \in \mathcal{K}_0(\Gamma)$ which is not the zero function. Then there exists a positive constant $\varepsilon > 0$ and a non-empty open set U such that $g \geq \varepsilon \mathbf{1}_U$, where $\mathbf{1}_U$ is the indicator function of U.[3] By the above, it then follows that $\int_\Gamma g \,\mathrm{d}\mu' \geq \varepsilon\mu'(U) > 0$.

Let $f \in \mathcal{K}_0(\Gamma)$ be an arbitrary function, and define the map $h : \Gamma \times \Gamma \to \mathbb{R}$ by:

$$h(x, y) := \frac{f(x)g(yx)}{\int_\Gamma g(tx)\,\mathrm{d}\mu'(t)},$$

for all $x, y, t \in \Gamma$. By the above, the denominator is non-zero, and so the ratio is defined everywhere.

We claim that h belongs to $\mathcal{K}_0(\Gamma \times \Gamma)$. Indeed, h is certainly compactly supported since f and g are, so it is sufficient to show that h is continuous. We know that f and g are continuous, and by Lem 1.21, the denominator

$$L(g)(x) := \int_\Gamma g(tx)\,\mathrm{d}\mu'(t)$$

is also continuous; hence $h \in \mathcal{K}_0(\Gamma \times \Gamma)$. Now, it follows from a version of *Fubini's Theorem* (Prop A.2) that

$$\int_\Gamma \int_\Gamma h(x, y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu'(y) = \int_\Gamma \int_\Gamma h(x, y)\,\mathrm{d}\mu'(y)\,\mathrm{d}\mu(x). \tag{12}$$

[3]Recall that the *indicator function* $\mathbf{1}_A : \Gamma \to \{0, 1\}$ of a subset $A \subseteq \Gamma$ is defined by:

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in \Gamma$.

Hence, by the translation-invariance of μ, it follows that

$$\begin{aligned}\int_\Gamma\int_\Gamma h(x,y)\,\mathrm{d}\mu'(y)\,\mathrm{d}\mu(x) &= \int_\Gamma\int_\Gamma h(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu'(y)\\ &= \int_\Gamma\int_\Gamma h(y^{-1}x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu'(y)\\ &= \int_\Gamma\int_\Gamma h(y^{-1}x,y)\,\mathrm{d}\mu'(y)\,\mathrm{d}\mu(x)\\ &= \int_\Gamma\int_\Gamma h(y^{-1},xy)\,\mathrm{d}\mu'(y)\,\mathrm{d}\mu(x).\end{aligned}$$

Now, substituting in our function h, we have that

$$\begin{aligned}\int_\Gamma f(x)\,\mathrm{d}\mu(x) &= \int_\Gamma f(x)\frac{\int_\Gamma g(yx)\,\mathrm{d}\mu'(y)}{\int_\Gamma g(tx)\,\mathrm{d}\mu'(t)}\,\mathrm{d}\mu(x)\\ &= \int_\Gamma\int_\Gamma h(x,y)\,\mathrm{d}\mu'(y)\,\mathrm{d}\mu(x)\\ &= \int_\Gamma\int_\Gamma h(y^{-1},xy)\,\mathrm{d}\mu'(y)\,\mathrm{d}\mu(x)\\ &= \int_\Gamma\int_\Gamma \frac{f(y^{-1})g(x)}{\int_\Gamma g(ty^{-1})\,\mathrm{d}t}\,\mathrm{d}\mu'(y)\,\mathrm{d}\mu(x)\\ &= \int_\Gamma g(x)\,\mathrm{d}\mu(x)\int_\Gamma \frac{f(y^{-1})}{\int_\Gamma g(ty^{-1})\,\mathrm{d}\mu'(t)}\,\mathrm{d}\mu'(y).\end{aligned}$$

Hence $\int_\Gamma f(x)\,\mathrm{d}\mu(x) = C\int_\Gamma g(x)\,\mathrm{d}\mu(x)$, where C is a constant which depends on f and g, but is independent of μ. Therefore we can write

$$\int_\Gamma f\,\mathrm{d}\mu' = k\int_\Gamma f\,\mathrm{d}\mu,$$

where

$$k := \frac{\int_\Gamma g\,\mathrm{d}\mu'}{\int_\Gamma g\,\mathrm{d}\mu},$$

and from this it follows that $\mu' = k\mu$ [5, pp290-1]. □

Corollary (Theorem 1.6). *The Lebesgue measure λ is the* unique *complete translation-invariant measure on $\mathcal{B}(\mathbb{R})$ such that $\lambda[0,1] = 1$ [31, p623].*

Proof. The Lebesgue measure $\lambda : \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ can clearly be seen to be a Haar measure, and as such, is unique up to multiplication by a constant. By normalising the measure of the interval $[0,1]$, we certify the uniqueness of λ. □

Definition 1.23 (Left and right translate). Let Γ be a group, and let A be an arbitrary set. Let $f : \Gamma \times A \to A$ be a continuous function, and write $f(x, a) = x \cdot a$, for all $x \in \Gamma$ and $a \in A$. We say that f is a *group action* of Γ on A if it has the following properties:

(a) $e \cdot a = a$, where $e \in \Gamma$ is the identity, and $a \in A$ is arbitrary, and

(b) $(xy) \cdot a = x \cdot (y \cdot a)$, for all $x, y \in \Gamma$ and $a \in A$ [23, p199].

For each $a \in A$, we define the Γ-*orbit* of a to be the set $\Gamma a := \{x \cdot a \,|\, x \in \Gamma\}$.

We write A^Γ to denote the set of functions on Γ whose values lie in A. Fix an element $x \in \Gamma$, and let $f \in A^\Gamma$. We define the following actions of the group Γ on A^Γ:

1. The *left translate* of f by x is the function ${}_xf \in A^\Gamma$ defined by ${}_xf(s) := f(x^{-1}s)$, for all $s \in \Gamma$.
2. The *right translate* of f by x is the function $f_x \in A^\Gamma$ defined by $f_x(s) := f(sx^{-1})$, for all $s \in \Gamma$.

Notice that the identity $e \in \Gamma$ has the property that ${}_ef(s) = f_e(s) = f(s)$, for all $s \in \Gamma$. Furthermore, if $x, y \in \Gamma$, then

$${}_{xy}f(s) = f((xy)^{-1}s) = f(y^{-1}x^{-1}s) = {}_yf(x^{-1}s) = {}_x({}_yf(s)),$$

and

$$f_{xy}(s) = f(s(xy)^{-1}) = f(sy^{-1}x^{-1}) = f_x(sy^{-1}) = (f_x)_y(s),$$

for all $s \in \Gamma$. Hence each of the above functions does indeed define a group action of Γ on A^Γ. We will also make use of the following function:

3. Given a function $f \in A^\Gamma$, we define the "symmetric" function $\check{f} \in A^\Gamma$ by $\check{f}(s) := f(s^{-1})$, for all $s \in \Gamma$ [25, p64].

Proposition 1.24. *Let Γ be a locally compact group, and let μ be a regular Borel measure on Γ. Then μ is a left (resp. right) Haar measure on Γ if and only if $\check{\mu}$ is a right (resp. left) Haar measure on Γ [5, p293].*

Proof. Firstly, we claim that $\check{\mu}$ is a Borel regular measure on Γ. Indeed, a subset $U \subseteq \Gamma$ is open if and only if $U^{-1} \subseteq \Gamma$ is open. Write $\mathcal{S}$ to denote the collection of subsets $S \subseteq \Gamma$ such that $S^{-1} \in \mathcal{B}(\Gamma)$. Then the sets in $\mathcal{S}$ are open and in one-to-one correspondence with the sets in $\mathcal{B}(\Gamma)$. Therefore $\mathcal{S}$ is a σ-algebra which contains all Borel subsets of Γ. Hence a subset $A \subseteq \Gamma$ is a Borel set if and only if $A^{-1} \subseteq \Gamma$ is a Borel set.

Since the map which sends $x \mapsto x^{-1}$, for $x \in \Gamma$, is continuous, it follows that a subset $K \subseteq \Gamma$ is compact if and only if K^{-1} is compact. Hence $\check{\mu}$ is finite on compact sets, and so is a Borel measure.

By the outer regularity of μ, it follows that

$$\check{\mu}(A) = \sup\{\mu(K) \mid K^{-1} \subseteq A, \text{ and } K \subseteq \Gamma \text{ is compact}\}.$$

But we have seen that K^{-1} is compact if and only if K is compact, and $K^{-1} \subseteq A^{-1}$ if and only if $K \subseteq A$, so

$$\check{\mu}(A) = \sup\{\check{\mu}(K) \mid K \subseteq A, \text{ and } K \subseteq \Gamma \text{ is compact}\},$$

which is to say that $\check{\mu}$ is inner regular. We can use a similar argument to show that $\check{\mu}$ is outer regular, and so $\check{\mu}$ is a Borel regular measure.

Now, let μ be left-invariant. Then $\check{\mu}$ is right-invariant, since $(Ax)^{-1} = x^{-1}A^{-1}$, and so

$$\check{\mu}(Ax) = \mu(x^{-1}A^{-1}) = \mu(a^{-1}) = \check{\mu}(A).$$

Similarly, if μ is right-invariant, it follows that

$$\check{\mu}(xA) = \mu(A^{-1}x^{-1}) = \mu(A^{-1}) = \check{\mu}(A),$$

which is to say that $\check{\mu}$ is left-invariant [5, p293] [25, p64]. □

1.5 Invariant Means on Locally Compact Groups

Definition 1.25 (Measurable functions). Let (X, Σ_X) and (Y, Σ_Y) be *measurable spaces*, that is, sets X and Y equipped with σ-algebras Σ_X and Σ_Y respectively [5, p8]. A function $f : X \to Y$ is said to be *measurable* if, for each set A in Σ_Y, the preimage $f^{-1}(A)$ belongs to Σ_X.

Now we endow the space X with a measure μ, and let $f : X \to \mathbb{R}$ be a measurable function. We say that f is *bounded* if there exists some constant $C \in \mathbb{R}$ such that $|f(x)| < C$ for all $x \in X$.

We call a function $g : X \to \mathbb{R}$ *essentially bounded* if it is *almost equal* to a bounded function f, that is, if

$$\mu(\{x \in X \mid g(x) \neq f(x)\}) = 0.$$

We write $\mathcal{L}^\infty(X)$ to denote the space of all such essentially bounded measurable functions, and write $L^\infty(X)$ for the quotient space $\mathcal{L}^\infty(X)/\sim$, where the relation $\sim$ identifies functions which are almost equal [34, p4].

We define the norm $\|g\|_\infty$ of a function $g \in L^\infty(X)$ as the *essential supremum* of g, that is, the infimum of all $C \in \mathbb{R}$ such that

$$\mu\left(\{x \in X \mid |g(x)| > C\}\right) = 0.$$

[33, p318]

Definition 1.26 (Means on locally compact groups). Let Γ be a locally compact group, and let $m : L^\infty(\Gamma) \to \mathbb{R}$ be a bounded linear functional. We call m a *left-invariant mean* if it has the following properties:

(a) $m(f) \geq 0$ whenever $f \geq 0$,

(b) $m(\mathbf{1}_\Gamma) = 1$, where $\mathbf{1}_\Gamma$ is the indicator function of Γ,

(c) $m({}_x f) = m(f)$, for all $x \in \Gamma$ and $f \in L^\infty(\Gamma)$.

A locally compact group Γ is said to be *amenable* if it admits a left-invariant mean.

Examples 1.27.

1. Let Γ be a discrete group, and suppose that Γ is amenable with invariant mean $m : L^\infty(\Gamma) \to \mathbb{R}$. Define the function $\mu : \mathcal{P}(\Gamma) \to \mathbb{R}$ by $\mu(A) := m(\mathbf{1}_A)$, for all $A \subseteq \Gamma$.

   Observe that, if $A, B \subseteq \Gamma$ are disjoint subsets, then

   $$\begin{aligned}\mu(A \cup B) &= m(\mathbf{1}_{A\cup B}) \\ &= m(\mathbf{1}_A) + m(\mathbf{1}_B) \quad \text{(since } m \text{ is linear)} \\ &= \mu(A) + \mu(B),\end{aligned}$$

   which is to say that μ is finitely-additive. Furthermore, since $m(\mathbf{1}_\Gamma) = 1$, it follows that $\mu(\Gamma) = 1$, and from the linearity of m it follows that $\mu(\emptyset) = 0$. Finally, for each $x \in \Gamma$ and $A \subseteq \Gamma$:

   $$\begin{aligned}\mu(xA) &= m(\mathbf{1}_{xA}) \\ &= m(x\mathbf{1}_A) \\ &= m(\mathbf{1}_A) = \mu(A),\end{aligned}$$

   and so μ is left-invariant. Hence μ is a left-invariant measure in the sense of Def 1.7. Conversely, if Γ has a left-invariant measure μ, then we may consider $L^\infty(\Gamma)$ as a measure space with respect to μ. We can then construct a mean in the sense of Def 1.26, and so these two definitions of amenability are equivalent [10, p4].

2. Let Γ be a finite group. Then we can equip Γ with the discrete topology and the counting measure ν. Hence Γ is amenable by Ex 1.8.

