Chapter #1

EEE 2002

Automatic Control

- **Introduction**
- **1st order dynamics**
- **2nd order dynamics**

Introduction

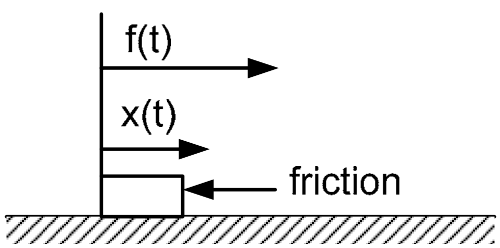
System: is a set of objects/elements that are connected or related to each other in such a way that they create and hence define a unity that performs a certain objective.

Control: means regulate, guide or give a command**.**

Automatic: something can change without the involvement or intervention of a human

Task: To study, analyse and ultimately to control the system to produce a "satisfactory" performance.

Model: Ordinary Differential Equations (ODE):



$$\Sigma F = ma \Leftrightarrow f - f_{friction} = ma \Leftrightarrow f - B\omega = ma \Leftrightarrow f - B\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

Dynamics: Properties of the system, we have to solve/study the ODE.

First order ODEs: $\frac{dx}{dt} = f(x,t)$

Analytic methods to study ODEs:

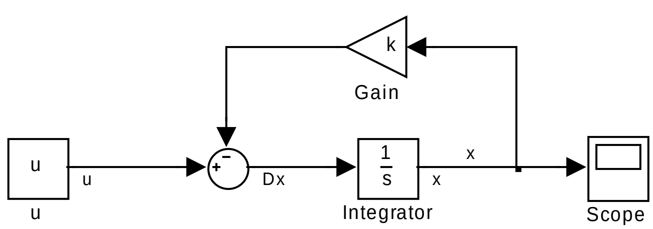
- **Analytic**: Explicit formula for x(t) (a solution – separate variables, integrating factor) which satisfies $\frac{dx}{dt} = f(x,t)$
  - $\frac{dx}{dt} = a \Leftrightarrow \int dx = \int adt \Leftrightarrow x(t) = at + C$ => INFINITE curves (for all Initial Conditions (ICs)).
  - x(t) is called solution of the system which is described by $\frac{dx}{dt} = a$.

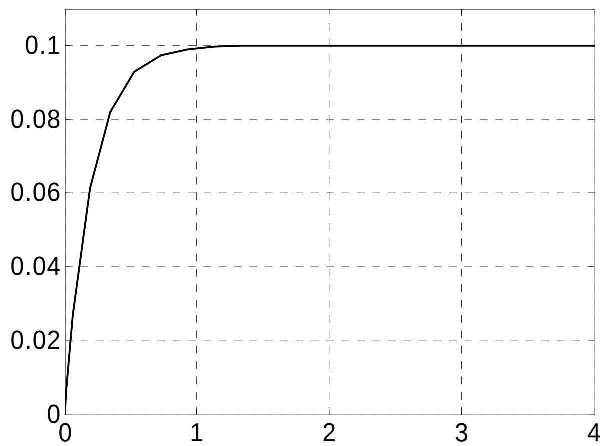
First order linear equations - (linear in *x* and *x'*)

General form: $\begin{cases} a(t)x' + b(t)x = c(t), & \textit{Non autonomous} \\ ax' + bx = c, & \textit{Autonomous} \end{cases}$

Example: $x' + kx = u$

<u>Numerical Solution: </u>k=5, u=0.5





Exercise: Find the response for k=0.5, u=1 and u=-1, initial conditions: 0, 1, -1. Describe the system's behaviour.

<u>Analytical Solutions:</u> (you do not have to know how to derive this)

Integrating factor:

$$e^{\int k dt} \overset{k=const}{=} e^{kt}$$

$$e^{kt}(x'+kx) = e^{kt}u \Rightarrow \left(e^{kt}x\right)' = e^{kt}u$$

$$\Rightarrow \int \left(e^{kt}x\right)' dt = \int e^{kt} u dt$$

$$\Rightarrow e^{kt}x = \int e^{kt}udt + c \Rightarrow x = e^{-kt}\int e^{kt}udt + e^{-kt}c$$

or: $x = e^{-kt}x(0) + e^{-kt}\int_0^t e^{kt_1}udt_1$

Assuming that k>0 the first part is called transient and the second is called steady state solution.

