Chapter #3

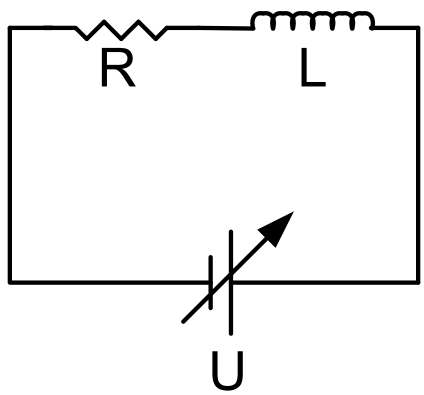
EEE 2002

Automatic Control

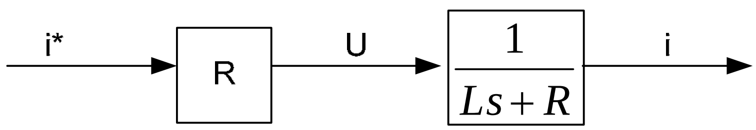
- **Closed loop systems**
- **Steady state error**
- **PID control**
- **Other controllers**

Introduction

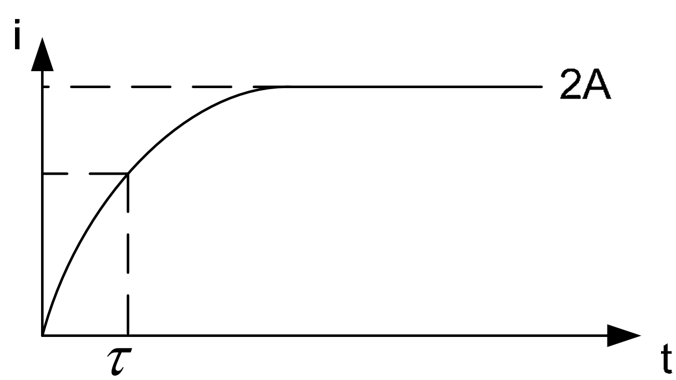
Assume the following electrical system:



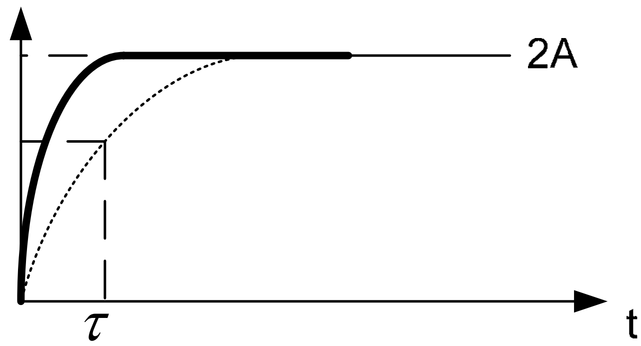
Task: Change/**control** U : i=i* (i*=demanded current), assume R=2 Ohms, L=1 H and i*=2 A:



Expected response:

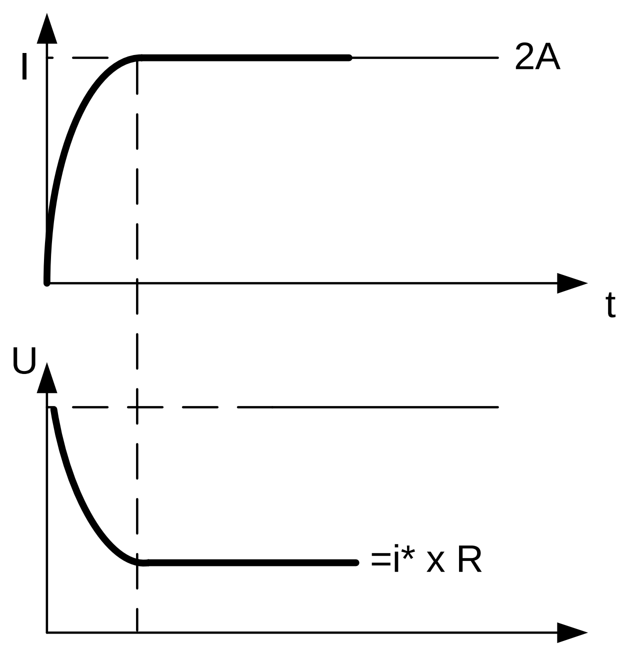


Assume that you want a faster system (same steady state):



L & R = constant hence we can ONLY change U but then $i_{ss} \neq i^*$

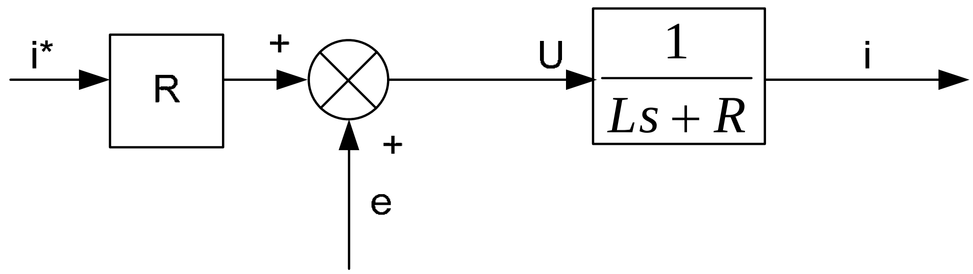
Ideally we would like:



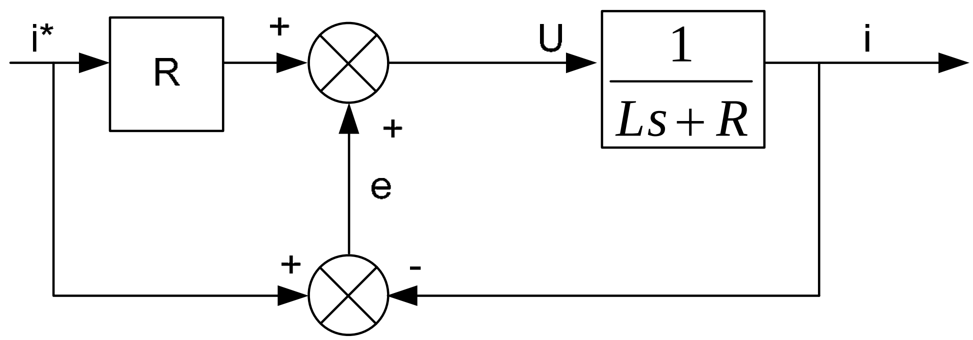
So U=f(i*,i),

- $i \to i^* \Rightarrow U \to i^* R$
- $i - i^* \to 0 \Rightarrow U \to i^* R$, the difference $i - i^*$: is called **ERROR**

So we would like:



Where $e \to 0 \Rightarrow i \to i^*$



Crosscheck:

$$G = \frac{1}{Ls + R}$$

$$\left.\begin{aligned} i &= UG \\ U &= e + i^* R = i^* - i + i^* R \end{aligned}\right\} \Rightarrow i = i^*(1+R)G - iG \Rightarrow$$

$$\Rightarrow i(1+G) = i^*(1+R)G \Rightarrow \frac{i}{i^*} = \frac{G(1+R)}{1+G} = \frac{1}{\frac{L}{(R+1)}s + 1}$$ which

implies that $\tau_{new} = \frac{L}{(R+1)} \Rightarrow \tau_{new} < \tau \Rightarrow$ faster system.

At the steady state:

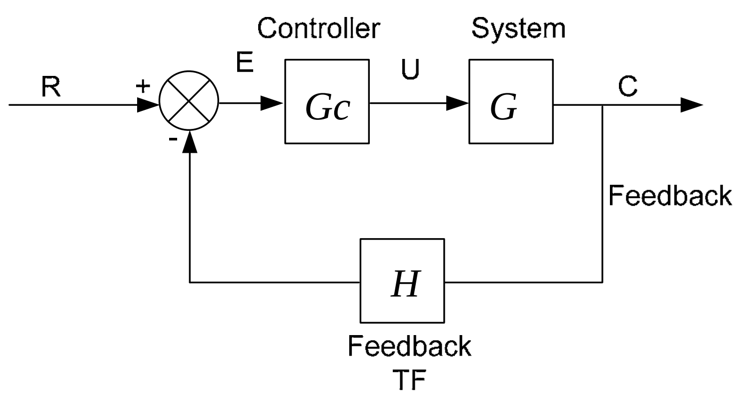
$$i = i^* \frac{1}{L/(R+1)\, s + 1} \Rightarrow i_{ss} = \lim_{s \to 0} s \frac{2}{s} \frac{1}{L/(R+1)\, s + 1} = 2.$$

Hence the input to the system sees the output: Feedback (cornerstone of control systems) this is a **CLOSED LOOP** system.