3. The additive group of integers $\mathbb{Z}$ is amenable. To show this, let $\varepsilon > 0$ be an arbitrary constant, and write M_ε to denote the set of means $\mu : \mathcal{P}(\mathbb{Z}) \to [0, 1]$ such that

   $$|\mu(A) - \mu(1 + A)| < \varepsilon, \tag{13}$$

   for all subsets $A \subseteq \mathbb{Z}$, and where $1 + A := \{1 + a \,|\, a \in A\}$. Pick a natural number N such that $N < \frac{2}{\varepsilon}$. Then, for each $\varepsilon > 0$, define the function $\mu_\varepsilon : \mathcal{P}(\mathbb{Z}) \to \mathbb{R}$ by:

   $$\mu_\varepsilon(A) := \frac{|\{i \in A, \,|\, 1 \leq i \leq N\}|}{N},$$

   for all $A \subseteq \mathbb{Z}$. Then $|\mu_\varepsilon(A) - \mu_\varepsilon(1 + A)| \leq \frac{2}{N} < \varepsilon$, and so $\mu_\varepsilon \in M_\varepsilon$; in particular, each M_ε is non-empty. It is not difficult to verify that each M_ε is closed, and that if $\varepsilon_1, \ldots, \varepsilon_n$ is a finite sequence of positive numbers, then

   $$\bigcap_{i=1}^{n} M_{\varepsilon_i} = M_{\min\{\varepsilon_i\}}.$$

Hence the space $\{M_\varepsilon \,|\, \varepsilon > 0\}$ of closed subsets of $[0,1]^{\mathcal{P}(\Gamma)}$ has the *finite intersection property* (Prop A.3), and so is compact. By *Tychonoff's Theorem* (Thm 1.11), the set $[0,1]^{\mathcal{P}(\mathbb{Z})}$ is compact, and since $\{M_\varepsilon \,|\, \varepsilon > 0\} \subseteq [0,1]^{\mathcal{P}(\mathbb{Z})}$, there exists a mean $\hat{\mu}$ in the intersection $\bigcap_{\varepsilon>0} M_\varepsilon$ which satisfies (13).

Since we can set ε to be arbitrarily small, this shows that $\hat{\mu}$ is left-invariant with respect to 1. But 1 is a generator of $\mathbb{Z}$ as an additive group, which is to say that each $k \in \mathbb{Z}$ can be formed by an arbitrary sum of 1 and -1 (see Def 2.1). Hence $\hat{\mu}$ is left-invariant with respect to $\mathbb{Z}$, and so is a left-invariant mean. Thus $\mathbb{Z}$ is amenable [10, p6]. □

4. Let Γ be a compact group (that is, a group which is compact and Hausdorff as a metric space) equipped with left Haar measure μ. Define the linear functional $m : L^\infty(\Gamma) \to \mathbb{R}$ by:

$$m(f) := \frac{1}{\mu(\Gamma)} \int_\Gamma f \, \mathrm{d}\mu,$$

which is well-defined since $0 < \mu(\Gamma) < \infty$. This can clearly be seen to be an invariant mean on Γ.

Lemma 1.28. *Let (Γ_n) be a sequence of discrete amenable groups. Then the disjoint union $\Gamma := \amalg_{n\in\mathbb{N}}\Gamma_n$ is also amenable [10, p6].*

Proof. Firstly, since each group Γ_n is amenable, we may find a left-invariant mean $\mu_n : \Gamma_n \to [0,1]$ for each $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, define the closed set

$$M_n := \{\mu \,|\, \mu \text{ is a mean on } \Gamma, \text{ and } \mu(xA) = \mu(A) \text{ for all } x \in \Gamma_n,\ A \subseteq \Gamma\}.$$

Define the measure $\hat{\mu}_n : \mathcal{P}(\Gamma) \to [0,1]$ by $\hat{\mu}_n(A) := \mu_n(A \cap \Gamma_n)$, such that $\hat{\mu}_n \in M_n$ for each n; in particular, note that each M_n is non-empty. Observe that, if $\Gamma_i, \Gamma_j \subseteq \Gamma_k$, then $M_k \subseteq M_i \cap M_j$, and that, by *Tychonoff's Theorem* (Thm 1.11), the set $[0,1]^{\mathcal{P}(\mathbb{Z})}$ is compact.

Hence the space $\{M_n \,|\, n \in \mathbb{N}\}$ of closed subsets of $[0,1]^{\mathcal{P}(\Gamma)}$ has the *finite intersection property* (Prop A.3), and so we can find a mean $\hat{\mu} \in \bigcap_{n\in\mathbb{N}} M_n$. Thus $\hat{\mu}(xA) = \hat{\mu}(A)$ for all $x \in \Gamma$ and $A \subseteq \Gamma$, and so Γ is amenable [10, p6]. □

Lemma 1.29. *Let Γ be a discrete group, and let $N \trianglelefteq \Gamma$ be a normal subgroup such that N and G/N are both amenable. Then Γ is amenable [10, p6].*

In particular, the direct product of a finite number of amenable groups is amenable.

Proof. Suppose that $N \trianglelefteq \Gamma$ and Γ/N are amenable with left-invariant means μ_1 and μ_2 respectively. For each subset $A \subseteq \Gamma$, define the function $f_A : \Gamma \to \mathbb{R}$ by:

$$f_A(x) := \mu_1(N \cap x^{-1}A).$$

Note that, since $\mu_1(xN) = \mu_1(N)$ for all $x \in \Gamma$, we may define f_A on Γ/N. Now, define the map $\mu : \mathcal{P}(\Gamma) \to [0,1]$ by $\mu(A) := \int_\Gamma f_A \, \mathrm{d}\mu_2$. Clearly $\mu(\emptyset) = 0$ and $\mu(\Gamma) = 1$ by the fact that μ_1 and μ_2 are means. Furthermore, if $A, B \subseteq \Gamma$ are disjoint subsets, then

$$\begin{aligned}
\mu(A \cup B) &= \int_\Gamma f_{A\cup B}(x) \, \mathrm{d}\mu_2(x) \\
&= \int_\Gamma \mu_1\left(N \cap x^{-1}(A \cup B)\right) \mathrm{d}\mu_2(x) \\
&= \int_\Gamma \mu_1\left(\left(N \cap x^{-1}A\right) \cup \left(N \cap x^{-1}B\right)\right) \mathrm{d}\mu_2(x) \\
&= \int_\Gamma \mu_1\left(N \cap x^{-1}A\right) + \mu_1\left(N \cap x^{-1}B\right) \mathrm{d}\mu_2(x) \\
&= \int_\Gamma f_A(x) + f_B(x) \, \mathrm{d}\mu_2(x) = \mu(A) + \mu(B),
\end{aligned}$$

which is to say that μ is finitely-additive. Finally, note that

$$\begin{aligned}
\mu(xA) &= \int_\Gamma f_{xA}(x) \, \mathrm{d}\mu_2(x) \\
&= \int_\Gamma \mu_1(N \cap x^{-1}xA) \, \mathrm{d}\mu_2(x) \\
&= \int_\Gamma {}_x(f_A)(x) \, \mathrm{d}\mu_2(x) \\
&= \int_\Gamma f_A(x) \, \mathrm{d}\mu_2(x) = \mu(A),
\end{aligned}$$

since μ_2 is left-invariant. Hence μ is a left-invariant mean on Γ, and Γ is amenable [10, p6].

As an immediate consequence, if Γ and Δ are amenable groups, then $\Gamma \times \Delta$ is amenable, since $\Gamma \cong (\Gamma \times \Delta)/\Delta$. Hence, by induction, the direct product of finitely-many amenable groups is amenable. $\square$

Theorem 1.30. *Every discrete Abelian group Γ is amenable [10, p6].*

Proof. Firstly, we consider the case where Γ is a finitely-generated Abelian group, that is, where each $x \in \Gamma$ can be produced by the combination (under the group action) of a finite number of elements (see Def 2.1). By the *Fundamental Theorem of Finitely-Generated Abelian Groups* [24, p24], Γ can be decomposed as a direct sum of the form:

$$\Gamma = \mathbb{Z}^n \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_m},$$

where each p_i is a power of a prime number. Hence $\Gamma = \mathbb{Z}^n \oplus \Gamma'$, where Γ' is a finite group. By Ex 1.27(2), Γ' is amenable, and by Ex 1.27(3) and Lem 1.29, $\mathbb{Z}^n$ is amenable. Hence by Lem 1.29 again, it follows that Γ is amenable.

Since every Abelian group can be written as the direct union of its finitely-generated subgroups, it follows from Lem 1.28 that every discrete Abelian group is amenable [10, p6]. □

Examples 1.31.

1. Consider the additive group of real numbers $\mathbb{R}$ with the discrete topology – not the usual topology. Since $\mathbb{R}$ is Abelian, it is also amenable by the above theorem. Thus, there exists a finitely-additive translation-invariant measure μ which is defined on *all* subsets of $\mathbb{R}$, and not just the Borel sets. Note that this measure μ will have to be different to the Lebesgue measure λ.

2. Similarly, recall that the *special orthogonal group* $SO_2(\mathbb{R})$ can be defined as the multiplicative subgroup of $\mathbb{R}^2$ consisting of all elements which lie on the unit circle:

   $$SO_2(\mathbb{R}) := \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}.$$

   Observe that $SO_2(\mathbb{R})$ is Abelian. Then we can equip $SO_2(\mathbb{R})$ with the discrete topology, such that $SO_2(\mathbb{R})$ is amenable.

2 Følner Sequences

Amenable groups can be characterised more combinatorially by way of *Følner sequences*, collections of sets which remain "almost fixed" when acted upon by elements of the group. The motivation for this comes from the measures μ_ε constructed in the proof of the amenability of $\mathbb{Z}$ (Ex 1.27(3)), which is "almost invariant," that is, invariant to within an arbitrarily small $\varepsilon > 0$. We then showed that the "limit" of these measures was indeed invariant.

It turns out that the existence of a Følner sequence guarantees that the group is amenable in the same way, and this geometric viewpoint will shed light on some fresh examples of amenable groups. Firstly, we will spend some time reviewing a few core concepts from geometric group theory.

2.1 Quasi-Isometries

Definition 2.1 (Word metric). Let Γ be an arbitrary group, and let $S \subseteq \Gamma$ be a subset. Define the sets $S^{-1} := \{x^{-1} \,|\, x \in S\}$, and $S^{\pm} := S \cup S^{-1}$. We define a *word* $w := x_1 x_2 \cdots x_n$ in $S^{\pm}$ to be the product (under the group operation) of a finite string of elements of $S^{\pm}$.

A word is said to be *reduced* if it contains no strings of the form xx^{-1}, and two words are said to be *equivalent* if they are the same up to addition or removal of such strings. We define the *length* of a word $w := x_1 \cdots x_n$ to be the natural number n, and sometimes denote this by $|w|$.

We say that Γ is *generated by* S if each element of Γ can be expressed as a word consisting of elements of $S^{\pm}$. The elements of S are called the *generators* of Γ. We say that Γ is *finitely-generated* if it can be generated by a finite set.

Since *every* element $x \in \Gamma$ can be expressed as a word consisting of elements of S, we can define a function $d_S : \Gamma \times \Gamma \to \mathbb{R}$ by the length of the *shortest* word representing the element $x^{-1}y$, for all $x, y \in \Gamma$. The function d_S is easily verified to be a metric, and is often called the *word metric* with respect to S [3, p139].

Example 2.2. The *free group on* S, denoted $\mathbb{F}_S$ can be viewed as the set of all reduced words in the "alphabet" of S: each reduced word defines a distinct element of $\mathbb{F}_S$. If S has n elements, we can call $\mathbb{F}_S$ the *free group on* n *generators*, and denote this by $\mathbb{F}_n$ [1, pp70-5].

Definition 2.3 (Cayley graph). Let Γ be a group generated by a finite subset $S \subseteq \Gamma$. The *Cayley graph* of Γ with respect to S, denoted $\mathrm{Cay}(\Gamma; S)$, is a connected metric graph constructed as follows:

1. Assign to each element x of Γ a vertex, such that the vertices of $\mathrm{Cay}(\Gamma; S)$ are in one-to-one correspondence with Γ.
2. Draw an edge s of length one between x and xs for each $x \in \Gamma$ and $s \in S^{\pm}$.

Notice that, if the identity e is not contained in the generating set S, then the Cayley graph contains no loops. Furthermore, since the set $S^{\pm}$ is symmetric, the edges of $\mathrm{Cay}(\Gamma; S)$ need not be directed. For brevity, we will assume that all finite generating sets S are symmetric and do not contain the identity, unless otherwise stated. [3, p8] [28].

Observe that the word metric d_S on a group Γ is closely related to the Cayley graph; any two distinct vertices $x, y \in \mathrm{Cay}(\Gamma; S)$ are adjacent (i.e. connected by an edge) if and only if $d_S(x, y) = 1$. Indeed, if an element $x^{-1}y \in \Gamma$ can be expressed as a word of length n in S, then the vertices $x, y \in \mathrm{Cay}(\Gamma; S)$ can be joined by a path of length n.

Conversely, if $x, y \in \mathrm{Cay}(\Gamma; S)$ are joined by a path of length n, then we can find a word of length n to express $x^{-1}y \in \Gamma$. Hence the word metric finds the length of a *geodesic*, or shortest path from x to y in $\mathrm{Cay}(\Gamma; S)$ [15].

Since the generating set S is presumed to be finite, the Cayley graph is locally finite, and hence defines a proper geodesic space, that is, one in which all closed and bounded subsets are compact [6, p84] [36, pp11-2].

The structure of a finitely-generated group Γ as a metric space in the form of a Cayley graph clearly depends on the choice of generating set. We would like to study the geometry of such groups independently of the generating set, and this becomes easier to do once we "zoom out" and consider the large-scale structure.

Definition 2.4 (Quasi-isometry). Let X and Y be proper metric spaces, and let $f : X \to Y$ be a function between them. We say that f is a *quasi-isometry* if there exist constants $A \geq 1$ and $B, C \geq 0$ such that:

(a) For any two elements $x, x' \in X$:

$$\frac{1}{A} \cdot d_X(x, x') - B \leq d_Y(f(x), f(x')) \leq A \cdot d_X(x, x') + B,$$

(b) For each $y \in Y$, we can find an $x \in X$ such that $d_Y(f(x), y) \leq C$.

If f satisfies (a) alone, then we call f a *quasi-isometric embedding.*

If there exists a quasi-isometry $f : X \to Y$, then we say the spaces X and Y are *quasi-isometric.*

If we need to be more precise, and specify the constants A and B, we can say that the map f is an (A,B)-*quasi-isometric embedding* or an (A,B)-*quasi-isometry* [3, p138] [6, p85].