Response to a sinusoidal input

$$y_1'+ky_1 = k\cos(\omega t)$$

$$y_2'+ky_2 = k\sin(\omega t) \Rightarrow jy_2'+kjy_2 = kj\sin(\omega t)$$

$$jy_2'+kjy_2 + y_1'+ky_1 = kj\sin(\omega t)+k\cos(\omega t)$$

$$(y_1'+jy_2')+k(y_1+jy_2) = k(\cos(\omega t)+j\sin(\omega t))$$

$$\tilde{y}'+k\tilde{y} = ke^{j\omega t}$$

Integrating factor: $e^{kt} => \left(\tilde{y}e^{kt}\right)' = ke^{(k+j\omega t)}$

$$\tilde{y}e^{kt} = \frac{k}{k+j\omega t}e^{(k+j\omega t)} \Rightarrow$$

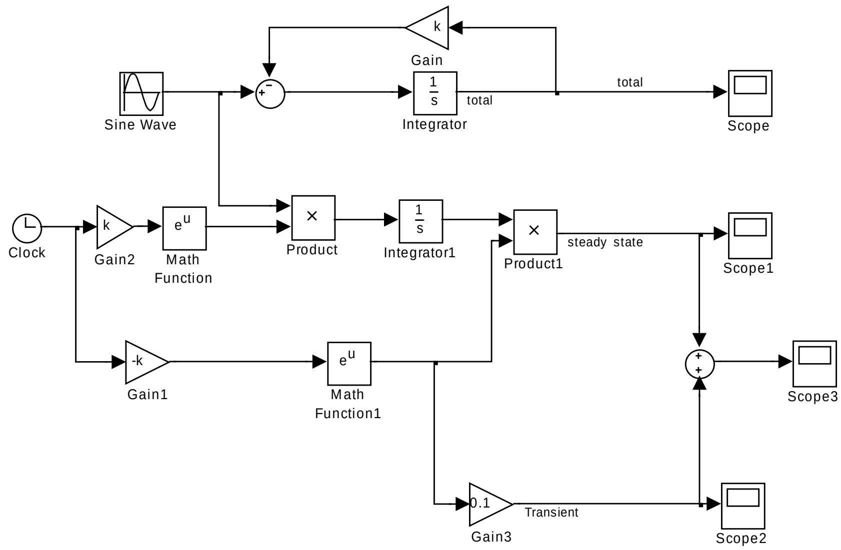
$$\tilde{y} = \frac{k}{k+j\omega t}e^{j\omega t} \Rightarrow$$

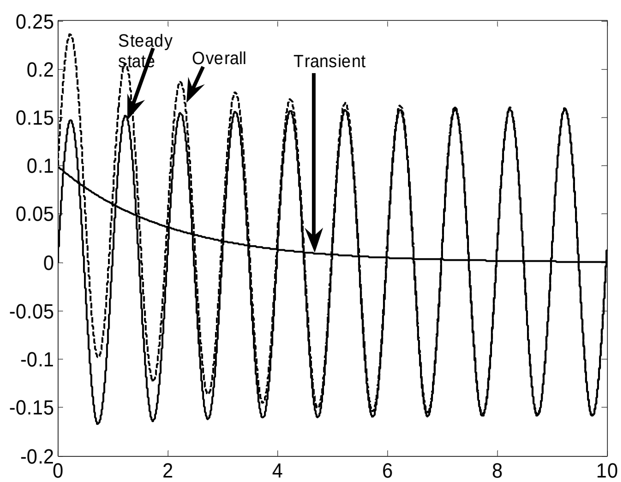
$$\tilde{y} = \frac{1}{\sqrt{1+\omega^2/k^2}}e^{j\left(\omega t-\tan^{-1}(\omega/k)\right)}$$

$$\mathrm{Re}(\tilde{y}) = \mathrm{Re}\left(\frac{1}{\sqrt{1+\omega^2/k^2}}e^{j\left(\omega t-\tan^{-1}(\omega/k)\right)}\right) = \frac{1}{\sqrt{1+\omega^2/k^2}}\cos\left(\omega t-\tan^{-1}(\omega/k)\right)$$

= magnified/attenuated amplitude and phase shifted.

