2 Graph Theory

2.1 Where possible draw the graphs below. If you can’t draw the graph say why not.

(a) A simple graph with 1 edge and 2 vertices.
(b) A simple graph with 2 edges and 2 vertices.
(c) A non-simple graph with no loops.
(d) A non-simple graph with no multiple edges.
(e) A graph with 6 vertices and degree sequence \(1, 2, 3, 4, 5, 5\).
(f) A simple graph with 6 vertices and degree sequence \(1, 2, 3, 4, 5, 5\).
(g) A simple graph with 6 vertices and degree sequence \(2, 3, 3, 4, 5, 5\).

Answer:

(a) \(\text{\includegraphics{graph.png}}\)

(b) Not possible. A simple graph with 2 vertices has at most 1 edge.
(f) Not possible. If $a$ and $b$ are the vertices of degree 5 then $a$ is adjacent to all the other vertices (except itself). Similarly $b$ is adjacent to all the other vertices. Therefore vertices all have degree at least 2.

(g) $\vdash 2$ Show, by labelling the vertices, that the graphs below are isomorphic.
Answer:
First label the vertices of the LH graph.
Then label the vertices \(g, i, k\) on the RH graph. (Vertices of degree 3 lying on a circuit of degree 3.)
Then label \(a, c, e\) adjacent to them.
then \(b, d, f\).
Then \(h, j, l\).

\[\textbf{I.3} \quad (a) \quad \text{Is it true that any two isomorphic graphs have the same number of vertices? If so why?}
(b) \quad \text{Let } G_1 \text{ and } G_2 \text{ be graphs and let } \phi \text{ be an isomorphism from } G_1 \text{ to } G_2. \text{ If } v \text{ is a vertex of } G_1 \text{ is it true that } \deg(v) = \deg(\phi(v)) \text{ and if so why? Show, using the Handshaking Lemma, that } G_1 \text{ and } G_2 \text{ have the same number of edges.}
(c) \quad \text{Show that isomorphic graphs have the same degree sequence.} \]
(d) If two graphs have the same degree sequence, need they be isomorphic?

**Answer:** Let \( \phi : G_1 = (V_1, E_1) \to G_2 = (V_2, E_2) \) be an isomorphism.

(a) Yes: \( \phi : V_1 \to V_2 \) is bijective.

(b) If the \( d \) vertices \( v_1, \ldots, v_d \) are adjacent to \( v \) in \( G_1 \) then (from the definition of isomorphism) the \( d \) vertices \( \phi(v_1), \ldots, \phi(v_d) \) are adjacent to \( \phi(v) \) in \( G_2 \). Also the number of loops incident to \( v \) and \( \phi(v) \) are equal. Hence \( v \) and \( \phi(v) \) have the same degree.

From the Handshaking Lemma,

\[
|E_1| = \frac{1}{2} \sum_{v \in V_1} \deg(v) = \frac{1}{2} \sum_{v \in V_2} \deg(v) = |E_2|.
\]

(c) If the degree sequence of \( G_1 \) is \( \langle d_1, \ldots, d_k \rangle \), where \( d_i = \deg(v_i) \), then from the previous part of the question, \( d_i = \deg(\phi(v_i)) \) so \( \langle d_1, \ldots, d_k \rangle \) is also the degree sequence of \( G_2 \).

(d) No. The graphs of Example 4.6 in the notes all have the same degree sequence but no two of them are isomorphic.

4.9 Let \( G \) be a graph and let \( u \) and \( v \) be vertices of \( G \) (which may or may not be the same). Suppose that \( G \) contains two distinct paths \( P \) and \( P' \) from \( u \) to \( v \). ("Distinct" means "not equal".) Show that \( G \) contains a cycle.

**Answer:** There are two paths from \( u \) to \( v \). Look at where they first diverge (at vertex \( a \) say) then where they first meet again (at vertex \( b \) say) then put together the two paths from \( a \) to \( b \) and from \( b \) to \( a \) together to form a cycle.