General closed loop systems

A more general form for a CL system:

$\frac{C}{R} = \frac{G'}{1+G'H}$, where $G' = Gc \times G$, Gc= TF of controller, G TF of plant or Open Loop (OL) TF:



Our Task: DESIGN Gc and H (if applicable).

Assume H=1 and Gc=K=const.

$$\frac{C}{R}=\frac{KG}{1+KG}=\frac{K}{LS+R+K}\Rightarrow C_{ss}=\lim_{s\to 0} s\frac{2}{s}\frac{K}{LS+R+K}=\frac{K}{R+K}$$

If K>>R then Css=2, also:

$$\frac{C}{R}=\frac{K}{LS+R+K}\Rightarrow\frac{K/(R+K)}{L/(R+K)S+1}$$ so

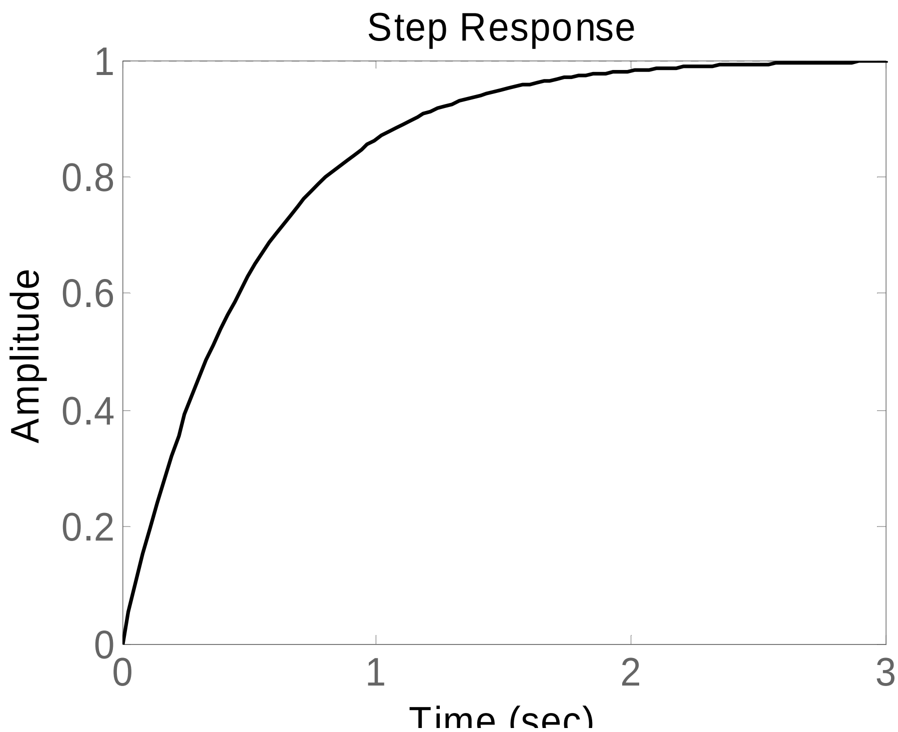
$$\tau_{new}=L/(R+K)\Rightarrow\tau_{new}<\tau\Rightarrow$$ faster system.

A closed loop can also achieve a predefined steady state:

OLFT: $G=\dfrac{2}{s+2}$

Open loop response (desired current=1A):

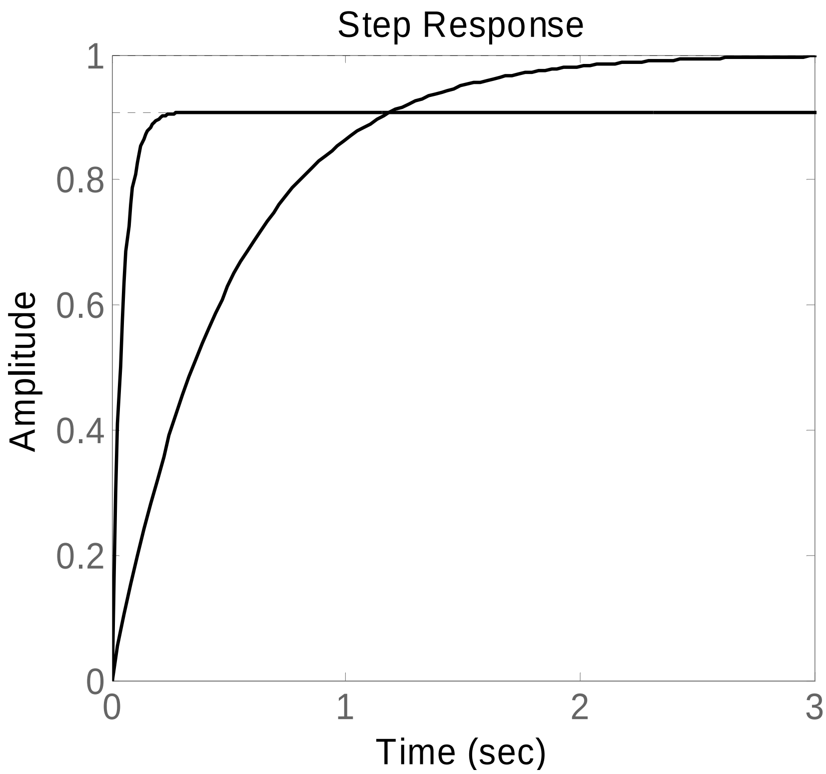
```
>> num=2;
>> den=[1 2];
>> g=tf(num,den);
>> step(g)
```



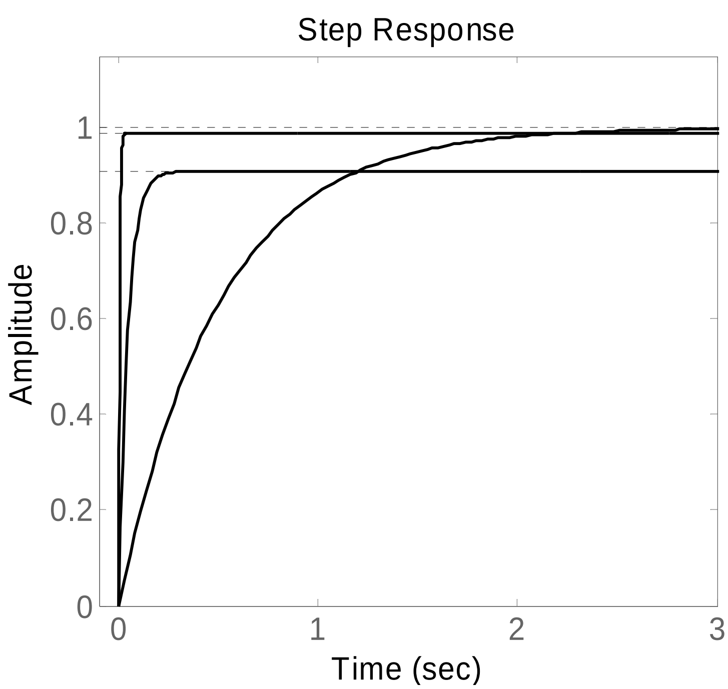
CLTF: $G_{CL} = \frac{C}{R} = \frac{K}{s+2+K} \Rightarrow Css = \frac{K}{2+K}$

Closed loop response (k=10):

```
>> numc=10;
>> denc=1;
>> gc=tf(numc,denc);
>> gol=series(g,gc);
>> h=tf(1,1);
>> gcl=feedback(gol,h);
>> hold on
>> step(gcl)
```


Step Response
Amplitude
Time (sec)
0
0.2
0.4
0.6
0.8
1
0
1
2
3

K=100:

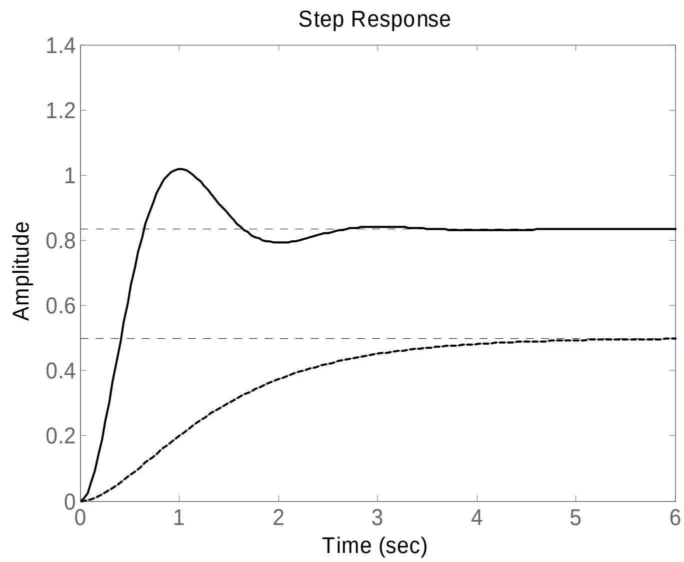

Step Response
Amplitude
Time (sec)
0
0.2
0.4
0.6
0.8
1
0
1
2
3

Higher order systems

Until now we have seen only 1st order systems with feedback. In these systems, the feedback and the controller influence the steady state and the time constant. The decrease of the time constant means that we moved the pole further into “minus infinity” area. Hence we changed the s-plane of the system. What is it going to happen if we use a 2nd order system in a feedback system?

$$G = \frac{1}{(s+1)(s+2)}$$

```
>> num=1;
>> den=conv([1 1],[1 2]);
>> g=tf(num,den);
>> step(g)
>> hold
Current plot held
>> numc=10;
>> denc=1;
>> gc=tf(numc,denc);
>> gol=series(g,gc);
>> h=tf(1,1);
>> gcl=feedback(gol,h);
```



So apart from faster system and smaller steady state error we have oscillations!

$$G_{CL} = \frac{k}{(s+1)(s+2)+k} = \frac{k}{s^2+3s+2+k}$$

CE: $s^2+3s+2+k=0$ but the general from: $s^2+2\zeta\omega_n s+\omega_n^2=0$ so $2\zeta\omega_n=3$ and $\omega_n^2=2+k$ and for k=10 $\omega_n^2=12 \Leftrightarrow \omega_n=\sqrt{12}$ and hence $2\zeta\sqrt{12}=3 \Rightarrow \zeta=0.433$:

```
>> [wn,z]=damp(gcl)

wn =

   3.46410161513775
   3.46410161513775

z =

   0.43301270189222
```

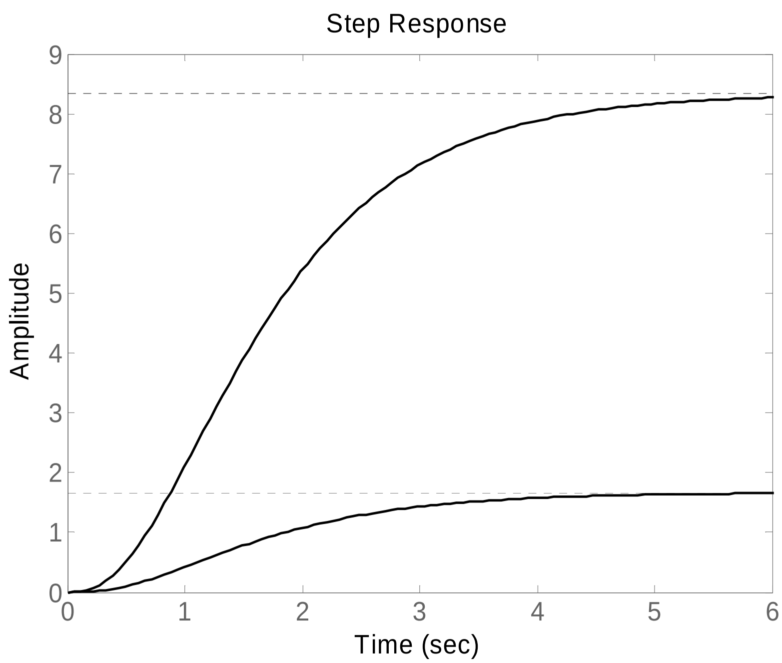
```
   0.43301270189222
```

So the feedback and the controller can completely change the location of the poles in the s-plane.

Example: $G = \dfrac{K}{(s+1)(s+2)(s+3)}$

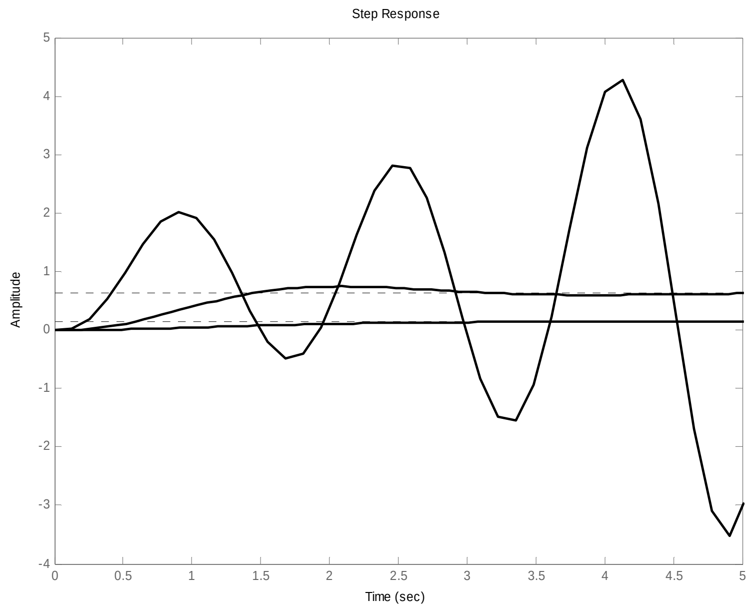
```
>> num=10;
>> den=conv(conv([1 1],[1 2]),[1 3]);
>> g=tf(num,den);
>> step(g)
>> hold
Current plot held
>> num=50;
>> g=tf(num,den);
>> step(g)
```

The open loop response for various gains is:



The open loop system will be stable for all values of K since they do not influence the poles of the system.

The response of closed loop system for K=1, 10, 100 is:



Hence the feedback may introduce instability.

To understand why we have these changes solve the CE. We will extensively study this at the “Root Locus” chapter.

Properties of feedback systems:

1. Minimise steady state error.
2. Faster system.
3. Less sensitive to system uncertainties.
4. Introduce instability (even for negative feedback).
5. Expensive (we need to feedback the signal, i.e. use a sensor).

Systems classification

Types

We already have seen some of these categories:

- Electrical
- Mechanical
- Hydraulic

- o 1st order
- o 2nd order
- o Higher order

- ➢ Overdamped
- ➢ Underdamped
- ➢ Critically damped

- ▪ Stable
- ▪ Unstable
- ▪ Marginally stable and others

Another way to classify control systems is to use the number of the poles at the origin of the OPEN LOOP system:

$$G_{OL}(s) = \frac{K(s+b_1)(s+b_2)\cdots(s+b_n)}{s^N(s+c_1)(s+c_2)\cdots(s+c_m)}$$

This implies a pole multiplicity N at the origin. This system is called type N.

- $G_{OL}(s) = \dfrac{s+10}{(s+3)(s+4)}$ => Type 0
- $G_{OL}(s) = \dfrac{(s-1)(s+5)}{s(s+3)(s+4)}$ => Type 1
- $G_{OL}(s) = \dfrac{1}{s^2(s+3)(s+4)}$ => Type 2
- $G_{OL}(s) = \dfrac{1}{s^2(s^2+5s-100)}$ => Type 2
- $G_{OL}(s) = \dfrac{1}{s^5(s^{10}-1)}$ => Type 5

Note: THE TYPE OF THE SYSTEM IS DIFFERENT FROM THE ORDER OF THE SYSTEM!!!