$u = \cos(2\pi t)$ k=1m, x(0)=0.1:





Autonomous 1st order ODEs => Liner Time Invariant (LTI) systems

$\frac{dx}{dt} = f(x)$ (not t on RHS).

Analytic solution:

Can be solved as before: Transient and steady state part.

Second order ODEs: $\frac{d^2x}{dt^2} = f(x', x, t)$

Second order linear ODEs with constant coefficients: $x'' + Ax' + Bx = u$

u=0 => Homogeneous ODE; I need two "representative solutions"

$x'' + Ax' + Bx = 0$, assume $x = e^{rt}$ => $x' = re^{rt}$ & $x'' = r^2 e^{rt}$ =>

$x'' + Ax' + Bx = 0 \Leftrightarrow r^2 e^{rt} + Are^{rt} + Be^{rt} = 0 \Leftrightarrow$

$r^2 + Ar + B = 0$; Characteristic equation => Check its roots.

$$r = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$$

- Roots are real and unequal: r_1 and r_2 ($A^2 > 4B$ =>Overdamped system)

$x_1 = e^{r_1 t}$ and $x_2 = e^{r_2 t}$ are solutions of the ODE =>

$x = C_1 x_1 + C_2 x_2 = C_1 e^{r_1 t} + C_2 e^{r_2 t}$. If r_1 and r_2<0 then $x \to 0$.

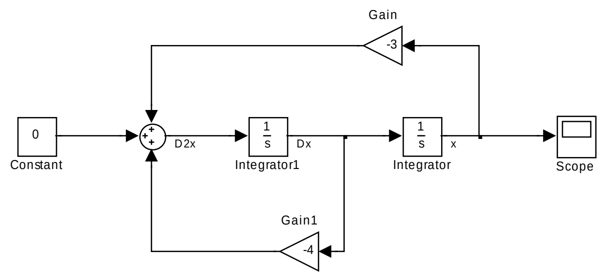
Example:

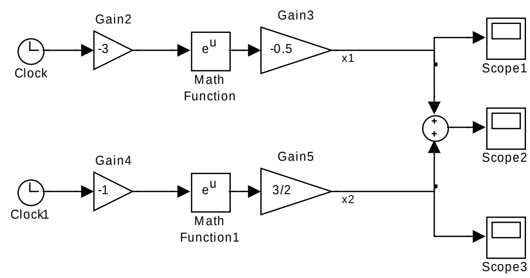
$x'' + 4x' + 3x = 0 \Leftrightarrow r^2 + 4r + 3 = 0 \Leftrightarrow (r+3)(r+1) = 0$

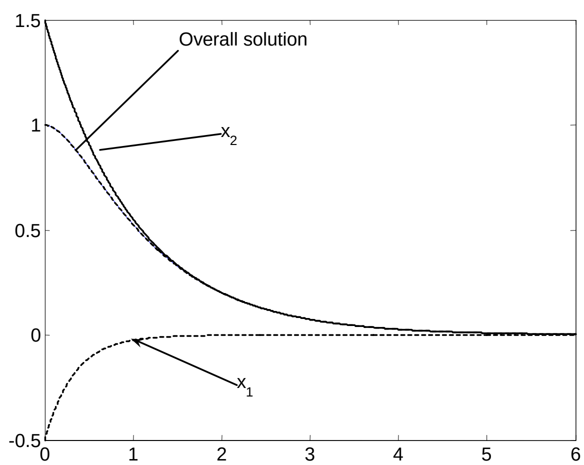
$x = C_1 e^{-3t} + C_2 e^{-t}$. Assume that $x(0) = 1$ and $x'(0) = 0$:

$x(0) = C_1 + C_2 = 1$ and $x' = -3C_1 e^{-3t} - C_2 e^{-t} \Rightarrow$ $x'(0) = -3C_1 - C_2 = 0$ =>

C_1=-0.5, C_2=3/2 => $x = -0.5e^{-3t} + \frac{3}{2} e^{-t}$:





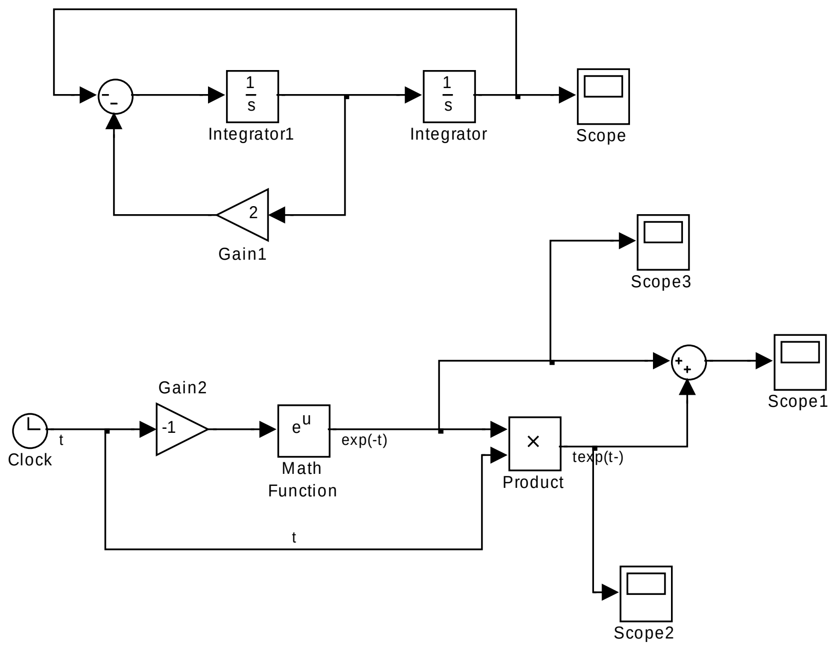


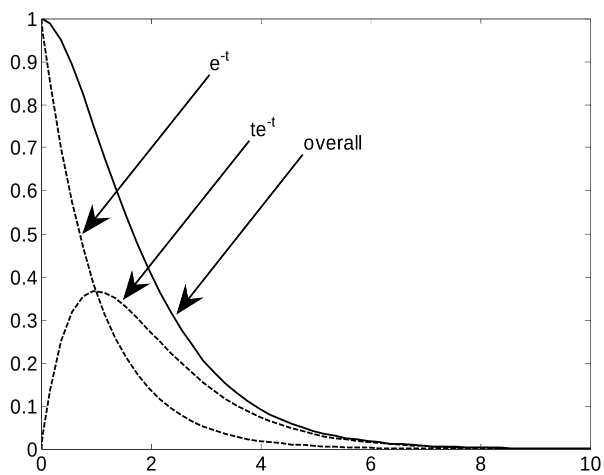
- Roots are real and equal: $r_1=r_2$ ($A^2 = 4B$ Critically damped system)

$x_1 = e^{rt}$ and $x_2 = te^{rt}$ => $x = C_1x_1 + C_2x_2 = C_1e^{rt} + C_2te^{rt}$

Example:

A=2, B=1, x(0)=1, x'(0)=0 => $c_1=c_2=1$





- Roots are complex: r=a+bj $A^2 < 4B$
  - o Underdamped system $A \neq 0$

So $x = e^{rt} = e^{(a+bj)t} = e^{at+bjt} = e^{at}e^{jbt} = e^{at}\left(\cos(bt) + j\sin(bt)\right)$=Re+jIm.

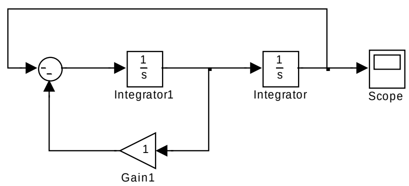
Theorem: If x is a complex solution to a real ODE then Re(x) and Im(x) are the real solutions of the ODE:

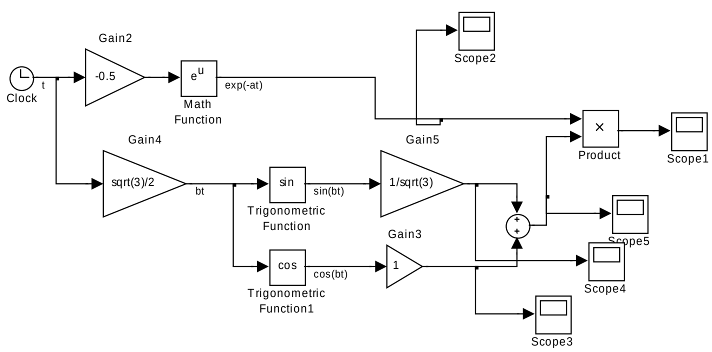
$$x_1 = e^{at}\cos(bt),\ x_2 = e^{at}\sin(bt)\ \Rightarrow$$

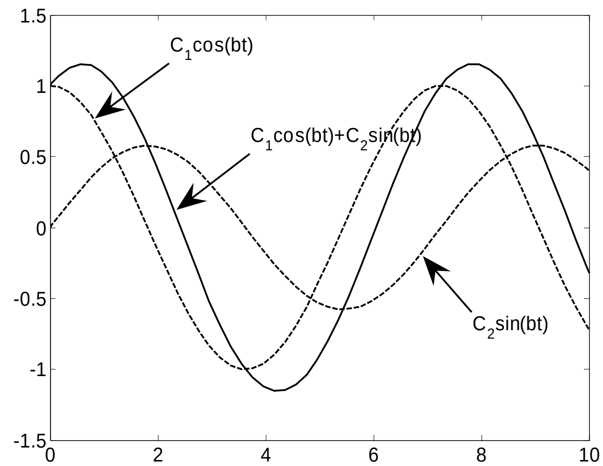
$$x = c_1 x_1 + c_2 x_2 = c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)$$
$$= e^{at}\left(c_1\cos(bt) + c_2\sin(bt)\right) = e^{at} G\cos(bt - \phi)$$

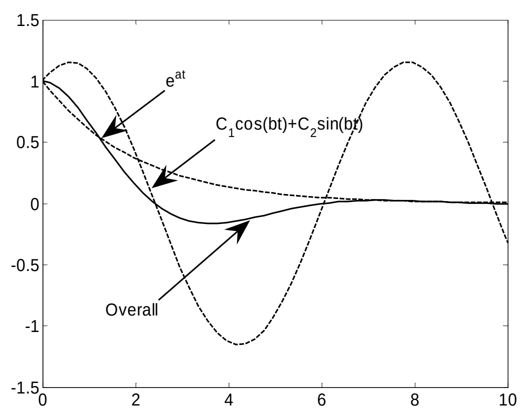
where $G = \dfrac{c_1}{\cos\left(\tan^{-1}\left(c_2/c_1\right)\right)}$, & $\phi = \tan^{-1}\left(c_2/c_1\right)$

A=1, B=1, x(0)=1, x'(0)=0 => c_1=1, c_2=1/sqrt(3)







