4.4 Let $G$ be a simple graph with $n \geq 2$ vertices. Can $G$ have a vertex of degree 0? Can $G$ have a vertex of degree $n - 1$? Can $G$ have a vertex of degree $n$ or more?

(a) Show that $G$ cannot have degree sequence $(0, 1, \ldots, n - 1)$. [Hint: Consider the vertex of degree 0 and that of $n - 1$.]
(b) Show that $G$ must have at least two vertices of the same degree.

**Answer:** YES $G$ can have a vertex of degree 0 and YES $G$ can have a vertex of degree $n - 1$. NO $G$ cannot have a vertex of degree $n$ or more.

(a) If $v$ is a vertex of degree $n - 1$ then $v$ is adjacent to all other vertices. So there cannot be a vertex of degree 0.

(b) Suppose that all vertices of $G$ have different degrees. Then the degree sequence is $(0, 1, \ldots, n - 1)$. Not possible by (a).

4.5 Which of the graphs (a), (b), (c) and (d) are isomorphic to subgraphs of the graph $G$ where $G$ is one of

(i) the complete graph $K_n$, where $n \geq 1$ (the answer may not be the same for all $n$);
(ii) the Petersen graph;
(iii) the graphs of Example 4.6 in the notes;
(iv) the graph $G$ below?

![Graphs](image)

**Answer:**

(i) (a), (b), (c) and (d) are all subgraphs of $K_n$, for $n \geq 6$. All except (d) are subgraphs of $K_5$. (a) and (b) are subgraphs of $K_4$, but (c) and (d) are not. Only (a) is a subgraph of $K_3$. None are subgraphs of $K_1$ or $K_2$.

(ii) (c) and (d) are subgraphs of the Petersen graph.

(iii) (b) and (d) are subgraphs of the graphs of Example 2.6.

(iv) (b), (c) and (d) are subgraphs of the given graph $G$. 

Answer: Use the cut-down algorithm, and begin by removing all loops.
4.11 Let $G$ be a connected graph and let $e$ be an edge of $G$.

(a) Show that if $e$ appears in every spanning tree for $G$ that $e$ is a bridge (i.e. $G - e$ is disconnected).

(b) Show that if $e$ appears in no spanning tree for $G$ then $e$ is a loop.

**Answer:**

(a) Suppose $e$ is not a bridge. Then $G - e$ is connected. Let $T$ be a spanning tree for $G - e$. Then $T$ is a spanning tree for $G$ which does not contain $e$. Therefore if $e$ belongs to all spanning trees for $G$ it must be a bridge.

(b) Suppose that $e$ is not a loop. Let $T$ be a spanning tree for $G$ which does not contain $e$. Then the graph $T_0$ obtained from $T$ by adding the edge $e$ is connected but is not a tree (as it contains too many edges, see Theorem 2.45). Therefore $T_0$ contains a cycle $C$ and $C$ must contain $e$ (otherwise it would be a cycle in $T$ which is a tree). As $e$ is not a loop it follows that $C$ contains some edge $f \neq e$ of $T$. Removing $f$ from $T_0$ leaves a connected graph $T_0 - f$. As $T_0 - f$ has the same number of vertices and edges as $T$ it is a tree (Theorem 2.45 again). Therefore $T_0 - f$ is a spanning tree for $G$ containing $e$. Hence if $e$ belongs to no spanning tree for $G$ it must be a loop.