Examples 2.5.

1. A metric space X is quasi-isometric to a one-point space if and only if it is bounded, that is, if and only if there exists a constant $R \geq 0$ such that $d(x,y) \leq R$ for all $x, y \in X$.

2. More generally, if $A \subseteq X$ is a subset, then the inclusion $A \hookrightarrow X$ is a quasi-isometry if and only if A is *quasi-dense* in X, that is, if and only if we can find a constant $C > 0$ such that each point $x \in X$ lies in the C-neighbourhood of some point of A.

   The natural inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$, for example, is a quasi-isometry, since each point of $\mathbb{R}$ is within $\frac{1}{2}$ of a point of $\mathbb{Z}$ [3, p139].

Proposition 2.6.

Let Γ be a group with finite generating sets S and S'. Then Cay$(G;S)$ is quasi-isometric to Cay$(G;S')$ [3, p139-40].

Proof. We consider the identity map $i : (\Gamma, d_S) \to (\Gamma, d_{S'})$, and define the value

$$\alpha := \max\{d_{S'}(e,s) \,|\, s \in S^{\pm}\}.$$

Observe that, since S is finite, α is finite. Let $x, y \in \Gamma$, and write $n := d_S(x,y)$, such that $x^{-1}y = s_1 \cdots s_n$ for some $s_1, \ldots, s_n \in S$. Since the word metric is left-invariant, it follows that

$$\begin{aligned} d_{S'}(x,y) &= d_{S'}(x, gs_1 \cdots s_n) \\ &\leq d_{S'}(x, xs_1) + d_{S'}(xs_1, xs_1s_2) + \cdots + d_{S'}(xs_1 \cdots s_{n-1}, xs_1 \cdots s_n) \\ &= d_{S'}(e, s_1) + \cdots + d_{S'}(e, s_n) \\ &\leq \alpha n = \alpha d_S(x,y). \end{aligned}$$

By a similar argument, we can show that $d_S(x,y) \leq \beta d_{S'}(x,y)$ for some constant $\beta > 0$, and hence that i is a quasi-isometry. Now consider the composition

$$\mathrm{Cay}(\Gamma; S) \xrightarrow{\varphi} (\Gamma, d_S) \xrightarrow{i} (\Gamma, d_{S'}) \xrightarrow{\psi} \mathrm{Cay}(\Gamma; S'),$$

where φ maps a point $a \in \mathrm{Cay}(\Gamma;S)$ to some element $x \in \Gamma$ such that $d_S(a,x) \leq \frac{1}{2}$, and ψ is the inclusion. Clearly φ and ψ are quasi-isometries, and it is easy to verify that the composition of quasi-isometries is a quasi-isometry. Hence Cay$(G;S)$ is quasi-isometric to Cay$(G;S')$ [36, p18]. $\square$

Henceforth we will refer to "the" Cayley graph of a group, and this is well-defined up to quasi-isometry.

2.2 The Følner Criterion

Definition 2.7 (Følner criterion for discrete groups). We define the *symmetric difference* of two sets A and B by $A \triangle B := (A \setminus B) \cup (B \setminus A)$, that is, the set of points in precisely one of A or B.

Let Γ be a discrete group. Then Γ is said to satisfy the *Følner criterion* if, for each finite subset $A \subseteq \Gamma$ and each $\varepsilon > 0$, there exists a finite non-empty subset $F \subseteq \Gamma$ such that

$$\frac{|aF \triangle F|}{|F|} \leq \varepsilon,$$

for all $a \in A$. If Γ is countable, we define a *Følner sequence* to be a sequence (F_n) of finite non-empty subsets of Γ such that

$$\bigcup_{n \in \mathbb{N}} F_n = \Gamma, \qquad \text{and} \qquad \lim_{n \to \infty} \frac{|xF_n \triangle F_n|}{|F_n|} = 0,$$

for all $x \in \Gamma$. The sets F_n are often called *Følner sets* [10, p8]. In fact, possessing a Følner sequence is equivalent to satisfying the Følner criterion:

Proposition 2.8. *A discrete group* Γ *contains a Følner sequence if and only if it satisfies the Følner criterion.*

Proof. Firstly, suppose that Γ satisfies the Følner criterion, and let $A_1 \subseteq A_2 \subseteq \cdots$ be an ascending chain of finite subsets of Γ such that $\Gamma = \bigcup_{n \in \mathbb{N}} A_n$. Define $\varepsilon := \frac{1}{n}$ for each $n \in \mathbb{N}$. Then for each n, there exists a finite non-empty subset $F_n \subseteq \Gamma$ such that

$$\frac{|aF_n \triangle F_n|}{|F_n|} \leq \frac{1}{n},$$

for all $a \in A_n$. But each $x \in \Gamma$ is contained in some A_n, and so is contained in all A_m with $m \geq n$. It follows that

$$\lim_{n \to \infty} \frac{|xF_n \triangle F_n|}{|F_n|} = 0.$$

Conversely, suppose that Γ contains a Følner sequence (F_n). Then for each $\varepsilon > 0$ we can find a natural number N such that

$$\frac{|xF_n \triangle F_n|}{|F_n|} < \varepsilon,$$

for all $n \geq N$ and $x \in \Gamma$. Hence Γ satisfies the Følner criterion [9] [10, p9]. $\square$

Følner sequences in a discrete group Γ are naturally related to the Cayley graph of Γ with respect to some finite generating set S.

Let F be a finite subset of the vertices of $\mathrm{Cay}(\Gamma; S)$. We define the *boundary* of F, denoted by ∂F, to be the set of vertices in F^c which are adjacent to some vertex in F. It is not difficult to see that $|xF \triangle F| \leq |\partial F|$, since we discard all edges which do not have endpoints in precisely one of xF or F [4].

The Følner criterion can then be reformulated for Cayley graphs by saying that for each $\varepsilon > 0$, there exists a finite non-empty subset F of the vertices of $\mathrm{Cay}(\Gamma; S)$ such that

$$\frac{|\partial F|}{|F|} < \varepsilon.$$

[13, p11]

We will shortly show that a group satisfies the Følner criterion if and only if it admits a left-invariant mean, and hence that the Følner criterion gives an equivalent characterisation of amenability. In order to do this, we recall some fundamental structures from from functional analysis, and an important consequence of the *Hahn-Banach Theorem* (Thm A.4).

Definition 2.9 (Topological vector space). Let V be a vector space over the field $\mathbb{R}$. We call V a *topological vector space* if it is equipped with a *vector topology*, that is, a topology τ such that:

(a) Each singleton in V is closed, and

(b) The operations of addition and scalar multiplication on V are continuous with respect to τ [33, p7].

It is not difficult to show that these two conditions together imply that each topological vector space is also a Hausdorff space [33, p10].

Theorem 2.10 (Hahn–Banach Separation). *Let V be a topological vector space over the field $\mathbb{R}$, and let $A, B \subseteq V$ be non-empty subsets which are convex and disjoint. If A is open, then there exists a continuous linear map $f : V \to \mathbb{R}$ and a constant $t \in \mathbb{R}$ such that*

$$f(a) < t \leq f(b),$$

for all $a \in A$ and $b \in B$. (For a proof, we direct the reader to [33, p59].) □

In order to show that $\mathbb{Z}$ is amenable in Ex 1.27(3), we defined the space M_ε of finitely-additive probability measures which remain "almost fixed" under left translation. In fact, this space is fairly standard, and has diverse applications including in the proof of the following theorem, so we formalise its definition here.

Definition 2.11. Let Γ be a locally compact Hausdorff space. A mean $\mu : \mathcal{P}(X) \to [0,1]$ is said to be *almost invariant* if there exists a constant $\varepsilon > 0$ such that:

$$|\mu(A) - \mu(xA)| < \varepsilon,$$

for all subsets $A \subseteq X$, and $x \in X$. If we need to specify the constant ε, we can say that μ is *ε-almost-invariant*, and we write M_ε to denote the set of all such means. We write M_ε^B to denote the set of means which are ε-almost-invariant under left translation by elements from a given subset $B \subseteq X$.

Notice that each of these sets is closed.

Theorem 2.12 (Følner). *Let Γ be a finitely-generated discrete group. Then Γ satisfies the Følner criterion if and only if it is amenable [10, p9].*

Proof. The first direction of the proof follows the arguments used in Ex 1.27(3). Firstly, suppose that Γ satisfies the Følner criterion. Let $B \subseteq \Gamma$ be a subset, let $\varepsilon > 0$ be an arbitrary constant, and consider the set M_ε^B. Since each M_ε^B is a closed subset of the compact set $[0,1]^{\mathcal{P}(\Gamma)}$, each M_ε^B is also compact.

Now, define the function $\mu_B : \mathcal{P}(\Gamma) \to [0,1]$ by:

$$\mu_B(A) := \frac{|A \cap F_B|}{|F_B|},$$

where F_B is a Følner set, that is:

$$\frac{|bF_B \bigtriangleup F_B|}{|F_B|} < \varepsilon,$$

for all $b \in B$. Then

$$|\mu_B(A) - \mu_B(bA)| = \frac{||A \cap F_B| - |bA \cap F_B||}{|F_B|} \leq \frac{|bF_B \bigtriangleup F_B|}{|F_B|} < \varepsilon,$$

since $(A \cap F_B) \subseteq (F_B \cup bF_B)$ and $(bA \cap F_B) \subseteq (F_B \cup bF_B)$. Hence $\mu_B \in M_\varepsilon^B$, and in particular, each M_ε^B is non-empty. Observe that if $B_1, \ldots, B_m$ is a finite sequence of subsets, and $\varepsilon_1, \ldots, \varepsilon_n$ is a finite sequence of positive numbers, then

$$\bigcap_{i=1}^{m} M_\varepsilon^{B_i} = M_\varepsilon^{(\bigcup B_i)},$$

for fixed $\varepsilon > 0$, and

$$\bigcap_{j=1}^{n} M^B_{\varepsilon_j} = M^B_{\min\{\varepsilon_j\}},$$

for a fixed subset $B \subseteq \Gamma$. Hence the space $\{M^B_\varepsilon \,|\, \varepsilon > 0, \text{ and } B \subseteq \Gamma\}$ of closed subsets of $[0,1]^{\mathcal{P}(\Gamma)}$ has the *finite intersection property* (Prop A.3), and so is compact. Since $[0,1]^{\mathcal{P}(\Gamma)}$ is compact by *Tychonoff's Theorem* (Thm 1.11), there exists an almost invariant mean $\hat{\mu}$ such that

$$\hat{\mu} \in \bigcap_{\substack{B \subseteq \Gamma \\ \varepsilon > 0}} M^B_\varepsilon = \bigcap_{\varepsilon > 0} M^\Gamma_\varepsilon.$$

Since we can set ε to be arbitrarily small, this shows that $\hat{\mu}$ is a left-invariant mean on Γ, and so Γ is amenable [10, p9].

⋆ ⋆ ⋆

The proof of the converse which we present here is due to [26]. Suppose that Γ is amenable with mean μ, and consider the space $l^1(\Gamma)$ of absolutely convergent series in Γ, that is,

$$l^1(\Gamma) := \left\{ f : \Gamma \to \mathbb{R} \,\middle|\, \sum_{x \in \Gamma} |f(x)| < \infty \right\}.$$

Equip $l^1(\Gamma)$ with the norm $\|f\|_1 := \sum_{x\in\Gamma} |f(x)|$, and define the space

$$\Phi := \left\{ f \in l^1(\Gamma) \,|\, f \geq 0,\ f \text{ is finitely-supported, and } \|f\|_1 = 1 \right\}.$$

We claim that, for each subset $A \subseteq \Gamma$ and each constant $\varepsilon > 0$, we can find a function $f \in \Phi$ such that $\|f - {}_af\|_1 < \varepsilon$ for all $a \in A$. Suppose, for a contradiction, that this is not true. Then we can find $A \subseteq \Gamma$ and $\varepsilon > 0$ such that $\sup\{\|f - {}_af\|_1 \,|\, a \in A\} \geq \varepsilon$ for all $f \in \Phi$. Hence the subset

$$B := \{f - {}_af \,|\, f \in \Phi, \text{ and } a \in A\} \subseteq l^1(\Gamma)$$

is convex and bounded away from zero by ε. We can therefore apply the *Hahn-Banach Separation Theorem* (Thm 2.10) on an open ε-neighbourhood of zero and B to find a linear functional $T \in l^1(\Gamma)^*$ and a constant $t > \varepsilon > 0$ such that $T(f - {}_af) \geq t$ for all $f \in \Phi$ and $a \in A$.