Error constants

Consider a **<u>unity</u>** feedback system with an OLTF, G(s), hence:

$$G_{CL}(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}.$$

$E(s) = R(s) - C(s)$, this implies that the transfer function between the error and the input is:

$$\frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = 1 - \frac{G(s)}{1+G(s)} = \frac{1}{1+G(s)}$$

And therefore $E(s) = R(s)\frac{1}{1+G(s)}$

From the final value theorem: $E_{ss} = \lim_{s\to 0}\left(\frac{sR(s)}{1+G(s)}\right)$

For a unit step input:

$$E_{ss} = \lim_{s\to 0}\left(\frac{s\,{}^{1}\!/\!{}_{s}}{1+G(s)}\right) = \frac{1}{1+G(0)}$$

We define as static position error constant as K_p: $K_p = \lim_{s\to 0}(G(s)) = G(0)$

Hence $E_{ss} = \frac{1}{1+K_p}$

For a unit ramp:

$$E_{ss} = \lim_{s\to 0}\left(\frac{s\,{}^{1}\!/\!{}_{s^2}}{1+G(s)}\right) = \lim_{s\to 0}\left(\frac{1}{s+sG(s)}\right) = \lim_{s\to 0}\left(\frac{1}{sG(s)}\right)$$

We define as static velocity error constant as K_u: $K_u = \lim_{s\to 0}(sG(s))$

And therefore $E_{ss} = \frac{1}{K_u}$

For a unit parabolic input:

$$E_{ss} = \lim_{s\to 0}\left(\frac{s\ 1/s^3}{1+G(s)}\right) = \lim_{s\to 0}\left(\frac{1}{s^2 + s^2G(s)}\right) = \lim_{s\to 0}\left(\frac{1}{s^2G(s)}\right)$$

We define as static acceleration error constant as K_a: $K_a = \lim_{s\to 0}\left(s^2\ G(s)\right)$

And therefore $E_{ss} = \dfrac{1}{K_a}$

Types and error constants

Another way to classify control systems is by their static error with a combination of their type.

<u>Error and unit step $r(t) = 1$:</u>

1. Type 0: $G_{OL}(s) = \dfrac{K(s+b_1)(s+b_2)\cdots(s+b_m)}{(s+c_1)(s+c_2)\cdots(s+c_n)}$ so

$$K_p = \lim_{s\to 0}(G(s)) = \lim_{s\to 0}\left(\frac{K(s+b_1)(s+b_2)\cdots(s+b_m)}{(s+c_1)(s+c_2)\cdots(s+c_n)}\right) = K'$$

So $E_{ss} = \dfrac{1}{1+K_p}$

2. Type N>0: $G_{OL}(s) = \dfrac{K(s+b_1)(s+b_2)\cdots(s+b_n)}{s^N(s+c_1)(s+c_2)\cdots(s+c_m)}$

$$K_p = \lim_{s\to 0}(G(s)) = \lim_{s\to 0}\left(\frac{K(s+b_1)(s+b_2)\cdots(s+b_n)}{s^N(s+c_1)(s+c_2)\cdots(s+c_m)}\right) = \infty$$

So $E_{ss} = 0$

Error and unit ramp $r(t) = t$:

1. Type 0: $G_{OL}(s) = \dfrac{K(s+b_1)(s+b_2)\cdots(s+b_n)}{s(s+c_1)(s+c_2)\cdots(s+c_m)}$ so

$$K_u = \lim_{s\to 0}(sG(s)) = \lim_{s\to 0}\left(s\frac{K(s+b_1)(s+b_2)\cdots(s+b_n)}{(s+c_1)(s+c_2)\cdots(s+c_m)}\right) = 0$$

So $E_{ss} = \infty$

2. Type 1: $G_{OL}(s) = \dfrac{K(s+b_1)(s+b_2)\cdots(s+b_n)}{s(s+c_1)(s+c_2)\cdots(s+c_m)}$

$$K_u = \lim_{s\to 0}(sG(s)) = \lim_{s\to 0}\left(s\frac{K(s+b_1)(s+b_2)\cdots(s+b_n)}{s(s+c_1)(s+c_2)\cdots(s+c_m)}\right) = K'$$

So $E_{ss} = \dfrac{1}{K_v}$

3. Type N>1: $G_{OL}(s) = \dfrac{K(s+b_1)(s+b_2)\cdots(s+b_n)}{s^N(s+c_1)(s+c_2)\cdots(s+c_m)}$

$$K_a = \lim_{s\to 0}(sG(s)) = \lim_{s\to 0}\left(s\,\frac{K(s+b_1)(s+b_2)\cdots(s+b_n)}{s^N(s+c_1)(s+c_2)\cdots(s+c_m)}\right) = \infty$$

So $E_{ss} = 0$

Error and unit parabolic input $r(t) = \frac{1}{2}t^2$**:**

Similarly we find that:

Type 0: $E_{ss} = \infty$

Type 1: $E_{ss} = \infty$

1. Type 2: $E_{ss} = \dfrac{1}{K_a}$
2. Type N>2: $E_{ss} = 0$

From the above the following table can be derived:

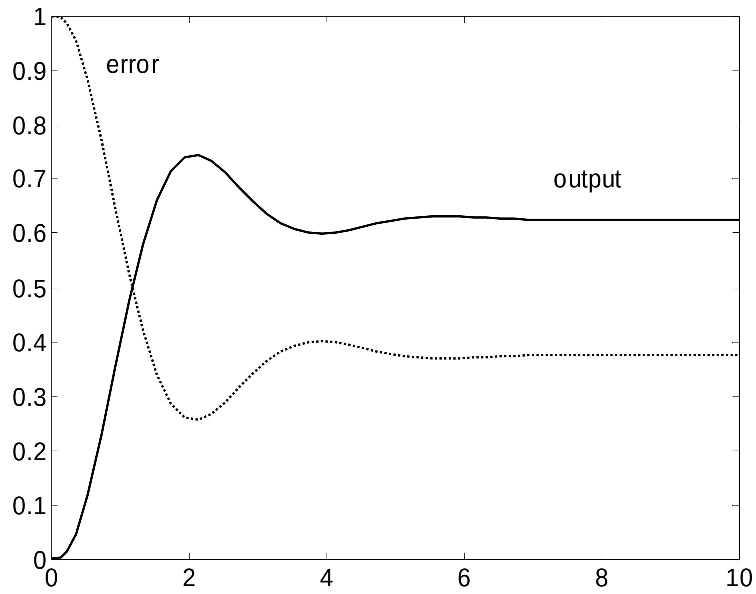
	$r(t)=1$	$r(t)=t$	$r(t)=\frac{1}{2}t^2$
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$

Hence if we want to decrease the error we have to increase the type of the system.

PID control

From previous table, if the type is 0 then we have a steady state error for a step input. Also by increasing K, we increase the oscillations and we may cause instability to the system. Hence we need another solution.

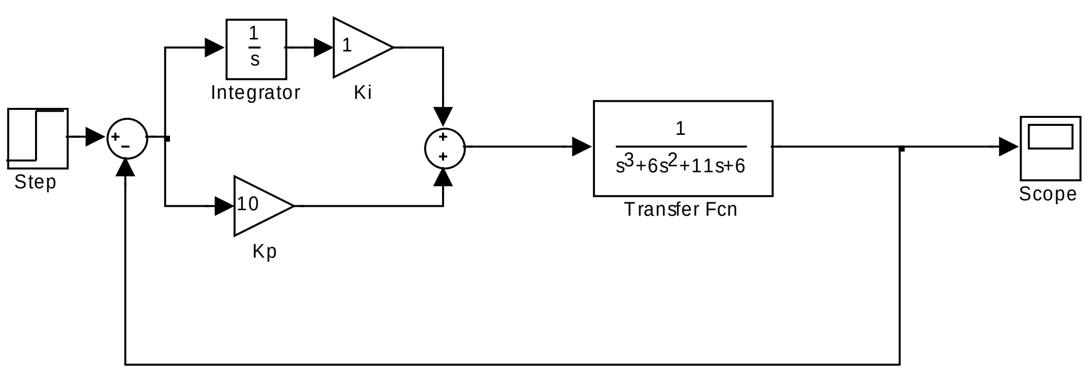
Consider the system: $G(s)=\frac{K}{(s+1)(s+2)(s+3)}$ for K=10:

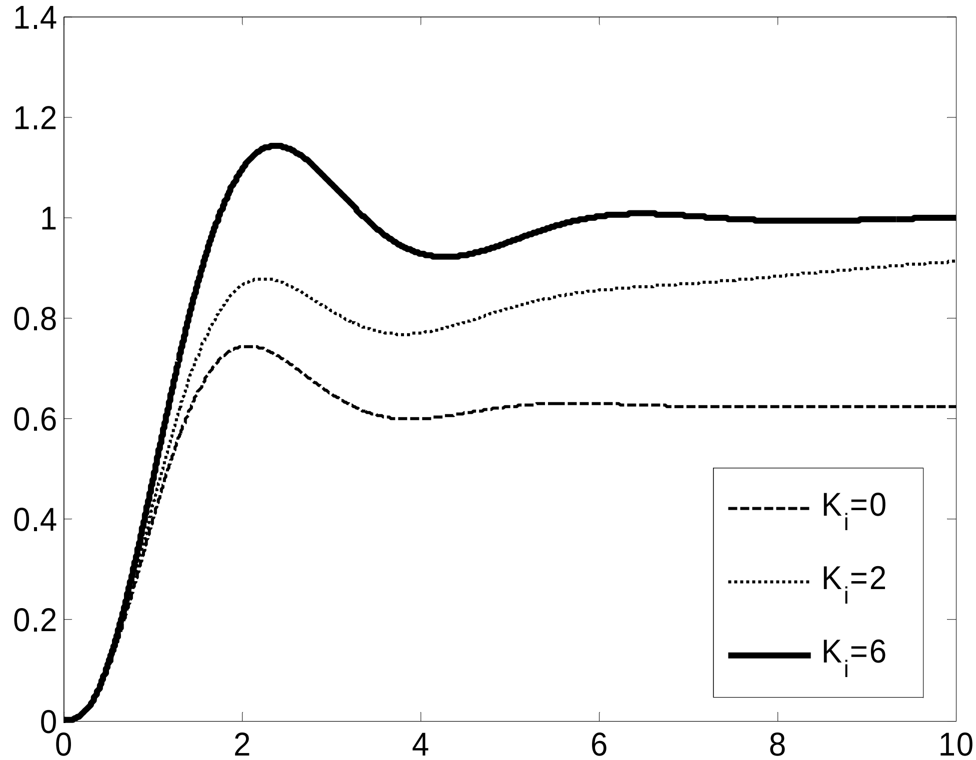


To increase the type of the OLTF (which is G_CxG) we add an integrator:

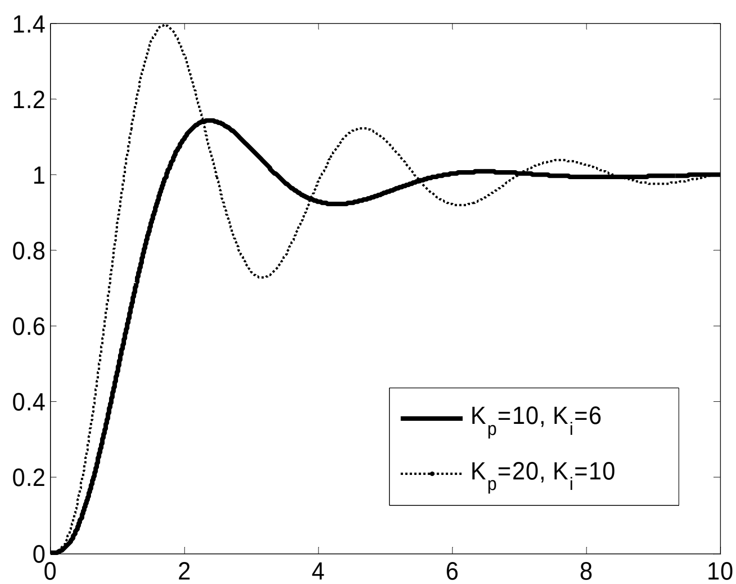
$$G_c(s) = K_p + K_i \frac{1}{s} \;\Rightarrow\; G_{OL}(s) = \frac{sK_p + K_i}{s} \frac{1}{(s+1)(s+2)(s+3)}$$

(This is the so-called PI controller).

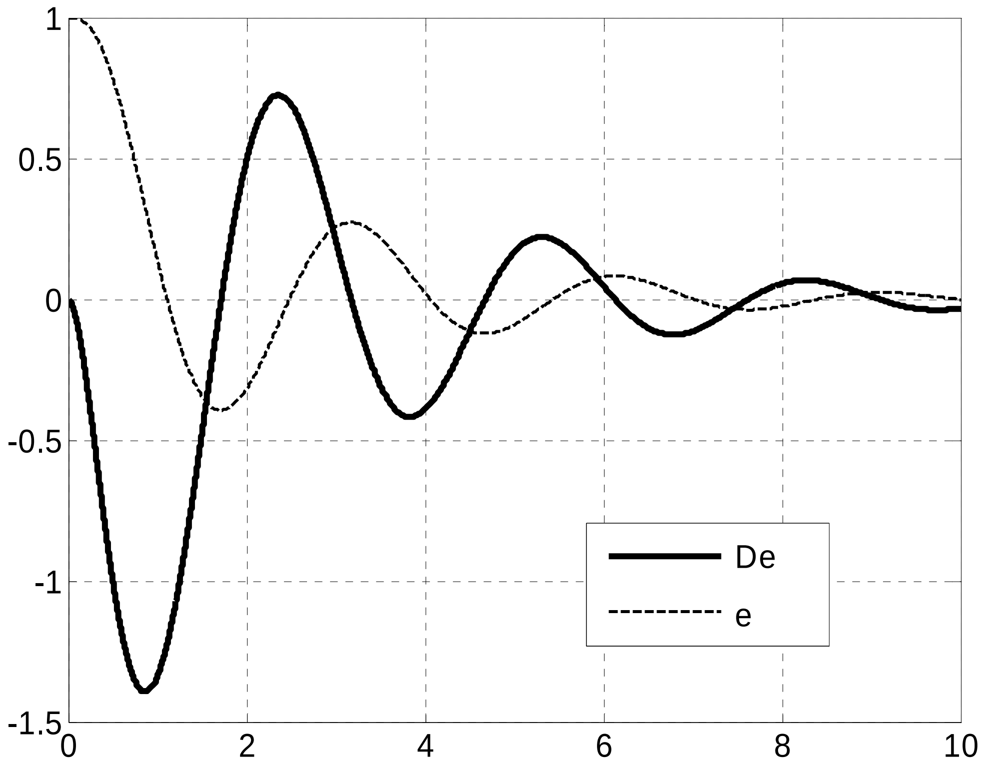




Further increase of Kp, Ki:



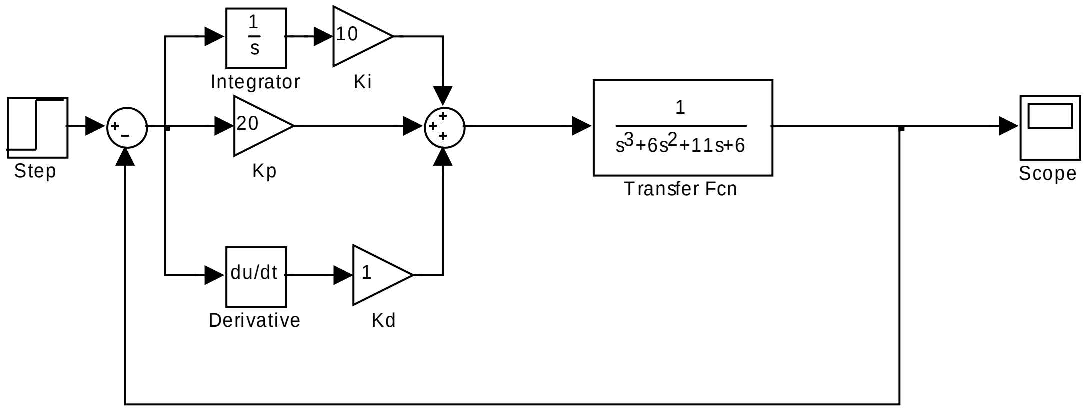
Check the derivative of error, i.e. the rate of change of e:

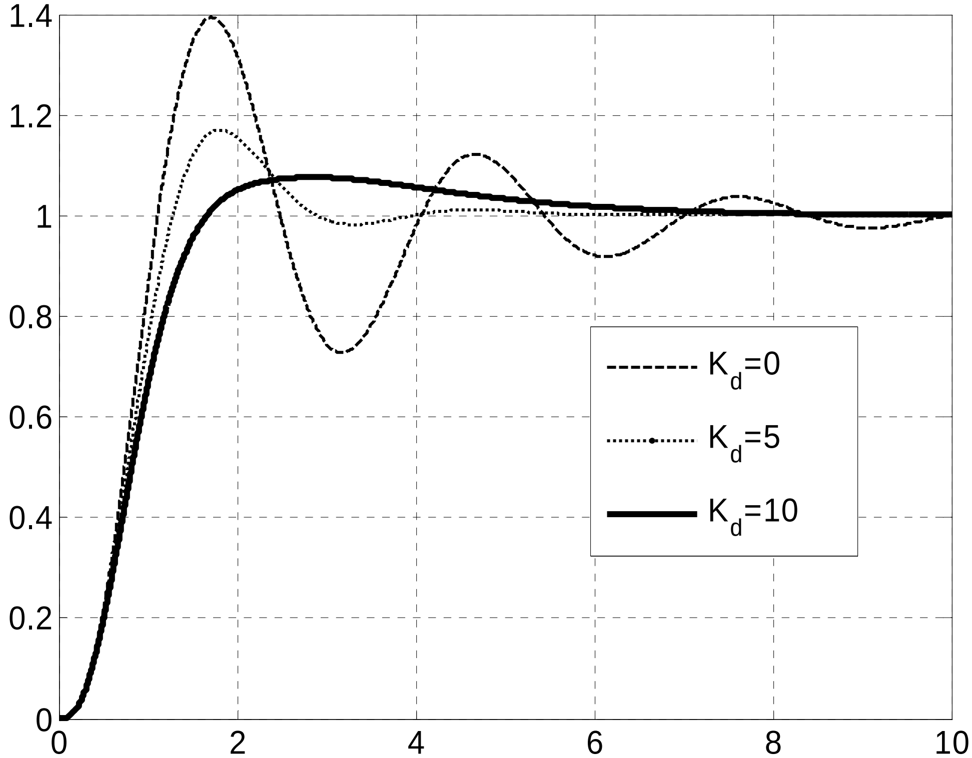


Maximum value of De just before e=0. So De can control the oscillations:

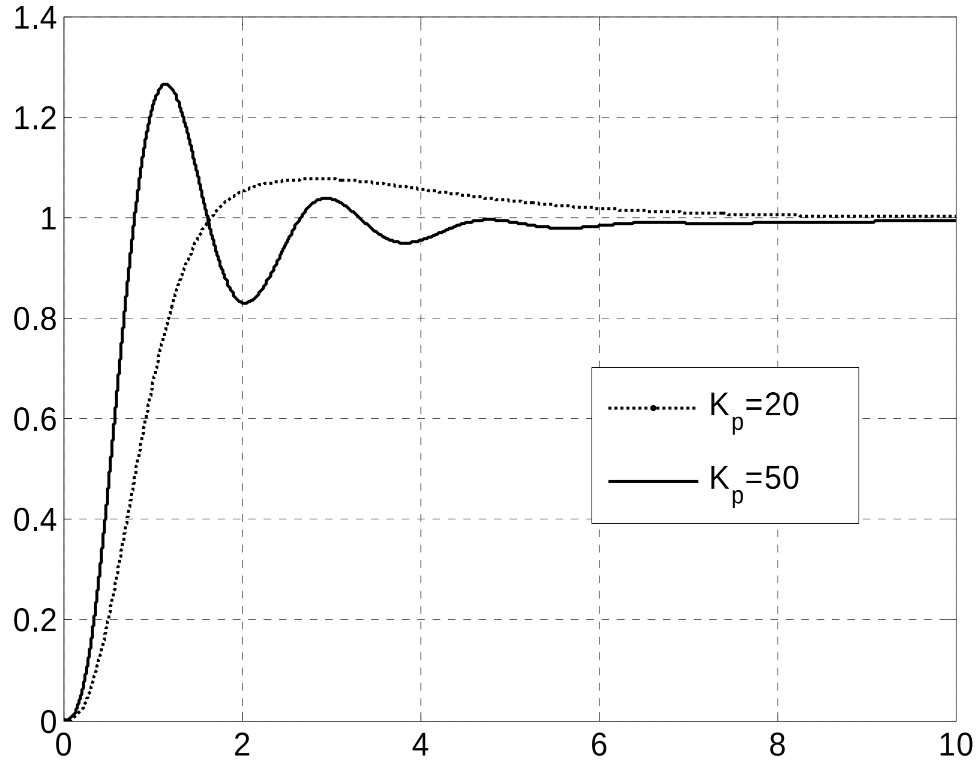
$$G_c(s) = K_p + K_i \frac{1}{s} + K_d s \;\; =>$$

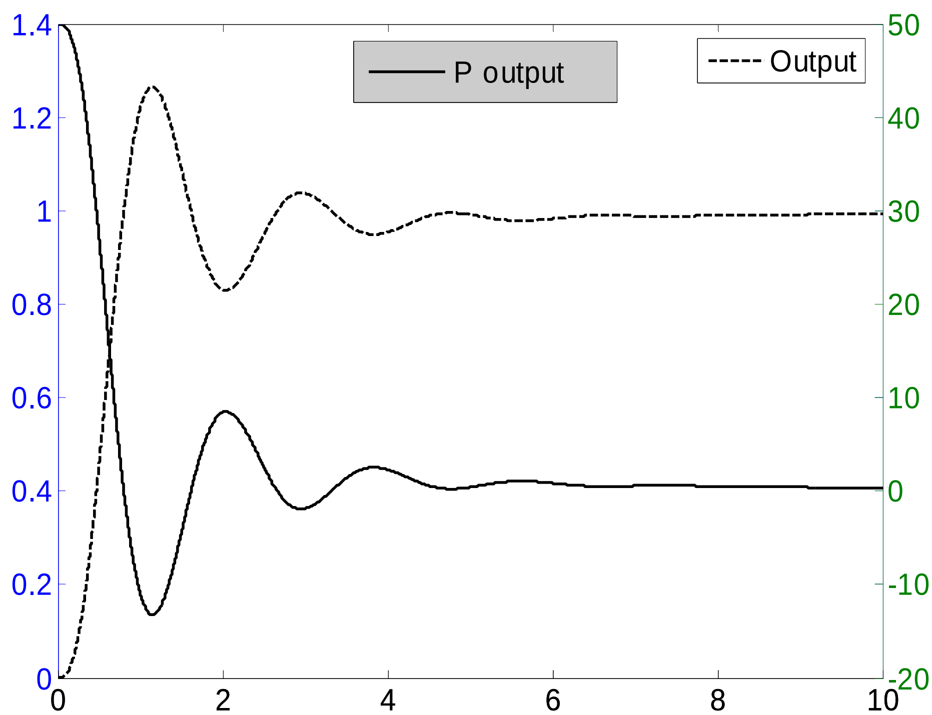
$$G_{OL}(s) = \frac{s^2 K_d + sK_p + K_i}{s} \frac{1}{(s+1)(s+2)(s+3)}$$

