2. Spectral graph theory

2.1 From graphs to Linear Algebra

The adjacency matrix

Given a graph $G$ with vertices $V$ and edges $E$, $G = (V, E)$

Enumerate the vertices of $G$ from 1 to $n$

$V = \{v_1, v_2, \ldots, v_n\}$

Vertices $x, y \in V$ are adjacent

if and only if they are connected

by an edge in $E$, we write $x \sim y$

Definition: Adjacency Matrix

$A_G = (a_{ij})_{1 \leq i, j \leq n}$ with

$a_{ij} = \begin{cases} 1 & \text{if } x_i \sim x_j \\ 0 & \text{otherwise} \end{cases}$

The degree of a vertex $x$ is $d_x$, for regular graphs

the degree of each vertex is $d$. 
Example 1

\[ X_1 \quad X_2 \quad X_3 \quad X_4 \]
\[
\begin{array}{cccc}
X_1 & X_2 & X_3 & X_4 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 \\
\end{array}
\]

Example 2

\[ A_e = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix} \]

Remark: The adjacency matrix of a graph is symmetric, \( A_e = A_e^T \).

The eigenvalues of \( A_e \) are \( \mu_j \), where \( \mu_j \) are the zeros of the characteristic polynomial:

\[
\det(tI_n - A_e) =
\]

The identity matrix \( I_n \) and \( tI_n \) are:

\[
I_n = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
\end{pmatrix}
\]

\[
tI_n = \begin{pmatrix}
t & 0 & \ldots & 0 \\
0 & t & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & t \\
\end{pmatrix}
\]

From the determinant:

\[
(t-1)(t-2) - (t)(t-2) = t^2 - 4t + 3 - t^2 + 2t = -2t + 3
\]

Solving for the roots:

\[
\mu_1 = 1, \quad \mu_2 = 2
\]

The characteristic polynomial:

\[
(t^2 - t + 1)^2 = (t - 1)^2
\]

Solutions for \( t \):

\[
t = 1, \quad t = 2 \\
\text{or} \quad t = 1 + \sqrt{3}, \quad t = 1 - \sqrt{3}
\]

Notice the pattern in the expressions. The context suggests a discussion of graph theory and matrix algebra.
As has \( n \) eigenvalues counted with multiplicities:

\[
\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n = -d_{\max} \mu_1 - \mu_2
\]

\( d_{\max} \) is the maximal degree of a unity in\( \mathcal{G} \):

\[
d_{\max} = \max_{x \in \mathcal{G}} d_x
\]

Def. The eigenvalues \( \mu_1, \ldots, \mu_n \) are called spectrum of \( G \)

\[
\det(tI_n - A_G) = (t+1)^{n-1} (t-n+1)
\]

\[
\begin{vmatrix}
  t & -1 & -1 \\
  -1 & t & -1 \\
  -1 & -1 & t
\end{vmatrix} = t^3 - 3t - 2 =
\]

\[
t^3 + t^2 - 3t - 2 =
\]

\[
t^2(t+1) - (t^2 + 3t + 2)
\]

\[
t^2(t+1) - (t^2 + 2t + 2)
\]

\[
t^2(t+1) - (t(t+1)+2(t+1))
\]

\[
t^2(t+1) - (t+1)(t+2)
\]

\[
t^2(t+1) - (t+1)(t-2)
\]

\[
t^2(t+1) - (t+1)(t^2 - t - 2)
\]

\[
t^2(t+1) - (t+1)(t^2 - t - 2)
\]