Now, let $g, h \in \Gamma^*$ be arbitrary maps, and define a partial inner product on Γ^* by:

$$\langle g, h \rangle := \sum_{x\in\Gamma} g(x)h(x),$$

whenever this sum is finite. We will also define the *convolution* operation $* : \Gamma^* \times \Gamma^* \to \Gamma^*$ by:

$$(g * h)(x) := \sum_{y\in\Gamma} g(y^{-1}x)h(x),$$

for all $x \in \Gamma$. Recall the space $l^\infty(\Gamma)$ of bounded sequences $f : \Gamma \to \mathbb{R}$. It can be shown (in [19, p121], for example) that $l^1(\Gamma)^* \cong l^\infty(\Gamma)$. Indeed, we can define an isometric isomorphism $\Theta : l^\infty(\Gamma) \to l^1(\Gamma)^*$ by:

$$\Theta(h)(g) := \langle g, h \rangle,$$

for all $g \in l^1(\Gamma)$ and $h \in l^\infty(\Gamma)$. It follows that for each functional $U \in l^1(\Gamma)^*$, there exists a *unique* function $h \in l^\infty(\Gamma)$ with

$$\langle g, h \rangle = \sum_{x \in \Gamma} g(x)h(x),$$

for all $g \in l^1(\Gamma)$. For each $a \in \Gamma$, consider the *Kronecker delta function*, $\delta_a : \Gamma \to \mathbb{R}$, defined by:

$$\delta_a(x) := \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{if } x \neq a, \end{cases}$$

for all $x \in \Gamma$. Clearly $\delta_a \in l^1(\Gamma)$, and so $f - \delta_a * f \in l^1(\Gamma)$. Therefore there exists a function $m^a \in l^\infty(\Gamma)$ such that

$$\sum_{a \in A} \langle f - \delta_a * f, m^a \rangle = \sum_{a \in A} \sum_{x \in \Gamma} (f - \delta_a * f)(x) m^a(x) \geq t,$$

for all $f \in \Phi$. Taking $f := \delta_y$ for $y \in \Gamma$, we see that

$$\begin{aligned} \sum_{a \in A} \langle \delta_y - \delta_a * \delta_y, m^a \rangle &= \sum_{a \in A} \sum_{x \in \Gamma} \delta_y(x) m^a(x) - \delta_y(a^{-1}x) m^a(x) \\ &= \sum_{a \in A} m^a(y) - m^a(ay) \geq t. \end{aligned}$$

Then, applying the mean μ, we obtain $\mu(m^a(y) - m^a(ay) \geq t > 0$ for all $y \in \Gamma$, which contradicts the left invariance of μ. Hence, for each subset $A \subseteq \Gamma$, we can find a function $f \in \Phi$ such that $\|f - {}_a f\|_1 < \varepsilon$ for all $a \in A$.

Now, fix a non-empty subset $A \subseteq \Gamma$, and some constant $\varepsilon > 0$. By the above, we can find a function $f \in \Phi$ such that $\|f - {}_a f\|_1 < \frac{\varepsilon}{|A|}$ for all $a \in A$. Then, since f is finitely-supported, we can use the *layer-cake* representation (see [21, p26]) to write

$$f = \sum_{i=1}^{n} c_i \mathbf{1}_{F_i},$$

for some chain $F_1 \supset \cdots \supset F_n$ of non-empty finite subsets of Γ, and some constants $c_i > 0$. Observe also that

$$\|f\|_1 = \sum_{i=1}^{n} c_i |F_i| = 1,$$

since $f \in \Phi$. But we can also see that $|f(x) - {}_af(x)| \geq c_i$ whenever $x \in (aF_i \triangle F_i)$, and so

$$\sum_{i=1}^{n} c_i |aF_i \triangle F_i| \leq \|f - {}_af\|_1 < \frac{\varepsilon}{|A|} \sum_{i=1}^{n} c_i |F_i|,$$

for each $a \in A$. Then

$$\sum_{a \in A} \sum_{i=1}^{n} c_i |aF_i \triangle F_i| < \varepsilon \sum_{i=1}^{n} c_i |F_i|,$$

and we can use a *pigeonhole* argument to show that there exists some i such that

$$\sum_{a \in A} |aF_i \triangle F_i| < \varepsilon |F_i|.$$

Since our choice of finite subset $A \subseteq \Gamma$ was arbitrary, this shows that Γ satisfies the Følner criterion [40]. $\square$

Examples 2.13.

1. If Γ is a finite group, then we may define a trivial Følner sequence by $F_n := \Gamma$ for all $n \in \mathbb{N}$. This affirms the amenability of finite groups.
2. Consider the additive group $\mathbb{Z}$, and define a sequence (F_n) of finite subsets of $\mathbb{Z}$ by $F_n := \{i \in \mathbb{Z} \,|\, -n \leq i \leq n\}$. Then

   $$x + F_n = \{i \in \mathbb{Z} \,|\, x - n \leq i \leq x + n\},$$

   for each $x \in \mathbb{Z}$. Hence

   $$\lim_{n \to \infty} \frac{|(x + F_n) \triangle F_n|}{|F_n|} = \lim_{n \to \infty} \frac{2x}{2n+1} = 0,$$

   which affirms that $\mathbb{Z}$ is amenable. Informally, this is because the boundary of a finite subset of $\mathbb{Z}$ is "small" compared to the size of the set [30, p5].

Now we are equipped to give the standard example of a finitely-generated group which is not amenable: the free group on two generators, $\mathbb{F}_2$. The Cayley graph for $\mathbb{F}_2$ with respect to the generating set $S := \{a^{\pm 1}, b^{\pm 1}\}$ is a regular tree of valency four, where each vertex x is adjacent to the vertices xa, xb, xa^{-1}, and xb^{-1}. Intuitively, the boundary of any finite subset of $\mathbb{F}_2$ will grow exponentially, and thus $\mathbb{F}_2$ will not admit a Følner sequence.

As an illustration of this, for each natural number n and $x_0 \in \mathbb{F}_2$, define the set $B(x_0, n) \subseteq \mathrm{Cay}(\mathbb{F}_2; S)$ by:

$$B(x_0, n) := \{x \in \mathrm{Cay}(\mathbb{F}_2; S) \,|\, d_S(x, x_0) \leq n\},$$

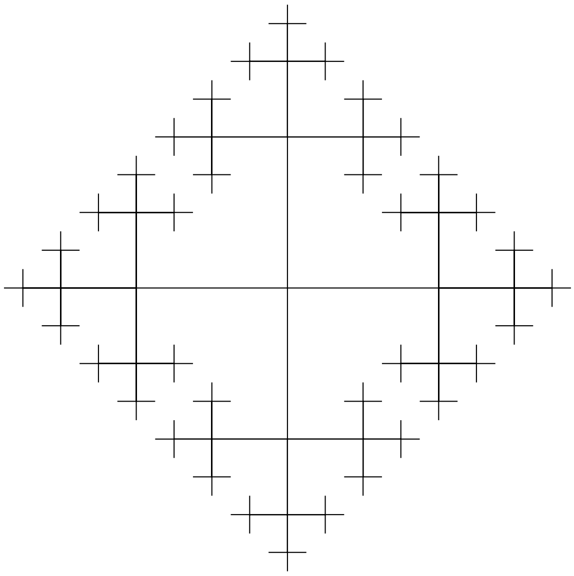

Figure 1: The Cayley graph of $\mathbb{F}_2$.

where d_S is the word metric on $\mathbb{F}_2$ with respect to S. Then the collection $\{B(x_0, n)\}_{n\in\mathbb{N}}$ forms an increasing chain of balls centred at x_0, and we can see from Figure 1 that $|B(x_0, 0)| = 1$, and

$$|B(x_0, n)| = 4\left(\sum_{i=0}^{n-1} 3^i\right) + 1,$$

for each integer $n \geq 1$. Furthermore, we see that $|\partial B(x_0, 0)| = 4$, and

$$|\partial B(x_0, n)| = 4\left(\sum_{i=0}^{n} 3^i - \sum_{i=0}^{n-1} 3^i\right) = 4\left(3^n\right),$$

for each integer $n \geq 1$. Hence

$$\lim_{n\to\infty} \frac{|\partial B(x_0, n)|}{|B(x_0, n)|} = \lim_{n\to\infty} \frac{4\left(3^n\right)}{4\left(3^{n-1}\right)} = 3,$$

and so the sequence $\{B(x_0, n)\}_{n\in\mathbb{N}}$ is not a Følner sequence [4]. In order to demonstrate that $\mathbb{F}_2$ is not amenable, however, we must show that *no* finite subset can be a Følner set. To do this, we use a simple edge-counting argument.

Proposition 2.14. *The free group on two generators, $\mathbb{F}_2$, is not amenable [13, p11].*

Proof. Let $F \subset \mathbb{F}_2$ be a finite subset, and consider its corresponding subgraph of $\mathrm{Cay}(\mathbb{F}_2; S)$, that is, the graph X_F such that:

(a) X_F has vertices corresponding to elements of F, and

(b) X_F has an edge joining x and xs for each $x \in F$ and $s \in \{a^{\pm 1}, b^{\pm 1}\}$.

Firstly, suppose that X_F is connected. Write V and E to denote the total number of vertices and edges, respectively, in X_F. Write v_i to denote the number of vertices of X_F of degree i, for $i = 1, \dots, 4$. Then, since $\mathrm{Cay}(\mathbb{F}_2; S)$ is a tree, it follows that $E = V - 1$, and by a handshaking argument, that

$$E = \frac{v_1 + 2v_2 + 3v_3 + 4v_4}{2}.$$

Hence

$$v_1 + v_2 + v_3 + v_4 - 1 = \frac{v_1 + 2v_2 + 3v_3 + 4v_4}{2},$$

and it follows that

$$|F| = V = \frac{3v_1 + 2v_2 + v_3 - 2}{2}.$$

Notice that the size of the boundary of F can be calculated by considering the number of edges which "leave" F, that is,

$$|\partial F| = 3v_1 + 2v_2 + v_3,$$

and so it follows that

$$\frac{|\partial F|}{|F|} = 2\left(\frac{3v_1 + 2v_2 + v_3}{3v_1 + 2v_2 + v_3 - 2}\right) > 2.$$

Hence F is not a Følner set.

Now, suppose that F is not connected, and write $F = F_1 \cup \cdots \cup F_n$ such that $X_{F_1}, \dots, X_{F_n}$ are the mutually disjoint connected components of X_F. Then at most $n - 1$ vertices of $\mathrm{Cay}(\mathbb{F}; S)$ can appear in the boundaries of more than one of the sets F_i, and so

$$\begin{aligned}
\frac{|\partial F|}{|F|} &\geq \frac{|\partial F_1| + \cdots + |\partial F_n| - (n-1)}{|F|} \\
&> \frac{2|F_1| + \cdots + 2|F_n| - n}{|F|} \\
&= \frac{2|F| - n}{|F|} = 2 - \frac{n}{|F|}.
\end{aligned}$$

But $|F| = |F_1| + \cdots + |F_n| \geq n$, so

$$\frac{|\partial F|}{|F|} > 1,$$

and F fails the Følner criterion. Since $F \subset \mathbb{F}_2$ was an arbitrary finite subset, this proves that $\mathbb{F}_2$ is not amenable [27]. $\square$

Definition 2.15 (Følner criterion for locally compact groups). Let Γ be a locally compact group equipped with Haar measure μ. Then Γ satisfies the *Følner criterion* if, for each compact subset $K \subseteq \Gamma$ and each $\varepsilon > 0$, there exists a Borel set $F \in \mathcal{B}(\Gamma)$ with $0 < \mu(F) < \infty$ such that

$$\frac{\mu(xF \triangle F)}{\mu(F)} < \varepsilon,$$

for all $x \in K$.

Recall that a topological space X is said to be *second-countable* if we can find some countable collection $\mathcal{W} := \{W_i \,|\, i \in \mathbb{N}\}$ such that:

(a) Each $W_i \in \mathcal{W}$ is open, and

(b) For each open subset $U \subseteq X$, we can find some finite sub-collection $W_{i_1}, \ldots, W_{i_k}$ of $\mathcal{W}$ such that $U = W_{i_1} \cup \cdots \cup W_{i_k}$ [5, pp390-1].

If Γ is second-countable, we define a *Følner sequence* to be a sequence (F_n) of compact subsets of Γ with $0 < \mu(F_n) < \infty$ for each $n \in \mathbb{N}$, and such that

$$\bigcup_{n \in \mathbb{N}} F_n = \Gamma, \qquad \text{and} \qquad \lim_{n \to \infty} \frac{\mu(xF_n \triangle F_n)}{\mu(F_n)} = 0,$$

for all $x \in \Gamma$ [9] [10, p8].

Example 2.16. If Γ is a compact group, then it has non-zero, finite measure. Hence we can trivially define the Følner set $F := \Gamma$, and so Γ is amenable.

2.3 Growth

In the discussion before Prop 2.14 on p41, we made use of balls of increasing radius centred around a vertex x_0 in the Cayley graph of a finitely-generated group Γ. This illustrated an argument for the (non-)existence of a Følner sequence in Γ, and in fact the amenability of a group is closely related to the rate of growth of such balls.

Definition 2.17 (Growth rate). Let Γ be a group generated by a finite subset $S \subseteq \Gamma$, let $x_0 \in \Gamma$, and let n be a non-negative integer. Then the *ball of radius n centred at x_0* is the set $B_S(x_0, n) \subseteq \mathrm{Cay}(\Gamma; S)$ defined by:

$$B_S(x_0, n) := \{x \in \mathrm{Cay}(\Gamma; S) \,|\, d_S(x, x_0) \leq n\},$$

where d_S is the word metric on Γ with respect to S. We define the *growth function*, γ_Γ^S of Γ with respect to S to be the number of elements in Γ which can be expressed as a word of length at most n, that is,

$$\gamma_\Gamma^S(n) := |B_S(e, n)|,$$

where $e \in \Gamma$ is the identity element. We will usually omit the subscript if there is no risk of ambiguity.

Given two increasing functions $\gamma_1, \gamma_2 : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we say that γ_1 *dominates* γ_2 whenever there exist constants $A, C > 0$ such that

$$A\gamma_1(Cn) \geq \gamma_2(n),$$

for all $n \in \mathbb{N}$. We write $\gamma_1 \succcurlyeq \gamma_2$ to denote this. We write $\gamma_1 \sim \gamma_2$ if $\gamma_1 \succcurlyeq \gamma_2$ and $\gamma_2 \succcurlyeq \gamma_1$. Clearly $\sim$ is reflexive, and it follows from the fact that the functions are increasing that $\sim$ is symmetric and transitive. Hence $\sim$ is an equivalence relation.

In fact, $\gamma^S \sim \gamma^{S'}$ for any two finite generating sets S and S'. To see this, define the value

$$\alpha := \max\{d_{S'}(e, s) \mid s \in S^{\pm}\},$$

and observe that, since S is finite, α is finite. Then for each $x \in \Gamma$, it follows that $d_S(e, x) \leq \alpha n$ whenever $d_{S'}(e, x) \leq n$. Hence $\gamma^{S'} \leq \gamma^S(\alpha n)$. By a similar argument, we can show that $\gamma^S \leq \gamma^{S'}(\beta n)$ for some $\beta > 0$, and by writing $C := \max\{\alpha, \beta\}$, the claim follows.

