Chapter #5

EEE 2002

Control Systems

- **Design Based on Root Locus**

Design Based Root Locus

Lead Control (equivalent to PD control)

Used when the steady state properties of the system are ok but there is poor performance, i.e. we want to move the poles of the system without greatly altering the gain.

General form: $G_c(s) = K_c \dfrac{s+a}{s+b}, \quad b > a$

Other forms of Lead controller: $G_c(s) = K_c \alpha \dfrac{Ts+1}{Tsa+1} = K_c \dfrac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}, \alpha < 1,$ so:

$a = 1/T, \quad 1/\alpha T = b$

Design strategy:

1. Express the design requirements as a desired pole location.
2. Plot the uncompensated root locus and check if there is a gain that brings the system to that location, i.e. if a P controller is enough.
3. If not, place the zero at the real part of the desired pole location, i.e. $a = \zeta \omega_n$.
4. Find the value of the pole by using the angle criterion:, i.e.
   $$\sum \arg(real\ poles) - \sum \arg(real\ zeros) = \pi$$
5. Find the value of K_c by using the magnitude criterion, i.e. $|GH| = 1$.
6. Check to see if the new compensated system satisfied the design specifications.
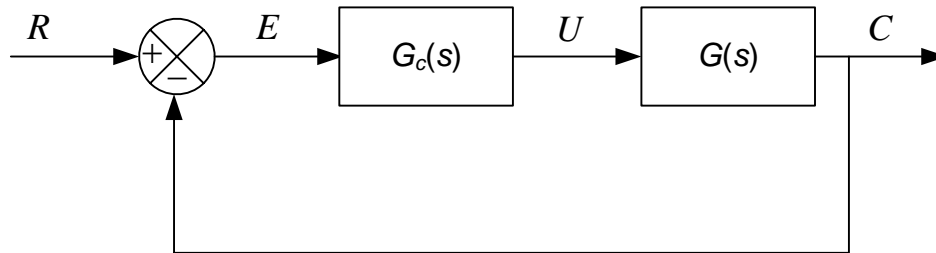7. If not repeat by placing the zero at a different location.

Example – Lead Control

Assume the system:

$$G(s) = \frac{4}{s(s+2)}$$

Create a closed loop (unity feedback strategy) such as the closed loop system will have $\omega_n = 4 \, rad/s$ and $\zeta = 0.5$.

Solution
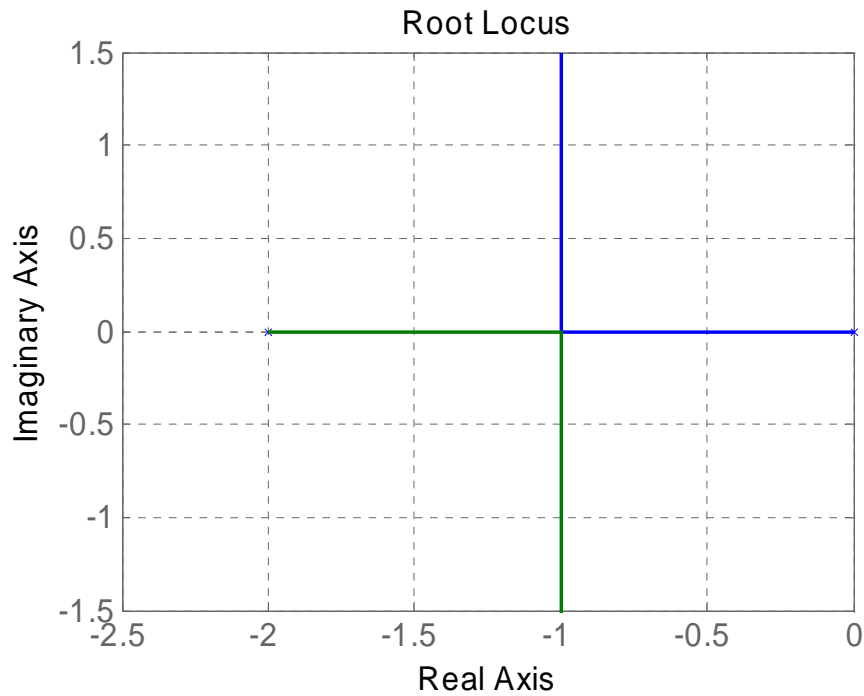
R ──→(+⊗−)──→ E ──→[$G_c(s)$]──→ U ──→[$G(s)$]──→ C