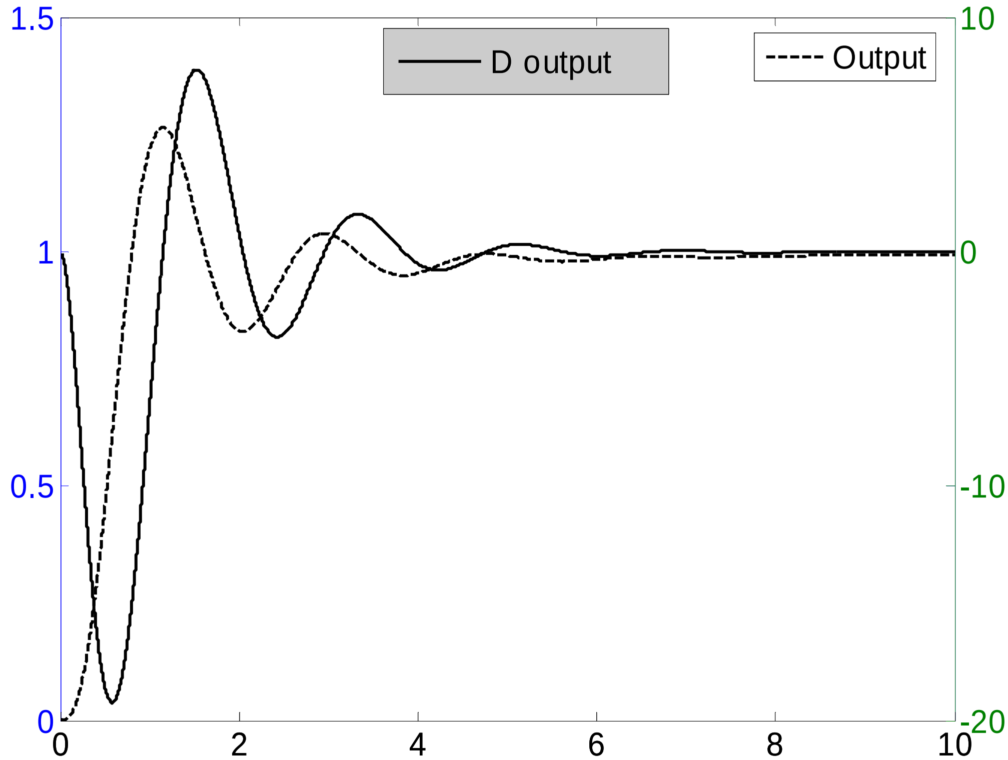


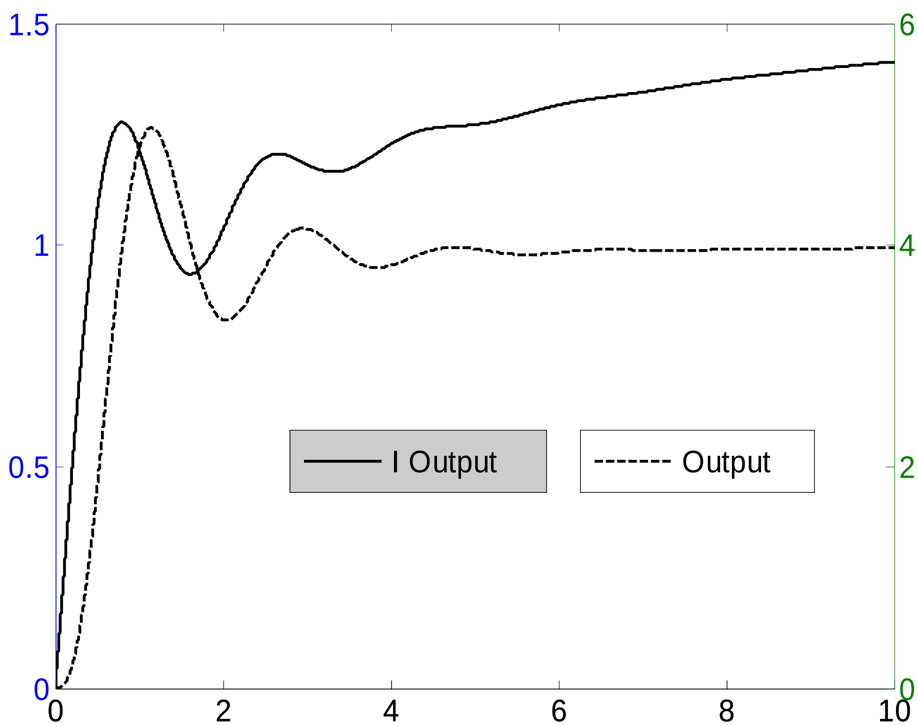


Since I have no oscillations I can increase K_p a little bit more to make the system faster:


1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
K_p=20
K_p=50


1.4
1.2
1
0.8
0.6
0.4
0.2
0
50
40
30
20
10
0
-10
-20
0
2
4
6
8
10
P output
Output


D output
Output
1.5
1
0.5
0
10
0
-10
-20
0
2
4
6
8
10


I Output
Output
1.5
1
0.5
0
6
4
2
0
0
2
4
6
8
10

Another way to write the PID controller:

$$G_{CL}(s) = K_p + \frac{K_i}{s} + K_d s = K_p\left(1 + \frac{K_i}{K_p}\frac{1}{s} + \frac{K_d}{K_p}s\right)$$

$$G_{CL}(s)K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

Tuning of PID controllers

1. Trial and error.
2. Ziegler Nichols I
3. Ziegler Nichols II
4. Root locus
5. Frequency response
6. Other advanced control methods

Trial and error:

P: Faster system, in some cases reduces the error (can cause instability).

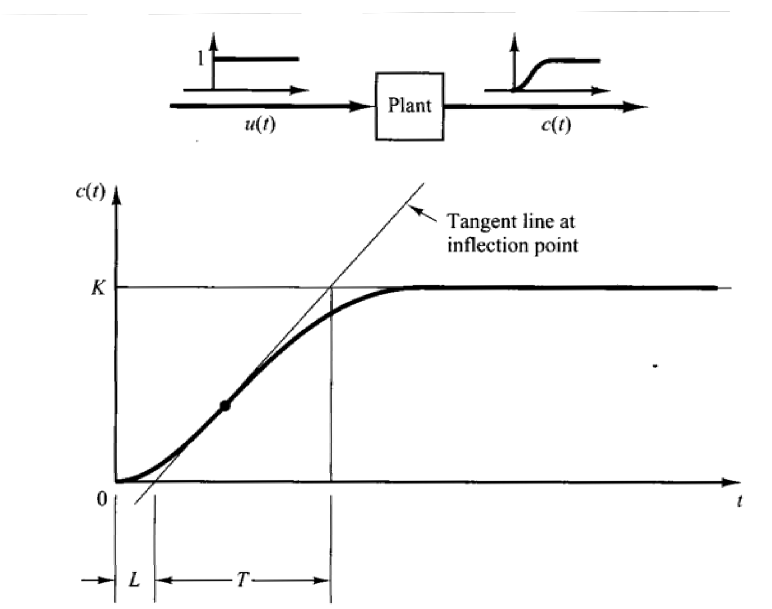
I: Reduces the steady state error, increases the number of oscillations.

D: Reduces the oscillations.

Ziegler Nichols I

Assume a system with no delays (we do not study these systems) and with no-complex conjugate poles.

Its open loop step response may look like (obtained experimentally of from simulations):



This can be modelled as:

$$\frac{C(s)}{U(s)} = \frac{Ke^{-Ls}}{Ts+1}$$

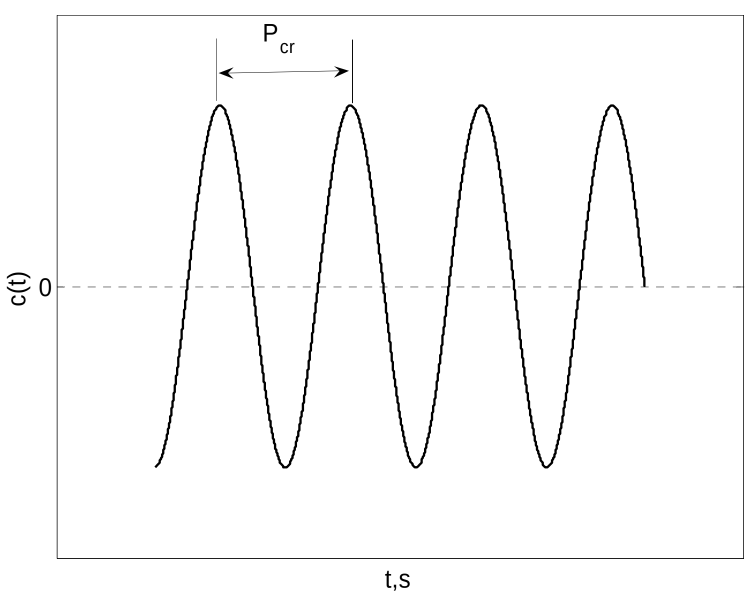
Based on that we have the following table:

Type of controller	K_p	T_i	T_d
P	T/L	∞	0
PI	$0.9T/L$	$L/0.3$	0
PID	$1.2T/L$	$2L$	$0.5L$

Ziegler Nichols II

Initially assume K_i=K_d=0.

Increase K_p until the system is marginally stable. Record the K_p=K_{cr} and the frequency of oscillations:



Type of controller	Kp	Ti	Td
P	$0.5K_{cr}$	∞	0
PI	$0.45K_{cr}$	$\frac{1}{1.2}P_{cr}$	0
PID	$0.6K_{cr}$	$0.5P_{cr}$	$0.125P_{cr}$

These methods aim at achieving an overshoot of 25%.

Root locus method

With the RL we specifically target a pole location at the s-plane, i.e. we target damping factors, natural and damped frequencies.