We then define the unique *growth rate* of Γ to be the equivalence class of growth functions. We denote this by γ, as we needn't specify a generating set. [6, pp171-2]

Definition 2.18 (Polynomial to exponential growth). Let Γ be a finitely-generated group. Then its growth rate γ can be described in one of three ways.

1. If $\gamma(n) \sim n^\alpha$ for some $\alpha > 0$, then γ is said to be a *polynomial growth rate*. The infimum of such constants α is called the *order* of growth.
2. If $\gamma(n) \succcurlyeq e^n$, then it is said to be an *exponential growth rate*.
3. If γ is neither polynomial nor exponential, then it is said to be an *intermediate growth rate* [29, p114].

Proposition 2.19. *Let Γ_1 and Γ_2 be groups generated by finite subsets $S_1 \subseteq \Gamma_1$ and $S_2 \subseteq \Gamma_2$ respectively, and suppose that the metric spaces (Γ_1, d_{S_1}) and (Γ_2, d_{S_2}) are quasi-isometric. Then Γ_1 and Γ_2 have the same growth rate.*

Proof. Write γ_1 and γ_2 to denote the growth rates of Γ_1 and Γ_2 respectively, and let $f : \Gamma_1 \to \Gamma_2$ be a quasi-isometric map. Then we can find constants $A \geq 1$ and $B \geq 0$ such that

$$\frac{1}{A} \cdot d_{S_1}(x, x') - B \leq d_{S_2}(f(x), f(x')) \leq A \cdot d_{S_1}(x, x') + B,$$

for all $x, x' \in \Gamma_1$. Write e_1 and e_2 to denote the identity elements in Γ_1 and Γ_2 respectively, and define $D := d_{S_2}(f(e), e')$.

Let $\alpha \in \mathbb{R}_{\geq 0}$. Write B^1_α to denote the ball with radius α centred at e_1 with respect to S_1, and B^2_α for that centred at e_2 with respect to S_2. Then for each $n \in \mathbb{N}$ and each $x \in B^1_n \subseteq \Gamma_1$, we have $f(x) \in B^2_{An+B+D} \subseteq \Gamma_2$.

Certainly $d_{S_1}(x, x') \leq AB$ whenever $f(x) = f(x')$, for all $x, x' \in \Gamma_1$, and so $|f^{-1}(y)| \leq \gamma_1(AB)$ for each $y \in \Gamma_2$. Hence there exists an $N \in \mathbb{N}$ such that

$$\gamma_2(2An) \geq \gamma_2(An + B + D) = \left|B^2_{An+B+D}\right|,$$

whenever $n \geq N$. But

$$\left|B^2_{An+B+D}\right| \geq \left|f\left[B^1_n\right]\right| \geq \frac{\left|B^1_n\right|}{\left|B^1_{AB}\right|} = \frac{\gamma_1(n)}{\gamma_1(AB)},$$

and so $\gamma_2 \succcurlyeq \gamma_1$. It can be shown that $\gamma_1 \succcurlyeq \gamma_2$ by an almost identical argument, and hence that $\gamma_1 \sim \gamma_2$ [43, p39]. $\square$

Not all functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ fit into one of the above categories, but we will show that the growth function of a finitely-generated group is always precisely one of these types.

Lemma 2.20 (Fekete). *Let (a_n) be a sequence of non-negative numbers which is subadditive, that is, such that $a_{m+n} \leq a_m + a_n$ for all $m, n \in \mathbb{N}$. Then the sequence $\left(\frac{a_n}{n}\right)$ converges, and*

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf\left\{\frac{a_n}{n} \,\middle|\, n \in \mathbb{N}\right\}.$$

Proof. Let b be a positive integer, and note that each $n \in \mathbb{N}$ can be expressed in the form $n = bq + r$, for some integers $q, r \geq 0$, with $r < b$. Then, by the subadditivity of (a_n), it follows that

$$\frac{a_n}{n} \leq \frac{a_b q + a_r}{bq + r} \leq \frac{a_b q + a_r}{bq},$$

for each $n \in \mathbb{N}$. But $\frac{a_r}{bq} \to 0$ as $n \to \infty$, and so

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_b}{b}.$$

Clearly

$$\frac{a_n}{n} \geq \inf\left\{\frac{a_b}{b} \,\middle|\, b \geq 1\right\},$$

for all $n \in \mathbb{N}$, and so the result follows. $\square$

Propositions 2.21.

1. *Let Γ be a finitely-generated group with growth rate γ. If γ is such that*

$$\lim_{n\to\infty} \frac{\log \gamma(n)}{\log n} = \infty,$$

then γ is super-polynomial*, that is, γ dominates all polynomial functions.*

2. *Similarly, if*

$$\lim_{n\to\infty} \frac{\log \gamma(n)}{n} = 0, \quad \textit{or, equivalently,} \quad \lim_{n\to\infty} \gamma(n)^{\frac{1}{n}} \leq 1,$$

then γ is sub-exponential*, that is, γ is dominated by all exponential functions.*

Proof.

1. Firstly, if $p(n)$ is a polynomial, then

$$\lim_{n\to\infty} \frac{\log p(n)}{\log n} < \infty,$$

and so γ dominates p.

2. If we define an exponential function $f(n) := e^{Cn}$ for some $C > 0$, then

$$\lim_{n\to\infty} \frac{\log e^{Cn}}{n} = C > 0,$$

and

$$\lim_{n\to\infty} \left(e^{Cn}\right)^{\frac{1}{n}} = e^{C} > 1,$$

and so γ is dominated by f. □

Theorem 2.22. *Let Γ be a finitely-generated group with growth rate γ. Then the limit*

$$\lim_{n\to\infty} \frac{\log \gamma(n)}{n}$$

exists and is finite. If Γ has exponential growth, then this limit is positive. If Γ has sub-exponential growth, then this limit is zero [11].

Proof. Firstly, observe that the growth function γ is *sub-multiplicative*, that is, $\gamma(m+n) \leq \gamma(m)\gamma(n)$, for all $m, n \in \mathbb{N}$. Then

$$\log(\gamma(m+n)) \leq \log(\gamma(m)\gamma(n)) = \log(\gamma(m)) + \log(\gamma(n)),$$

and so we can apply *Fekete's Lemma* (Lem 2.20) on the function $\log \gamma$. If the limit is some positive constant C, then we can see that $\gamma(n) \sim e^{Cn}$, and so Γ has exponential growth rate. If the limit is zero, then we can see that γ is dominated by all exponential functions, and so Γ has sub-exponential growth rate. $\square$

Example 2.23.

1. Every finite group trivially has polynomial growth of degree zero.

2. The lattice $\mathbb{Z}^k$ as an additive group has polynomial growth of degree k. To see this, consider the generating set $S := \{v_1, \ldots, v_k\}$ of standard basis vectors for $\mathbb{R}^k$. Then each ball $B_S(e, n)$ consists of precisely those words of the form $e_1^{i_1} \cdots e_k^{i_k}$, for some $i_j \in \mathbb{Z}$ such that

$$\sum_{j=1}^{k} |i_j| \leq n.$$

Then it is clear that $|B_S(e, n)| \leq n^k$, and so $\gamma_{\mathbb{Z}^k}(n) \leq n^k$. On the other hand, consider the additive group $\mathbb{Z}$ generated by the set $\{1, -1\}$. It is easy to verify that the growth rate for $\mathbb{Z}$ is given by $\gamma_{\mathbb{Z}}(n) = 2n + 1$. It follows then that

$$\gamma_{\mathbb{Z}^k}(kn) = |B(e, kn)| \geq (\gamma_{\mathbb{Z}}(n))^k = (2n + 1)^k,$$

and so $\gamma_{\mathbb{Z}^k}$ is bounded below by a polynomial of degree k. Hence $\mathbb{Z}^k$ has polynomial growth of degree k.

3. The free group on two generators, $\mathbb{F}_2$, has exponential growth. As discussed on p42, each time we increase the radius of $B(e, n)$ by one, we add $4(3^n)$ distinct words to the set. Hence $\gamma_{\mathbb{F}_2}(n) \sim 3^n$, and $\mathbb{F}_2$ has exponential growth rate.

Now we present an extremely powerful condition for the amenability of a finitely-generated group.

Theorem 2.24. *Let Γ be a finitely-generated group with sub-exponential growth. Then Γ is amenable.*

Proof. Firstly, write B_n to denote the ball of radius n centred at the identity e with respect to a finite generating set S. Since Γ is assumed to have sub-exponential growth rate γ, we know from Prop 2.21(2) that

$$\lim_{n \to \infty} |B_n|^{\frac{1}{n}} = \lim_{n \to \infty} \gamma(n)^{\frac{1}{n}} = C \leq 1.$$

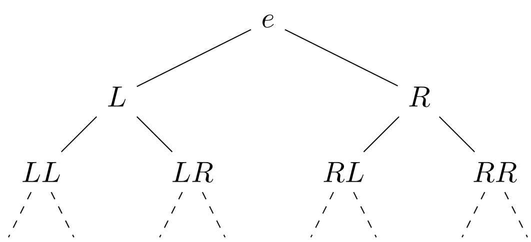

Figure 2: A binary tree T with non-symmetric generating set $S := \{L, R\}$.

Let $\varepsilon > 0$ be an arbitrary constant. We claim that there exists some $N \in \mathbb{N}$ such that

$$\frac{|B_{n+1}|}{|B_n|} < 1 + \frac{\varepsilon}{2},$$

whenever $n \geq N$. Indeed, assume for a contradiction that we can find a constant $\varepsilon > 0$ such that $|B_{n+1}| \geq (1 + \frac{\varepsilon}{2})\,|B_n|$, for all $n \in \mathbb{N}$. Then $|B_{n+1}| \geq (1 + \frac{\varepsilon}{2})^n |B_1|$, and so

$$\lim_{n\to\infty} \gamma(n)^{\frac{1}{n}} = \lim_{n\to\infty} |B_n|^{\frac{1}{n}} \geq 1 + \frac{\varepsilon}{2},$$

which contradicts the fact that γ is a sub-exponential growth rate. So, we can find some $N \in \mathbb{N}$ such that

$$\frac{|sB_n \bigtriangleup B_n|}{|B_n|} \leq \frac{2\left(|B_{n+1}| - |B_n|\right)}{|B_n|} < 2\left(1 + \frac{\varepsilon}{2} - 1\right) = \varepsilon,$$

for all $n \geq N$, and all $s \in S$. Since each $x \in \Gamma$ is a word in S, the sequence (B_n) forms a Følner sequence, and so Γ is amenable [10, p10] [18, pp32-3]. $\square$

Examples 2.25.

1. The *Grigorchuk group* G was designed in 1980 [12] as an example of a group with intermediate growth. It is a group of automorphisms of a rooted binary tree T (Figure 2), generated by four specific automorphisms $a, b, c, d : G \to G$. We can consider T as a finitely-generated group generated by the (non-symmetric) set $S := \{L, R\}$. Hence we can define the automorphisms according to where they send elements of S:

   - $a(L) := R$, $a(R) := L$, $a(Lx) := Rx$, and $a(Rx) := Lx$,
   - $b(L) := L$, $b(R) := R$, $b(Lx) := La(x)$, and $b(Rx) := Rc(x)$,
   - $c(L) := L$, $c(R) := R$, $c(Lx) := La(x)$, and $c(Rx) := Rd(x)$,
   - $d(L) := L$, $d(R) := R$, $d(Lx) := Lx$, and $d(Rx) := Rb(x)$,

for all $x \in S$. It can be shown that G has intermediate growth, and as such is amenable by Thm 2.24 [6, chpVIII].

2. Recall that the *discrete Heisenberg group* H is defined to be the group of 3×3 matrices of the form:

$$H := \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, a, b, c \in \mathbb{Z} \right\},$$

with the group operation defined as matrix multiplication. Since matrix multiplication is not commutative, the group H is not Abelian, and so we cannot use Thm 1.30 to show that H is amenable. It can be shown, however, that H has polynomial growth of order 4, and so H is amenable by Thm 2.24 [15, pp7-8].

3 The Banach–Tarski Paradox

The work of Hausdorff and Banach in the 1920s has inspired mathematicians from many fields, including analysts like Giuseppe Vitali, and applied mathematicians like John von Neumann [44]. Von Neumann's idea of amenability has its roots in the *Banach–Tarski Paradox*, a surprising and unintuitive result which arises in $\mathbb{R}^n$ when $n \geq 3$. According to the paradox, one may duplicate and alter the size of balls by slicing them into a finite number of sections and reassembling the pieces; a common visualisation is to imagine a pea being chopped up and rearranged into a ball the size of the Sun.

Conditions for the paradox are closely tied to the study of amenability, growth, and the existence of a *paradoxical decomposition*. In fact, we shall see in this chapter that a group Γ is amenable if and only if it is not paradoxical, thus giving us yet another characterisation of amenability [45, ppxiii-xv,157].

3.1 Paradoxical Decomposition

Definition 3.1 (Equidecomposition). Let Γ be a group which acts on a set X, and let $A, B \subseteq X$ be subsets. We say that A and B are Γ-*equidecomposable* if there exist subsets $A_1, \ldots, A_n, B_1 \ldots, B_n \subseteq \Gamma$ and elements $x_1, \ldots, x_n \in \Gamma$ such that:

(a) $A = A_1 \cup \cdots \cup A_n$ and $B = B_1 \cup \cdots \cup B_n$,

(b) $A_i \cap A_j = \emptyset$ and $B_i \cap B_j = \emptyset$, for all $i \neq j$,

(c) $x_i(A_i) = B_i$ for all $1 \leq i \leq n$.