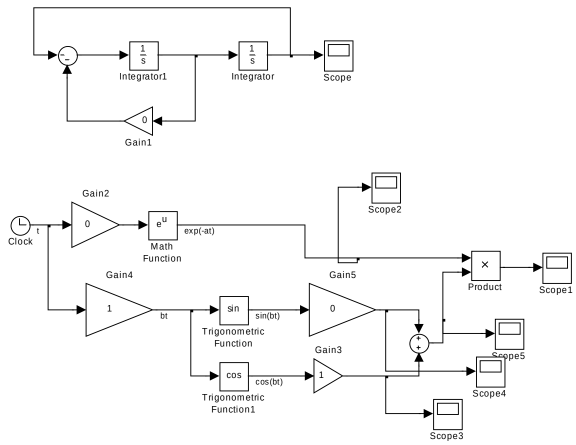


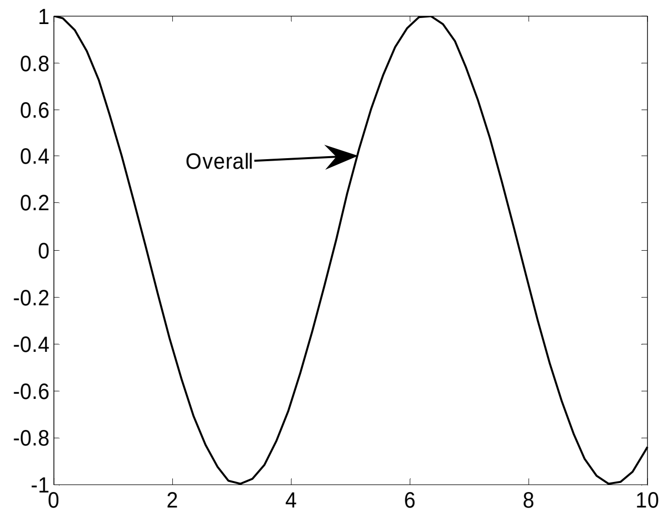
- Undamped system $A = 0$

$x'' + 0 + Bx = 0 \Leftrightarrow r^2 e^{rt} + 0 + Be^{rt} = 0 \Rightarrow r^2 = -B$ => Imaginary roots (If B<0 then I would have two equal real roots).

So $r = jb$ => $x = c_1 \cos(bt) + c_2 \sin(bt) = G\cos(bt - \phi)$

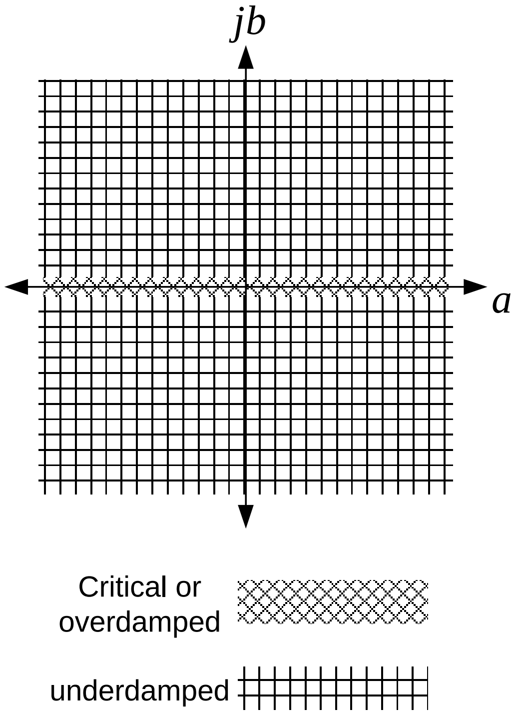
A=0, B=1, x(0)=1, x'(0)=0 =>c_1=1, c_2=0:

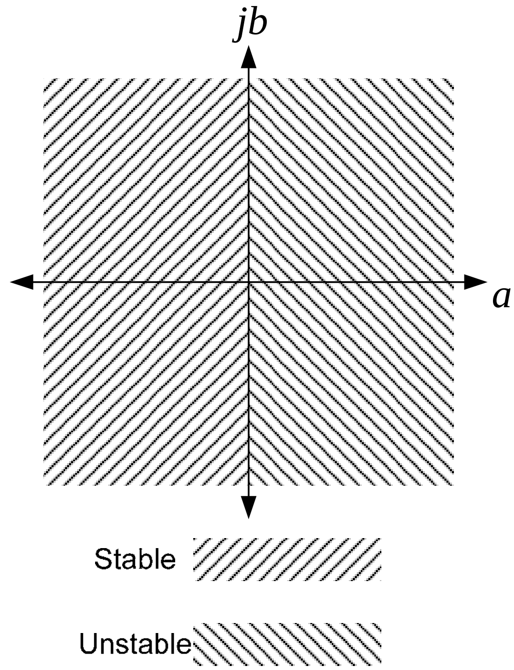




In all previous cases if the real part is positive then the solution will diverge to infinity and the ODE (and hence the system) is called unstable.

Space of roots





Natural frequency, damping frequency, damping factor

2nd order systems very important with rich dynamic behaviour

So $A = 2\zeta\omega_n, \quad B = \omega_n^2$ => $x'' + 2\zeta\omega_n x' + \omega_n^2 x = 0$

ζ is the damping factor and ω_n is the natural frequency of the system.