Example:

The OLTF is $G(s)=\frac{1}{s^2+11s-34}$ use a PI controller.

$$G_{OL}(s)=\left(K_p+\frac{K_i}{s}\right)\frac{1}{s^2+11s-34}=\frac{sK_p+K_i}{s\left(s^2+11s-34\right)}$$

$$G_{CL}(s)=\frac{sK_p+K_i}{s\left(s^2+11s-34\right)+sK_p+K_i}$$

This is a 3rd order system = 2nd order x 1st order:

CE: $s\left(s^2+11s-34\right)+sK_p+K_i=(s+a)\left(s^2+2\zeta\omega_n s+\omega_n^2\right)$

$$s^3+11s^2+s\left(-34+K_p\right)+K_i=s^3+(2\zeta\omega_n+a)s^2+\left(2\zeta\omega_n a+\omega_n^2\right)s+a\omega_n^2$$

$$\begin{Bmatrix} 11=2\zeta\omega_n+a \\ -34+K_p=2\zeta\omega_n a+\omega_n^2 \\ K_i=a\omega_n^2 \end{Bmatrix}$$

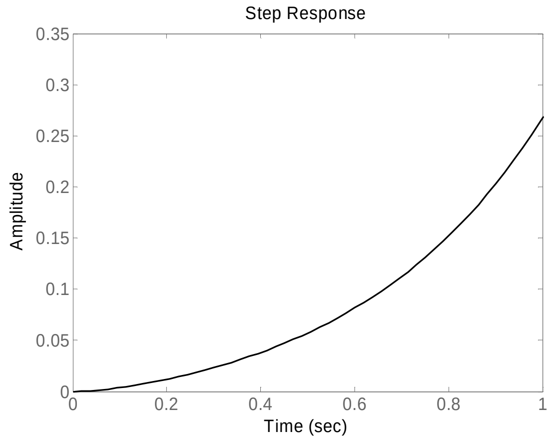
Assume that the design specs are: $\begin{Bmatrix} \omega_n=6 \\ \zeta=0.5 \end{Bmatrix}$

So: $\begin{Bmatrix} 11=6+a \\ -34+K_p=6a+36 \\ K_i=36a \end{Bmatrix} \Rightarrow \begin{Bmatrix} a=5 \\ K_p=100 \\ K_i=180 \end{Bmatrix}$

Matlab:

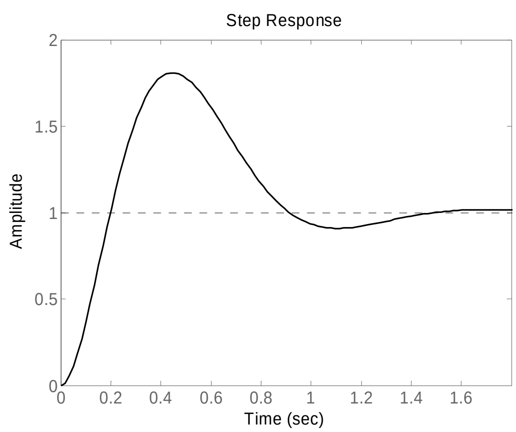
CL response, no controller:

```
>> num=1;
>> den=[1 11 -34];
>> g=tf(num,den);
>> gcl=feedback(g,1);
>> step(gcl)
```



CL, response, PI controller:

```
>> kp=100;
>> ki=180;
>> gc=tf([kp ki],[1 0]);
>> gol=series(gc,g);
>> gcl=feedback(gol,1);
>> step(gcl)
```



Homework:

Find the PID gains : the CLTF of $G(s)=\dfrac{1}{s^2+6s+16}$ has $\begin{Bmatrix}\omega_n = 6\\ \zeta = 0.5\end{Bmatrix}$

and a real pole at a=-5.

Solution: $K_p=50$, $K_i=180$, $K_d=5$.

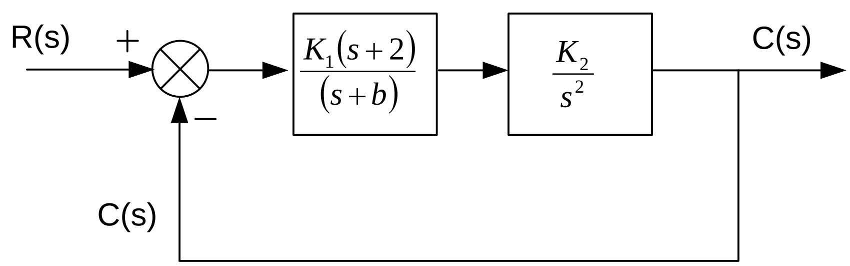
Frequency response – Advanced methods

We will not study these.

Other controllers

By using the previous (root locus) method we can design more general controllers.

Example:



$$K_1K_2=? \quad \omega_n=6$$
$$b=? \quad \theta=60^o$$

$$\left.\begin{matrix}\omega_n=6\\ \zeta=\cos(60^o)\end{matrix}\right\}\Rightarrow\left\{\begin{matrix}\omega_n=6\\ \zeta=0.5\end{matrix}\right\}$$

$$G(s)=\frac{K_1(s+2)}{s+b}\frac{K_2}{s^2}$$

$$\frac{C(s)}{R(s)} \overset{K_1K_2=K}{=} \frac{K(s+2)}{(s+b)s^2+K(s+2)}$$

$$CE: s^3+s^2b+Ks+2K=0$$

$$CE: s^3+s^2b+Ks+2K=(s+a)\left(s^2+2s\zeta\omega_n+\omega_n^2\right)$$

$$CE: s^3+s^2b+Ks+2K=(s+a)\left(s^2+2s\zeta\omega_n+\omega_n^2\right)$$

$$CE: s^3+s^2b+Ks+2K=s^3+(2\zeta\omega_n+a)s^2+\left(2\zeta a\omega_n+\omega_n{}^2\right)s+a\omega_n{}^2$$

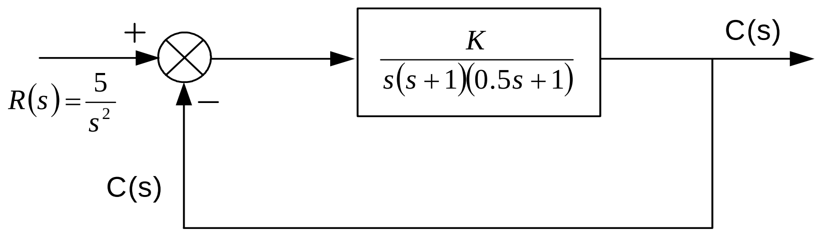
$$\left\{\begin{matrix} b=2\zeta\omega_n+a \\ K=2\zeta a\omega_n+\omega_n{}^2 \\ 2K=a\omega_n{}^2 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} b=9 \\ a=3 \\ K=54 \end{matrix}\right\}$$

Example:

$$G_{OL}(s)=\frac{K}{s(s+1)(0.5s+1)}$$, Unity feedback and input: r(t)=5t

a) If K=1.5, find the steady state error

b) The system must have steady state error, Ess<0.1 find the value of K



$$\frac{C(s)}{R(s)} = \frac{K}{s(s+1)(0.5s+1)+K} \Rightarrow C(s) = \frac{5}{s^2}\frac{K}{s(s+1)(0.5s+1)+K}$$

$$E(s) = R(s) - C(s) \Rightarrow E(s) = \frac{5}{s^2}\left(1 - \frac{K}{s(s+1)(0.5s+1)+K}\right)$$

$$E(s) = \frac{5}{s}\left(\frac{(s+1)(0.5s+1)}{s(s+1)(0.5s+1)+K}\right)$$

$$E_{ss} = \lim_{s\to 0}\left(s\frac{5}{s}\left(\frac{(s+1)(0.5s+1)}{s(s+1)(0.5s+1)+K}\right)\right)$$

$$E_{ss} = \frac{5}{K}$$

a) $E_{ss} = \frac{5}{1.5} = 3.33...$

b) $E_{ss} < 0.1 \Rightarrow \frac{5}{K} < 0.1 \Rightarrow K > 50$

Example:

$$G_{OL}(s) = \frac{K}{s(0.02s+1)(0.01s+1)}$$

a) Find the value of K such as the system is marginally stable

b) Find the frequency of oscillations at that point

$$\frac{C(s)}{R(s)} = \frac{K}{s(0.02s+1)(0.01s+1)+K}$$

$$CE: \quad s(0.02s+1)(0.01s+1)+K = 0$$

For marginally stable system: $s = 0 + j\omega$

$$CE: \quad j\omega(0.02j\omega+1)(0.01j\omega+1)+K = 0$$

$$CE: \quad j\left(\omega - 0.02\times 0.01\omega^3\right) + \left(-0.01\omega^2 - 0.02\omega^2 + K\right) = 0 + 0j$$

$$\left.\begin{array}{l}\omega - 0.02\times 0.01\omega^3 = 0\\ -0.03\omega^2 + K = 0\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\omega = 0\\ \omega = 70.71\,rad/s\end{array}\right\} \text{ and } K = 150$$