We write $A \sim_\Gamma B$ whenever A and B are Γ-equidecomposable, but we usually omit the subscript when there is no risk of ambiguity. If we wish to stress the value n, we say that A and B are Γ-equidecomposable *using n pieces*, and we sometimes write $A \sim_n B$ to convey this.

If $A \sim C$, for some subset $C \subseteq B$, then we write $A \preccurlyeq B$ [45, pp23-4].

It is not difficult to show that being Γ-equidecomposable is an equivalence relation. Certainly $\sim$ is symmetric and reflexive, and to show that it is transitive, suppose that $A, B, C \subseteq \Gamma$ are subsets such that $A \sim_m B$ and $B \sim_n C$. Then we can find subsets $A_i, B_i^1, B_j^2, C_j \subseteq \Gamma$ and elements $x_i, y_j \in \Gamma$ such that:

(a) $A = \bigcup_{i=1}^m A_i$, $B = \bigcup_{i=1}^m B_i^1 = \bigcup_{j=1}^n B_j^2$, and $C = \bigcup_{j=1}^n C_j$,

(b) The A_i are mutually disjoint, as are the B_i^1, etc.,

(c) $x_i(A_i) = B_i^1$, and $y_j(B_j^2) = C_j$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Then the sets $B_{ij} := B_i^1 \cap B_j^2$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ partition B, and we can define $A_{ij} := x_i^{-1}(B_{ij})$ such that the A_{ij} partition A. Write $x_{ij} := x_i|_{A_{ij}}$, and $y_{ij} := y_j|_{B_{ij}}$. Then the sets $C_{ij} := y_{ij}x_{ij}(A_{ij})$ partition C, and so $A \sim_{mn} C$ [10, p1].

Theorem 3.2 (Banach–Schröder–Bernstein). *Let Γ be a group which acts on a set X, and let $A, B \subseteq X$ be subsets. If $A \preccurlyeq B$ and $B \preccurlyeq A$, then $A \sim B$ [45, p25].*

Proof. Firstly, since $A \preccurlyeq B$ and $B \preccurlyeq A$, we can find bijections $f : A \to B'$ and $g : A' \to B$ for some subsets $A' \subseteq A$ and $B' \subseteq B$. Define $C_0 := A \setminus A'$, and $C_{n+1} := g^{-1}f(C_n)$ for each $n \in \mathbb{N}$. Write

$$C := \bigcup_{n=1}^{\infty} C_n.$$

Let $a \in A \setminus C$. Then for each n we have that $a \notin C_n$, and hence that $g(a) \notin f(C_n)$. Therefore $g(A \setminus C) \subseteq B \setminus f(C)$. Conversely, let $b \in B \setminus f(C)$. Then for each n we have that $b \notin f(C_n)$, and hence that $g^{-1}(b) \notin C_n$. So $g(A \setminus C) = B \setminus f(C)$, and $A \setminus C \sim B \setminus f(C)$ by our choice of g. But $C \sim f(C)$ by our choice of f, and it is easy to check that

$$((A \setminus C) \cup C) \sim ((B \setminus f(C)) \cup f(C)),$$

from which it follows that $A \sim B$ [45, p25]. □

Definition 3.3 (Paradoxical decomposition). Let Γ be a group which acts on a set X, and let $E \subseteq X$ be a subset. We say that E is Γ-*paradoxical* (or simply *paradoxical*) if there exist pairwise disjoint subsets $A_1, \ldots, A_m, B_1, \ldots B_n \subseteq E$ and elements $x_1, \ldots, x_m, y_1, \ldots y_n \in \Gamma$ such that:

$$\bigcup_{i=1}^{m} x_i(A_i) = \bigcup_{j=1}^{n} y_j(B_j) = E.$$

We call a group action f of Γ on X *paradoxical* if X is paradoxical. We therefore permit ourselves to call a group Γ *paradoxical* if the action of Γ on itself (by the group operation) is paradoxical [45, p4].

Corollary 3.4 (to Theorem 3.2). *Let Γ be a group which acts on a set X, and let $E \subseteq X$ be a subset. Then E is Γ-paradoxical if and only if there exists a proper subset $A \subset E$ such that $A \sim E \sim (E \setminus A)$ [45, p25].*

Proof. Firstly, suppose that E is Γ-paradoxical, such that we can find disjoint subsets $A, B \subset E$ with $A \sim E \sim B$. Then, since $B \subseteq (E \setminus A)$, it follows that $E \preccurlyeq (E \setminus A)$. But we also know that $(E \setminus A) \subset E$, and so $(E \setminus A) \preccurlyeq E$. Hence $(E \setminus A) \sim E$.

The converse is immediate [10, p2]. $\square$

Example 3.5 (Hilbert's Grand Hotel). Let M be the group of all bijective functions $f : \mathbb{Z} \to \mathbb{Z}$. Then the set of natural numbers $\mathbb{N}$ is an M-paradoxical subset of $\mathbb{Z}$.

To see why, consider the decomposition $\mathbb{N} := A \cup B$, where A is the set of even natural numbers, and B the odd natural numbers. Define the function $g : A \to \mathbb{N}$ by $g(n) := \frac{n}{2}$, and note that, since $\mathbb{Z} \setminus A$ and $\mathbb{Z} \setminus \mathbb{N}$ are both countable, we may extend g to a bijective function $g' \in M$. Then $g'A = \mathbb{N}$, and by a similar method we can find a function $h' \in M$ such that $h'B = \mathbb{N}$. Hence $\mathbb{N}$ is M-paradoxical [7, p730].

Proposition 3.6. *Let Γ be a paradoxical group which acts freely on a set X.*[4] *Then X is Γ-paradoxical [45, p11].*

Proof. Since Γ is paradoxical, we can use Corl 3.4 to find proper disjoint subsets $A, B \subset \Gamma$ such that $A \cup B = \Gamma$ and $A \sim \Gamma \sim B$. Then, by definition, we can write

$$A = \bigcup_{i=1}^{m} A_i, \qquad \text{and} \qquad B = \bigcup_{j=1}^{n} B_j,$$

for some pairwise disjoint subsets $A_1 \ldots, A_m, B_1, \ldots, B_n \subset \Gamma$, such that

$$\bigcup_{i=1}^{m} x_i(A_i) = \bigcup_{j=1}^{n} y_j(B_j) = \Gamma,$$

for some $x_1, \ldots, x_m, y_1, \ldots, y_n \in \Gamma$. Then, by the *Axiom of Choice*, we can construct a set M by selecting precisely one element from each Γ-orbit of X. Then $\{sM \mid s \in \Gamma\}$ partitions X, since the group action is free and hence does not fix any (non-trivial) points in X.

Now, write $A_i^* := \{aM \mid a \in A_i\}$, and $B_j^* := \{bM \mid b \in B_j\}$, and define

$$A^* := \bigcup_{i=1}^{m} A_i^*, \qquad \text{and} \qquad B^* := \bigcup_{j=1}^{n} B_j^*.$$

Then $A^*, B^* \subset X$ are proper disjoint subsets such that $A^* \cup B^* = X$ and $A^* \sim X \sim B^*$. Hence, using Corl 3.4, this shows that X is Γ-paradoxical [45, p11]. $\square$

[4]Recall that a group action $f : \Gamma \times X \to X$ is said to be *free* if no element of Γ (except the identity e) fixes a point in X.

Proposition 3.7. *The free group on two generators, $\mathbb{F}_2$, is paradoxical. Hence if $\mathbb{F}_2$ acts freely on a set X, then X is $\mathbb{F}_2$-paradoxical [45, p5].*

Proof. Suppose that $S := \{a, b\}$ is a generating set for $\mathbb{F}_2$, and write $W(x)$ to denote the set of reduced words in $S^{\pm}$ which begin (on the left) with x. Then

$$\mathbb{F}_2 = \{e\} \cup W(a) \cup W(a^{-1}) \cup W(b) \cup W(b^{-1}),$$

and

$$(W(a) \cup aW(a^{-1})) = \mathbb{F}_2 = (W(b) \cup bW(b^{-1})).$$

Writing $A := (W(a) \cup W(a^{-1}))$ and $B := (W(b) \cup W(b^{-1}))$, we see that $A \cup B = \mathbb{F}_2$, and $A \sim \mathbb{F}_2 \sim B$, and so $\mathbb{F}_2$ is paradoxical.

It follows from Prop 3.6 that a set X is $\mathbb{F}_2$-paradoxical whenever $\mathbb{F}_2$ acts freely on X [45, p5]. $\square$

Proposition 3.8 (Hausdorff). *The group of rotations about the origin in $\mathbb{R}^3$, denoted by $SO_3(\mathbb{R})$, contains $\mathbb{F}_2$ as a subgroup [45, p15].*

Proof. Consider the elements $p, q \in SO_3(\mathbb{R})$ which represent the anticlockwise rotations through an angle of $\arccos(\frac{1}{3})$ around the z-axis and x-axis, respectively. We can write p and q explicitly as:

$$p^{\pm 1} := \begin{bmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad q^{\pm 1} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} \\ 0 & \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}. \tag{14}$$

We claim that the set $S := \{p, q\}$ generates $\mathbb{F}_2$. In order to do this, let w be a non-trivial reduced word in $S^{\pm}$, and assume for a contradiction that w represents the identity in $SO_3(\mathbb{R})$. Without loss of generality, we may conjugate w by p, and hence assume that w ends (on the right) with $p^{\pm 1}$.

We will show that w *has* to be of the form

$$w \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3^n} \begin{bmatrix} a \\ b\sqrt{2} \\ c \end{bmatrix},$$

for some integers a, b, c with $3 \nmid b$, and where $n := |w|$. In particular, this will show that $w(1,0,0)^T \neq (1,0,0)^T$, which contradicts our assumption that w represents the identity. We proceed by induction on the length n.

Firstly, if $n = 1$, then $w = p^{\pm 1}$, and so

$$w \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ \pm 2\sqrt{2} \\ 0 \end{bmatrix},$$

which coheres to our hypothesis. Now, suppose that $w = sw'$ for some $s \in S^{\pm}$, and where w' satisfies

$$w' \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3^{n-1}} \begin{bmatrix} a' \\ b'\sqrt{2} \\ c' \end{bmatrix},$$

for some integers a', b', c' with $3 \nmid b'$, and where $n := |w| = |w'| + 1$. By computing sw' for each $s \in S^{\pm}$, we arrive at four results; in each case, w satisfies

$$w \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3^{n}} \begin{bmatrix} a \\ b\sqrt{2} \\ c \end{bmatrix},$$

for some constants a, b, c, and where $n := |w|$.

(i) If $s = p$ then $w = pw'$, and so $a = a' - 4b'$, $b = 2a' + b'$, and $c = 3c'$.

(ii) If $s = p^{-1}$ then $a = a' + 4b'$, $b = -2a' + b'$, and $c = 3c'$.

(iii) If $s = q$ then $a = 3a'$, $b = b' - 2c'$, and $c = 4b' + c'$.

(iv) If $s = q^{-1}$ then $a = 3a'$, $b = b' + 2c'$, and $c = -4b' + c'$.

Notice that in each case, a, b, c are all integers, and so it remains to show that b is never divisible by 3. Again, four cases arise as we can write w as $p^{\pm 1}q^{\pm 1}v$, $q^{\pm 1}p^{\pm 1}v$, $p^{\pm 1}p^{\pm 1}v$, or $q^{\pm 1}q^{\pm 1}v$, for some (possibly empty) word v. Write

$$v \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3^{n-2}} \begin{bmatrix} a'' \\ b''\sqrt{2} \\ c'' \end{bmatrix},$$

for some constants a'', b'', c'', and where $n := |w|$.

(i) If $w = p^{\pm 1}q^{\pm 1}v$, then

$$\begin{aligned} w \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= p^{\pm 1} \left(\frac{1}{3^{n-1}} \begin{bmatrix} 3a'' \\ (b'' \mp 2c'')\sqrt{2} \\ \pm 4b'' + c'' \end{bmatrix} \right) \\ &= p^{\pm 1} \left(\frac{1}{3^{n-1}} \begin{bmatrix} a' \\ b'\sqrt{2} \\ c' \end{bmatrix} \right) \\ &= \frac{1}{3^{n}} \begin{bmatrix} a' \mp 4b' \\ (\pm 2a' + b')\sqrt{2} \\ 3c' \end{bmatrix} \\ &= \frac{1}{3^{n}} \begin{bmatrix} a \\ b\sqrt{2} \\ c \end{bmatrix}. \end{aligned}$$

Hence $b = \pm 2a' + b' = \pm 6a'' + b'$, and since $3 \nmid b'$ by assumption, it follows that $3 \nmid b$.

(ii) If $w = q^{\pm 1}p^{\pm 1}v$, then we carry out a similar calculation to show that $b = b' \mp 6c''$, and so again $3 \nmid b$.

(iii) If $w = p^{\pm 1}p^{\pm 1}v$, then

$$\begin{aligned} b &= \pm 2a' + b' \\ &= \pm 2(a'' \mp 4b'') + b' \\ &= \pm 2a'' - 8b'' + b' \\ &= (\pm 2a'' + b'') - 9b'' = 2b' - 9b'', \end{aligned}$$

and so $3 \nmid b$.

(iv) If $w = q^{\pm}q^{\pm}v$, then we find that $b = 2b' - 9b''$, and so again $3 \nmid b$.