$$r = \frac{-A \pm \sqrt{A^2 - 4B}}{2} = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2}$$

$$r = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$$

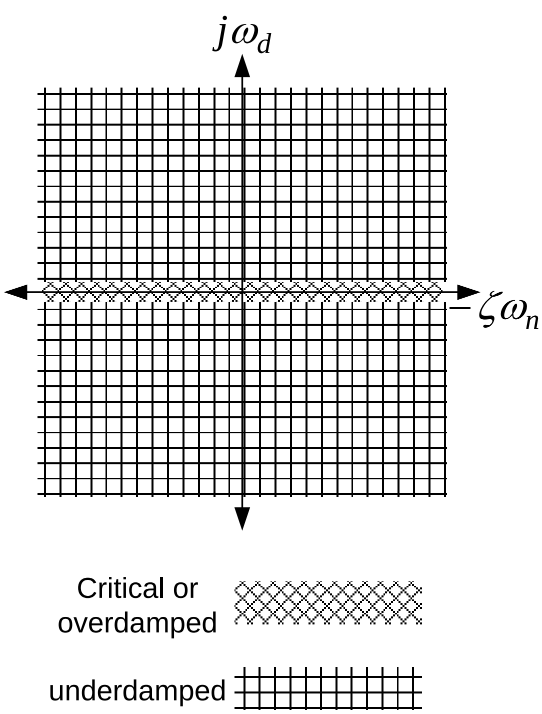
1. Real and unequal $\zeta^2\omega_n^2 > \omega_n^2 \Leftrightarrow \zeta^2 > 1 \overset{\zeta>0}{\Rightarrow} \zeta > 1$ => Overdamped system implies that $\zeta > 1$; $r_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$ => replace at $x = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.
2. Real and equal $\zeta^2\omega_n^2 = \omega_n^2 \Leftrightarrow \zeta^2 = 1 \Rightarrow \zeta = 1$ => Critically damped system implies that $\zeta = 1$; $r = -\omega_n$ => $x = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t}$.
3. Complex $\zeta^2\omega_n^2 < \omega_n^2 \Leftrightarrow \zeta^2 < 1 \overset{\zeta>0}{\Rightarrow} \zeta < 1$ => Underdamped systems implies $\zeta < 1$;

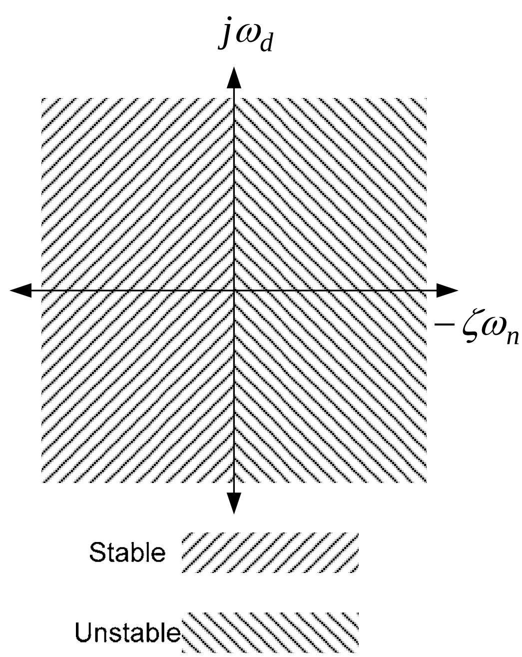
   $r_{1,2} = -\zeta\omega_n \pm j\sqrt{\omega_n^2 - \zeta^2\omega_n^2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

   $\overset{\omega_d = \omega_n\sqrt{1-\zeta^2}}{=} -\zeta\omega_n \pm j\omega_d$ => $x = e^{-\zeta\omega_n t} G\cos(\omega_d t - \phi)$; ω_d is called damped frequency or pseudo-frequency.
4. Imaginary roots $\zeta = 0$ and therefore the solution is $r = \pm j\omega_n$ => $x = G\cos(\omega_n t - \phi)$; so when there is no damping the frequency of the oscillations = natural frequency. $x = G\cos(\omega_n t - \phi)$
5. In all the previous cases if $\zeta > 0$ then the transient part tends to zero. If $\zeta < 0$ then the system will diverge to infinity with or without oscillations.