We have shown that b cannot be divisible by 3, and hence that $w(1,0,0)^T \neq (1,0,0)^T$, a contradiction. Therefore w cannot represent the identity in $SO_3(\mathbb{R})$, and so the subset $\{p, q\} \subset SO_3(\mathbb{R})$ generates a copy of the free group $\mathbb{F}_2$ [45, pp15-6]. □

3.2 The Pea and the Sun

Theorem 3.9 (Hausdorff Paradox). *There exists a countable subset D of the sphere S^2 such that $S^2 \setminus D$ is $SO_3(\mathbb{R})$-paradoxical [45, p18].*

Proof. Consider $\mathbb{F}_2$ as a subset of $SO_3(\mathbb{R})$, as constructed in Prop 3.8, and consider the action of $\mathbb{F}_2$ on the sphere S^2. Each non-trivial element of $\mathbb{F}_2$ fixes precisely two points of S^2, at the intersection of the axis of rotation with the sphere. Write D to be the union of all such points, that is:

$$D := \{P \in S^2 \,|\, xP = x, \text{ for some } x \in \mathbb{F}_2, \text{ with } x \neq e\}.$$

Since $\mathbb{F}_2$ is countable, we know that D is countable. Furthermore, if $P \in S^2 \setminus D$, then $xP \in S^2 \setminus D$ for all $x \in \mathbb{F}_2$. To see why, suppose that $yP = P$ for some $y \in \mathbb{F}_2$. Then P would be a fixed point of the element $x^{-1}yx \in \mathbb{F}_2$, and so $P \in D$, which is a contradiction.

Hence $\mathbb{F}_2$ acts freely on $S^2 \setminus D$, and since $\mathbb{F}_2$ embeds in $SO_3(\mathbb{R})$, it follows from Prop 3.6 that $S^2 \setminus D$ is $SO_3(\mathbb{R})$-paradoxical [45, pp17-8]. □

Proposition 3.10. *Let D be a countable subset of the sphere S^2. Then S^2 and $S^2 \setminus D$ are $SO_3(\mathbb{R})$-equidecomposable [45, p27].*

Proof. Firstly, since D is countable and S^2 is uncountable, we are able to find a line L through the origin which does not intersect D. Let $r_\theta \in SO_3(\mathbb{R})$

denote an anticlockwise rotation through an angle θ about the line L, and let $n \in \mathbb{Z}_{\geq 0}$. Write A to denote the set of angles θ such that $r_\theta^n(P) \in D$ whenever $P \in D$.

Clearly A is countable, and so we can find an angle φ which is not in A. Then $r_\varphi^n(D)$ and $r_\varphi(D)$ are disjoint for all n, and hence $r_\varphi^m \cap r_\varphi^n(D) = \emptyset$ for all $m, n \in \mathbb{Z}_{\geq 0}$ whenever $m \neq n$. Define the union

$$\bar{D} := \bigcup_{n=0}^{\infty} r_\varphi^n(D).$$

Then

$$S^2 = (\bar{D} \cup (S^2 \setminus \bar{D})) \sim (r_\varphi(\bar{D}) \cup (S^2 \setminus \bar{D})) = S^2 \setminus D,$$

and so $S^2 \sim (S^2 \setminus D)$ as required [45, p27]. $\square$

Corollary 3.11. *The sphere S^2 is $SO_3(\mathbb{R})$-paradoxical.*

Proof. By the *Hausdorff Paradox* (Thm 3.9), we know that $S^2 \setminus D$ is $SO_3(\mathbb{R})$-paradoxical for some countable subset $D \subset S^2$. But the above Prop 3.10 tells us that S^2 and $(S^2 \setminus D)$ are $SO_3(\mathbb{R})$-equidecomposable. Then, by Corl 3.4 and the transitivity of equidecomposition (see p51), it follows that S^2 is $SO_3(\mathbb{R})$-paradoxical. $\square$

Note that, as none of the results in this section have depended on the *size* of the sphere involved, we can conclude that 2-spheres of *any* radius have paradoxical decompositions. In fact, it can be shown that an n-sphere S^n of any radius is $SO_{n+1}(\mathbb{R})$-paradoxical for all $n \geq 2$ (refer to [45, p53] for details).

Theorem 3.12 (Banach–Tarski Paradox). *Let $E(3)$ denote the group of all isometries (that is, Euclidean transformations) of $\mathbb{R}^3$. Then every solid ball in $\mathbb{R}^3$ is $E(3)$-paradoxical, and $\mathbb{R}^3$ is $E(3)$-paradoxical [45, p27].*

Proof. We will prove the result for the unit ball $B \subset \mathbb{R}^3$ centred at the origin. Since $E(3)$ contains all translations in $\mathbb{R}^3$, and the proof does not rely on the radius of the ball, this will be sufficient to prove the result for any ball in $\mathbb{R}^3$.

By the fact that S^2 is $SO_3(\mathbb{R})$-paradoxical (Corl 3.10), we can find some pairwise disjoint subsets $A_1, \ldots, A_m, B_1, \ldots B_n \subseteq S^2$ and some rotations $x_1, \ldots, x_m, y_1, \ldots y_n \in SO_3(\mathbb{R})$ such that:

$$\bigcup_{i=1}^{m} x_i(A_i) = \bigcup_{j=1}^{n} y_j(B_j) = S^2.$$

Now, each subset X of S^2 is in one-to-one correspondence with the subset $X' := \{tx \mid x \in X, t \in (0,1]\}$ of $B \setminus \{\mathbf{0}\}$, and so the sets

$$A_i' := \{ta \mid a \in A_i, t \in (0,1]\}, \quad \text{and} \quad B_j' := \{tb \mid b \in B_j, t \in (0,1]\}$$

form a paradoxical decomposition for B. We claim that $(B \setminus \{\mathbf{0}\}) \sim B$, whence by Corl 3.4 it will follow that B is $SO_3(\mathbb{R})$-paradoxical.

Let $P := (0, 0, \frac{1}{2})$, and let L be a line through P which does not cross the origin. Let $r \in SO_3(\mathbb{R})$ be a rotation about L of infinite order, and define the set $\bar{D} := \{r^n(\mathbf{0}) \mid n \in \mathbb{Z}_{\geq 0}\}$. Then $r(\bar{D}) = (\bar{D} \setminus \{\mathbf{0}\})$ and, similarly to in the proof of Prop 3.10,

$$B = (\bar{D} \cup (B \setminus \bar{D})) \sim (r(\bar{D}) \cup (B \setminus \bar{D})) = B \setminus \{\mathbf{0}\},$$

and so $B \sim (B \setminus \{\mathbf{0}\})$. So B is $E(3)$-paradoxical, and hence $SO_3(\mathbb{R})$-paradoxical.

If instead we used the one-to-one correspondence of subsets X in S^2 to the subset $X' := \{tx \mid x \in X, t > 0\}$ of $\mathbb{R}^3 \setminus \{\mathbf{0}\}$, we would obtain a paradoxical decomposition for $\mathbb{R}^3 \setminus \{\mathbf{0}\}$. Using the same arguments as above, this would show that $\mathbb{R}^3$ is also $E(3)$-paradoxical [45, pp27-8]. $\square$

Recall that the Lebesgue measure λ of the union of a finite number of intervals $\mathcal{I} := I_1 \cup \cdots \cup I_n$ in $\mathbb{R}$ is defined to be:

$$\lambda(\mathcal{I}) := \ell(I_1) + \cdots + \ell(I_n).$$

It is natural to define the Lebesgue measure in $\mathbb{R}^k$ in a similar way. Let $\mathcal{J}$ be a *k-dimensional box*, that is, a subset of $\mathbb{R}^k$ of the form

$$\mathcal{J} := \{(x_1, \ldots, x_k) \mid x_i \in I_i, 1 \leq i \leq k\},$$

where $I_1, \ldots, I_k$ are intervals in $\mathbb{R}$. Then we define the *Lebesgue measure* of $\mathcal{J}$ to be the product of the measures of the intervals:

$$\lambda(\mathcal{J}) := \lambda(I_1) \times \cdots \times \lambda(I_k).$$

We can construct the Lebesgue measurable sets in $\mathbb{R}^k$ using the same process as in Chapter 1 [5, p14].

The Banach–Tarski Paradox implies that we may partition a solid ball in such a way that, through a series of rotations, we end up with two solid balls identical to the first. Since rotations are measure-preserving, this result sounds absurd, but when we consider what it means for a set to be measurable, we can resolve the mystery.

Indeed, the result implies that the sets comprising the paradoxical decomposition of a ball $B \subset \mathbb{R}^3$ cannot be measurable, and hence that there is no finitely-additive measure defined on every subset of $\mathbb{R}^3$ which gives the unit ball non-zero measure. This in turn implies that there are sets in $\mathbb{R}^3$ which are not Lebesgue measurable.

The Banach–Tarski Paradox has a stronger form, whereby any two bounded sets with non-empty interior are equidecomposable. Hence, in addition to duplicating balls, we can also decompose and reassemble them into balls of arbitrary radius! This result is often referred to as the *Pea-to-Sun Paradox.*

Lemma 3.13. *Let $B_0, B_1, \ldots, B_N$ be a finite sequence of balls in $\mathbb{R}^3$ of the same radius, such that $B_1, \ldots, B_N$ are pairwise disjoint. Then*

$$\bigcup_{k=1}^{N} B_k \sim_{E(3)} B_0.$$

Proof. Clearly $B_0 \preccurlyeq \bigcup_{k=1}^{N} B_k$, since $B_0 \preccurlyeq B_i$ for each k. Hence by the *Banach–Schröder–Bernstein Theorem* (Thm 3.2), it remains to show that $\bigcup_{k=1}^{N} B_k \preccurlyeq B_0$.

Firstly, since B_0 is $E(3)$-paradoxical by the *Banach–Tarski Paradox*, we can write

$$B_0 = \bigcup_{i=1}^{m} x_i(C_i) = \bigcup_{j=1}^{n} y_j(D_j),$$

for some pairwise disjoint subsets $C_1, \ldots, C_m, D_1, \ldots, D_n \subset B_0$ and some $x_1, \ldots, x_m, y_1, \ldots, y_n \in E(3)$. Then we have

$$B_1 \sim B = \bigcup_{i=1}^{m} x_i(C_i) \preccurlyeq \bigcup_{i=1}^{m} C_i,$$

and

$$B_2 \sim B = \bigcup_{j=1}^{n} y_j(D_j) \preccurlyeq \bigcup_{j=1}^{n} D_j.$$

Hence $B_1 \cup B_2 \preccurlyeq B_0$. By induction on k, we can extend this argument to show that $\bigcup_{k=1}^{N} B_k \preccurlyeq B_0$, as required [18, pp26-7]. $\square$

Theorem 3.14 (Pea-to-Sun Paradox). *Let $A, B \subset \mathbb{R}^3$ be bounded subsets with non-empty interior. Then $A \sim_{E(3)} B$ [45, p29].*

Proof. We begin by showing that any two bounded subsets $A, B \subset \mathbb{R}^3$ with non-empty interior satisfy $A \preccurlyeq B$. Since A is bounded, we can find a solid ball K such that $A \subseteq K$. Furthermore, since B has non-empty interior, we can find a solid ball L such that $L \subseteq B$.

Using the *Banach–Tarski Paradox* (Thm 3.12), we are able to duplicate L and translate the copies as many times as needed to cover K. Let $x_1, \ldots, x_n \in E(3)$ be translations such that

$$K \subseteq x_1 L \cup \cdots \cup x_n L,$$

and let $y_1, \ldots, y_n \in E(3)$ be such that $y_i L \cap y_j L = \emptyset$ whenever $i \neq j$. Then by Lem 3.13 it follows that

$$\bigcup_{i=1}^{n} y_i L \preccurlyeq L,$$

and hence that

$$A \subseteq K \subseteq \bigcup_{i=1}^{n} x_i L \preccurlyeq \bigcup_{i=1}^{n} y_i L \preccurlyeq L \subseteq B.$$

Therefore $A \preccurlyeq B$, and we can use the same argument, interchanging A and B, to show that $B \preccurlyeq A$. Hence by Thm 3.2, it follows that $A \sim B$ [18, p27] [45, p29]. □

In the original formulation of the Banach–Tarski Paradox [2], the result remains true in $\mathbb{R}^k$ for all $k \geq 3$. It can also be shown that there is a similar result in $\mathbb{R}^k$ for $k = 1, 2$, permitting decomposition into *countably-many* sets [2, pp257-63] [10, p7].

In general, then, we can attest that a paradoxical group *cannot be amenable*, else the measure μ of some arbitrary paradoxical subset B will be at once $\mu(B)$ and $2\mu(B)$. A theorem of Tarski implies the converse, thus giving us another equivalent definition of amenability.

Theorem 3.15 (Tarski). *Let Γ be a group which acts on a set X, and let $E \subseteq X$ be a subset. Then there exists a finitely-additive left-invariant measure $\mu : \mathcal{P}(X) \to [0, \infty]$ with $\mu(E) = 1$ if and only if E is* not *Γ-paradoxical [45, p128].*

Proof. One direction of the proof is clear: if X is Γ-paradoxical with decomposition $X = A \cup B$ and mu is such a measure, then

$$1 = \mu(X) = \mu(A) + \mu(B) = 2,$$

by the finite additivity and left-invariance of μ. This is clearly a contradiction. For a proof of the converse, we refer the reader to [45, p128]. □

Corollary 3.16. *A group is amenable if and only if it is not paradoxical.* □

Examples 3.17.

1. Using Props 3.7 and 3.8, we can see that that the special orthogonal group $SO_3(\mathbb{R})$ is paradoxical, and hence not amenable.

2. We have shown that the class of amenable groups is closed under the operations of taking quotients, extensions, and countable unions (Prop 1.28 and Lem 1.29). It can also be shown that any closed subgroup of an amenable group is amenable [9] [18, p33]. In particular, any group which contains $\mathbb{F}_2$ as a subgroup cannot be amenable – again we cite $SO_3(\mathbb{R})$ as an example.

3. Each of the special orthogonal groups $SO_k(\mathbb{R})$ for $k \geq 3$ contain $SO_3(\mathbb{R})$ as a subgroup, and hence are not amenable by (2). This give a hint as to why the Banach–Tarski Paradox is true in Euclidean space $\mathbb{R}^k$ for all $k \geq 3$.