ζ	Oscillations?	Name	Components of solution
$\zeta > 1$	No	Overdamped	Two exponentials: $e^{k_1 t}, e^{k_2 t}$, $k_1, k_2 < 0$
$\zeta = 1$	No	Critically damped	Two exponentials: e^{kt}, te^{kt}, $k < 0$
$\zeta < 1$	Yes	Underdamped	One exponential and one cosine $e^{kt}, \cos(\omega t)$, $k < 0$
$\zeta = 0$	Yes	Undamped	one cosine $\cos(\omega t)$

If $\zeta < 0$ then cases 1-3 are the same but with $k > 0$





NonHomogeneous (NH) differential equations

$x''+Ax'+Bx=u$

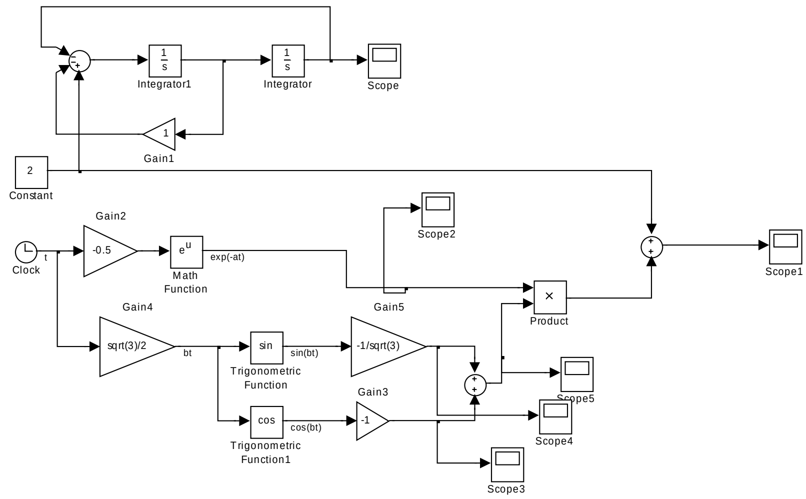
- u=0 => Homogeneous => x_1 & x_2.
- Assume a particular solution of the nonhomogeneous ODE: x_p
  - If u(t)=R=cosnt => $x_P = \frac{R}{B}$
- Then all the solutions of the NHODE are $x = x_P + c_1 x_1 + c_2 x_2$
- So we have all the previous cases for under/over/un/critically damped systems plus a constant R/B.
- If complementary solution is stable then the particular solution is called steady state.

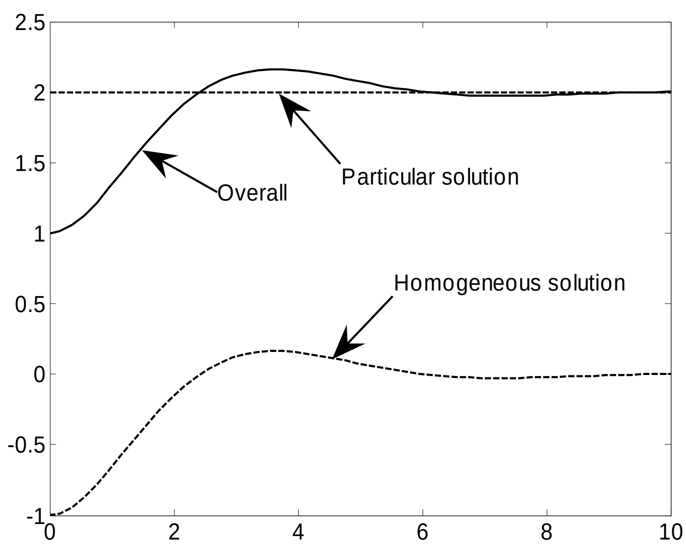
Example:

$x''+x'+x=2 \Rightarrow x_P = 2$

$$x = 2 + c_1 x_1 + c_2 x_2 = 2 + e^{at}\left(c_1 \cos(bt) + c_2 \sin(bt)\right)$$

x(0)=1, x'(0)=0 => c_1=-1, c_2=-1/sqrt(3)


Integrator1
Integrator
Scope
Gain1
Constant
Gain2
-0.5
Math
Function
exp(-at)
Scope2
Scope1
Clock
Product
Gain4
sqrt(3)/2
bt
sin
sin(bt)
Trigonometric
Function
Gain5
-1/sqrt(3)
Scope5
Gain3
cos
cos(bt)
-1
Trigonometric
Function1
Scope4
Scope3


Overall
Particular solution
Homogeneous solution