4. In Ex 1.31(2), we demonstrated that $SO_2(\mathbb{R})$ is amenable as a discrete group. Hence $SO_2(\mathbb{R})$ cannot be paradoxical, and this gives us a hint as to why the Banach–Tarski Paradox does not hold in $\mathbb{R}^2$.

3.3 Alternative Formulations

We arrive with three distinct representations of the amenability of various groups. A locally compact group Γ is *amenable* if it satisfies any (and hence all) of the following criteria:

(a) Γ admits a *left-invariant mean* (Def 1.26),

(b) Γ satisfies the *Følner criterion* (Def 2.15),

(c) Γ is not *paradoxical* (Def 3.3).

In fact, there are a vast number of equivalent characterisations of amenability: [45, p157] lists a further five, and [29] posits at least nine more. One of these is *Reiter's criterion*, which we have come across already, although not by name.

Theorem 3.18 (Reiter's criterion). *Let Γ be a discrete group, let $A \subseteq \Gamma$ be a finite subset, and let $\varepsilon > 0$ be an arbitrary constant. Then there exists an almost invariant mean $\mu \in l^1(\Gamma)$, that is, a mean such that*

$$\|\mu(A) - \mu(xA)\|_1 < \varepsilon,$$

if and only if Γ is amenable [18, p29]. □

We have already proved one direction of Reiter's criterion during the proof of Thm 2.12 (p39), as we found the "limit" of a sequence of almost invariant means to construct a fully invariant mean.

Another formulation from the viewpoint of a functional analyst makes use of the *Markov-Katukani Theorem* (see [29, p113]) and the *Hahn–Banach Theorem* (Thm A.4) to find fixed points in affine maps between compact sets in Γ. If the criteria of the theorems are satisfied, then such fixed points can be shown to exist, and can be used to construct an invariant mean on Γ (see [33, pp140-2]).

The existence of a Haar measure on a locally compact group and the construction of a paradoxical decomposition for S^2 both rely on the *Axiom of Choice*, the right to be able to form the Cartesian product of a collection of non-empty sets (Thm A.5). Until the Banach–Tarski Paradox came to light, there was little controversy surrounding the Axiom of Choice, but the Paradox is so counter-intuitive that many mathematicians have chosen to reject the Axiom.

It is possible to demonstrate the existence of Haar measure without employing the Axiom of Choice [22, pp112-3], but such proofs are less intuitive. It is not, however, possible to prove the Banach–Tarski Paradox without the Axiom, but we are able to reduce our reliance on it somewhat – again, such proofs are less instructive [45, chp13].

A Loose Ends

In this appendix, we detail a few technical results from measure theory and functional analysis which the reader may find useful for understanding some of the finer details of the proofs above, but which would interrupt the flow of the text if included in-line.

Proposition A.1. *Let* (A_n) *be a sequence of subsets of* $\mathbb{R}$*. Then*

$$\lambda^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \lambda^*(A_n).$$

[35, p37]

Proof. Firstly, we may assume that $\sum_{n=1}^{\infty} \lambda^*(A_n) < \infty$; if it were infinite then the result would trivially hold. Let $\varepsilon > 0$ be an arbitrary constant, and for each $n \in \mathbb{N}$, let $(I_m^n)_{m=1}^{\infty}$ be a covering of A_n, where $I_m^n \subseteq \mathbb{R}$ are open intervals.

Since

$$\lambda^*(A_n) = \inf \sum_{m=1}^{\infty} \ell(I_m^n),$$

and each I_m^n is an open interval, we have that

$$\sum_{m=1}^{\infty} \ell(I_m^n) < \lambda^*(A_n) + \frac{\varepsilon}{2^n},$$

for each n. Then, as $(I_m^n)_{n,m=1}^{\infty}$ forms an open covering of $\bigcup_{n=1}^{\infty} A_n$, it follows that:

$$\begin{aligned}
\lambda^* \left(\bigcup_{n=1}^{\infty} A_n \right) &\leq \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \ell\left(I_m^n\right) \right) \\
&\leq \sum_{n=1}^{\infty} \left(\lambda^*(A_n) + \frac{\varepsilon}{2^n} \right) \\
&= \sum_{n=1}^{\infty} \lambda^*(A_n) + \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} \\
&= \sum_{n=1}^{\infty} \lambda^*(A_n) + \varepsilon.
\end{aligned}$$

Since we can set ε to be arbitrarily small, the result follows [35, p37]. □

Proposition A.2 (Fubini). *Let X and Y be locally compact Hausdorff spaces endowed with regular Borel measures μ_1 and μ_2 respectively, and let $f \in \mathcal{K}_0(X \times Y)$.*[5] *Then the maps $f_x : Y \to X \times Y$ and $f^y : X \to X \times Y$ defined by:*

$$f_x(y) := f(x,y), \qquad \textit{and} \qquad f^y(x) := f(x,y)$$

are continuous and have compact support. Furthermore:

$$\int_X \int_Y f(x,y)\, d\mu_2(y)\, d\mu_1(x) = \int_Y \int_X f(x,y)\, d\mu_1(x)\, d\mu_2(y),$$

for all $x \in X$ and $y \in Y$ [5, pp144,221]. (First used in the proof of Thm 1.22, Eqn (12)).

Proof. Firstly, write K_1 and K_2 for the projections of $\mathrm{supp}(f)$ onto X and Y respectively, and observe that K_1 and K_2 are compact. Define the maps $i_x : Y \to X \times Y$ and $i^y : X \to X \times Y$ by:

$$i_x(y) := (x,y), \qquad \text{and} \qquad i^y(x) := (x,y),$$

for all $x \in X$ and $y \in Y$. Then i_x and i^y are continuous, and since $f_x = f \circ i_x$ and $f^y = f \circ i^y$, it follows that f_x and f^y are both continuous.

By definition, the support of a function is closed, and since $\mathrm{supp}(f_x) \subseteq K_2$ and $\mathrm{supp}(f^y) \subseteq K_2$, it follows that $\mathrm{supp}(f_x)$ and $\mathrm{supp}(f^y)$ are compact. Hence $f_x \in \mathcal{K}_0(Y)$ and $f^y \in \mathcal{K}_0(X)$ [5, p221].

Now, define the function $I_X : X \to \mathbb{R}$ by

$$I_X(x) := \int_Y f(x,y)\, \mathrm{d}\mu_2(y),$$

for all $x \in X$ and $y \in Y$. This integral exists by the fact that f_x is continuous. Clearly $\mathrm{supp}(I_X) = K_1$, and we claim that I_X is continuous. To show this, let $x_0 \in X$, and let $\varepsilon > 0$ be an arbitrary constant.

For each $t \in K_2$, we may choose open sets U_t and V_t which contain x_0 and t respectively, such that $|f(x,y) - f(x_0,t)| < \frac{\varepsilon}{2}$ whenever $(x,y) \in U_t \times V_t$. Hence, if $x \in U_t$ and $y \in V_t$, then

$$|f(x,y) - f(x_0,y)| \leq |f(x,y) - f(x_0,t)| + |f(x_0,t) - f(x_0,y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since K_2 is compact, there is a finite sequence of elements $t_1, \ldots, t_n$ of K_2 such that the neighbourhoods $V_{t_1}, \ldots, V_{t_n}$ form an open covering of K_2.

[5] Recall that we write $\mathcal{K}_0(X)$ to denote the set of all continuous functions $f : X \to \mathbb{R}$ with compact support (Def 1.19).

Write $U := \bigcap_{i=1}^{n} U_{t_i}$, and observe that $x_0 \in U$. Then

$$\begin{aligned}
&\left| \int_Y f(x,y)\,\mathrm{d}\mu_2(y) - \int_Y f(x_0,y)\,\mathrm{d}\mu_2(y) \right| \\
&\leq \int_{K_2} |f(x,t) - f(x_0,t)|\,\mathrm{d}\mu_2(t) \\
&\leq \varepsilon \mu_2(K_2).
\end{aligned}$$

Since we can set ε to be arbitrarily small, this proves that I_X is a continuous map, and hence that $I_X \in \mathcal{K}_0(X)$. Similarly, it can be shown that the function $I_Y : Y \to \mathbb{R}$ defined by

$$I_Y(y) := \int_X f(x,y)\,\mathrm{d}\mu_1(x)$$

belongs to $\mathcal{K}_0(Y)$. Hence the integrals

$$\int_X \int_Y f(x,y)\,\mathrm{d}\mu_2(y)\,\mathrm{d}\mu_1(x) \quad \text{and} \quad \int_Y \int_X f(x,y)\,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_2(y)$$

exist for all $x \in X$ and $y \in Y$, and it remains to show that they are equal. To do this, for each $s \in K_1$, use the above argument to choose a neighbourhood U_s such that $|f(x,t) - f(s,t)| < \varepsilon$ whenever $x \in U_s$ and $t \in K_2$. Then, since K_1 is compact, there is a finite sequence of elements $s_1, \ldots, s_n$ in K_1 such that the neighbourhoods $U_{s_1}, \ldots, U_{s_n}$ form an open covering of K_1. Now, define $A_1 := U_{s_1} \cap K_1$, and

$$A_i := \big(U_{s_i} \setminus \big(U_{s_1} \cup \cdots \cup U_{s_{i-1}}\big)\big) \cap K_1,$$

for $i = 2, \ldots, n$. Then the sets $A_1, \ldots A_n$ are disjoint Borel subsets of X, and satisfy $K_1 = \bigcup_{i=1}^{n} A_i$. Notice also that $A_i \subseteq U_{s_i}$ for each i. Let $g : X \times Y \to \mathbb{R}$ be the function defined by:

$$g(x,y) := \sum_{i=1}^{n} \mathbf{1}_{A_i}(x) f(s_i, y),$$

for all $x \in X$ and $y \in Y$. This function approximates f on each A_i, and with this we will show the required equality. By the above, we have that $|f(s,t) - g(s,t)| < \varepsilon$ for all $(s,t) \in K_1 \times K_2$, and since f and g both take value zero outside of $K_1 \times K_2$, it follows that the values

$$\alpha := \left| \int_Y \int_X f(x,y)\,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_2(y) - \int_Y \int_X g(x,y)\,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_2(y) \right|,$$

and

$$\beta := \left| \int_X \int_Y f(x,y)\,\mathrm{d}\mu_2(y)\,\mathrm{d}\mu_1(xy) - \int_X \int_Y g(x,y)\,\mathrm{d}\mu_2(y)\,\mathrm{d}\mu_1(x) \right|$$

satisfy $\alpha, \beta \leq \varepsilon\mu_1(K_1)\mu_2(K_2)$. The double integrals of g, however, are equal to each other, and so

$$\gamma := \left| \int_Y \int_X f(x,y)\,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_2(y) - \int_X \int_Y f(x,y)\,\mathrm{d}\mu_2(y)\,\mathrm{d}\mu_1(x) \right|$$

satisfies $\gamma \leq 2\varepsilon\mu_1(K_1)\mu_2(K_2)$. Since we can set ε to be arbitrarily small, this completes the proof [5, pp144,221]. $\square$

Proposition A.3 (Finite intersection property). *Let X be a topological space, and let $\mathcal{S}$ be a collection of subsets of X. Recall that $\mathcal{S}$ has the* finite intersection property *if, given a finite sub-collection $\{S_1, \ldots, S_n\}$ of $\mathcal{S}$, the intersection $\bigcap_{i=1}^n S_i$ is non-empty.*

The space X is compact if and only if, for every collection $\mathcal{S}$ of closed subsets of X which has the finite intersection property, the (potentially infinite) intersection $\bigcap_{S \in \mathcal{S}} S$ is non-empty [23, pp169-70].

Proof. Let $\mathcal{U}$ be a collection of open subsets of X, and write $\mathcal{S}$ to be the collection of their complements. Notice that each $A \in \mathcal{U}$ is open if and only if each $S \in \mathcal{S}$ is closed.

Suppose that X is compact, which is equivalent to saying that if no finite sub-collection $U_1, \ldots, U_n$ of $\mathcal{U}$ forms an open covering of X, then $\mathcal{U}$ is not a covering of X. But $\mathcal{U}$ covers X if and only if the intersection $\bigcap_{S \in \mathcal{S}} S$ is empty, and the collection $U_1, \ldots, U_n$ covers X if and only if the intersection $\bigcap_{i=1}^n S_i$, where $S_i := U_i^c$, is empty. This last statement is the converse of the finite intersection property.

Backtracking, it follows that X is compact if and only if $\bigcap_{S \in \mathcal{S}} S$ is non-empty whenever $\mathcal{S}$ has the finite intersection property [23, pp169-70]. $\square$

Theorem A.4 (Hahn–Banach). *Let V be a normed vector space over the field $\mathbb{R}$. Recall that a linear map $T : V \to \mathbb{R}$ is said to be* bounded *if there exists some constant $M \geq 0$ such that $\|Tv\| \leq M\|v\|$ for all $v \in V$.*

Let $W \subseteq V$ be a vector subspace, and let $f : W \to \mathbb{R}$ be a bounded linear map. Then there exists a bounded linear map $F : V \to \mathbb{R}$ such that:

(a) $F(w) = f(w)$, for all $w \in W$, and

(b) $\|F\| = \|f\|$ [33, p57].

Proof. For a proof, we refer the reader to [33, p57]. Note that an immediate consequence of this result is the *Hahn–Banach Separation Theorem* (Thm 2.10), which is employed in the proof of the equivalence of the invariant mean and Følner criteria (Thm 2.12). $\square$

Theorem A.5 (Axiom of Choice). *Let $\mathcal{A}$ be a collection of disjoint non-empty sets. Then there exists a set C comprising precisely one element from each set $A \in \mathcal{A}$ [23, p59].* $\square$

Note that the Axiom of Choice is equivalent to saying that the Cartesian product $A_1 \times A_2 \times \cdots$ of all sets $A_i \in \mathcal{A}$ is non-empty. In this article we take the Axiom of Choice to be a fact.

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