If a simple proportional controller is used then the CLTF is:

$$G_{CL}(s) = \frac{4K}{s(s+2)+4K} = \frac{K'}{s^2 + 2s + K'}$$
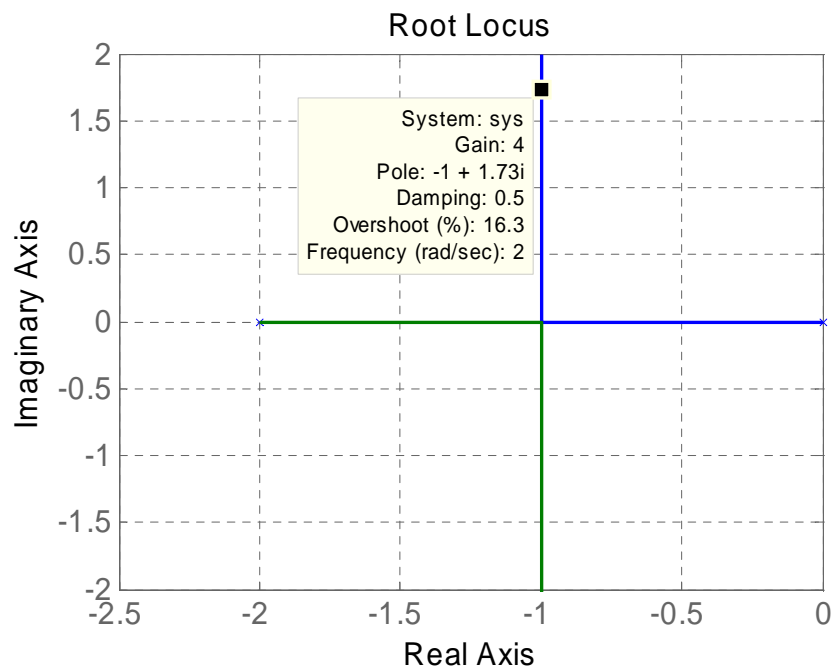
$$\frac{E(s)}{R(s)} = \frac{s^2 + 2s}{s^2 + 2s + K'} \Rightarrow E(s) = R(s)\frac{s^2 + 2s}{s^2 + 2s + K'} \Rightarrow E_{ss} = \lim_{s \to 0} s \frac{1}{s} \frac{s^2 + 2s}{s^2 + 2s + K'} = 0$$

And the root locus is:

```
num=[1]
den=[1 2 0]
rlocus(num,den)
```

Since the required damping factor is 0.5 this implies an angle of 60° which will give $\omega_n = 2 \Rightarrow \omega_d = 2\sqrt{1-0.5^2} = 1.73$ and this is achieved for K'=2:
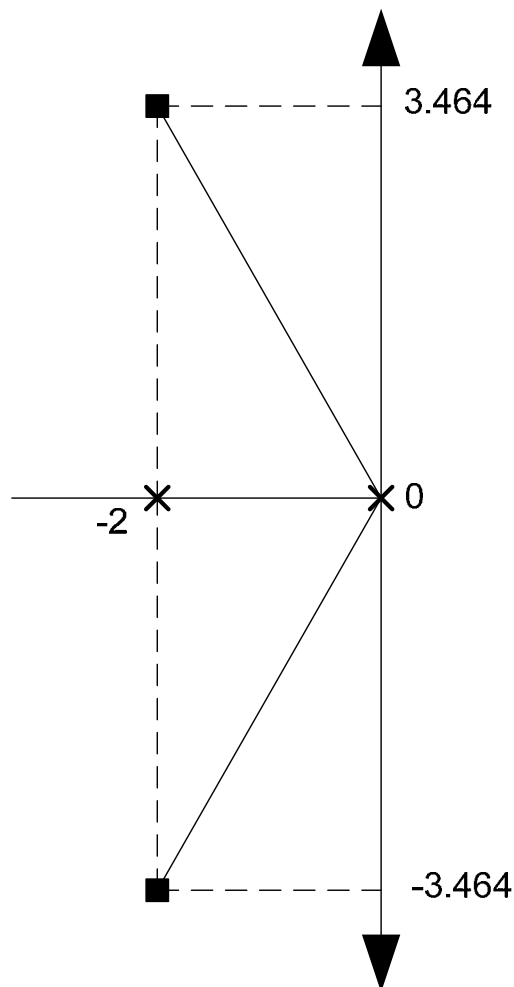
But for this gain and damping factor the natural frequency is not the required, i.e. the system is slower.

So we replace the proportional controller with a more general:

The new open loop TF is: $G = K \dfrac{s+a}{s+b} \dfrac{4}{s(s+2)}$

The location of the open loop poles and zeros and the desired closed loop poles $\left(-2 \pm j2\sqrt{3}\right)$ is:

One strategy is to place the open loop zero such as it will be equal to the real part of the desired closed loop, i.e. $a=-2$ (in this example it just happens to coincide with the open loop pole). This will leave us with the location of the pole. Let's assume that the controller pole is placed at x=-b and create an angle ψ with the real axis:
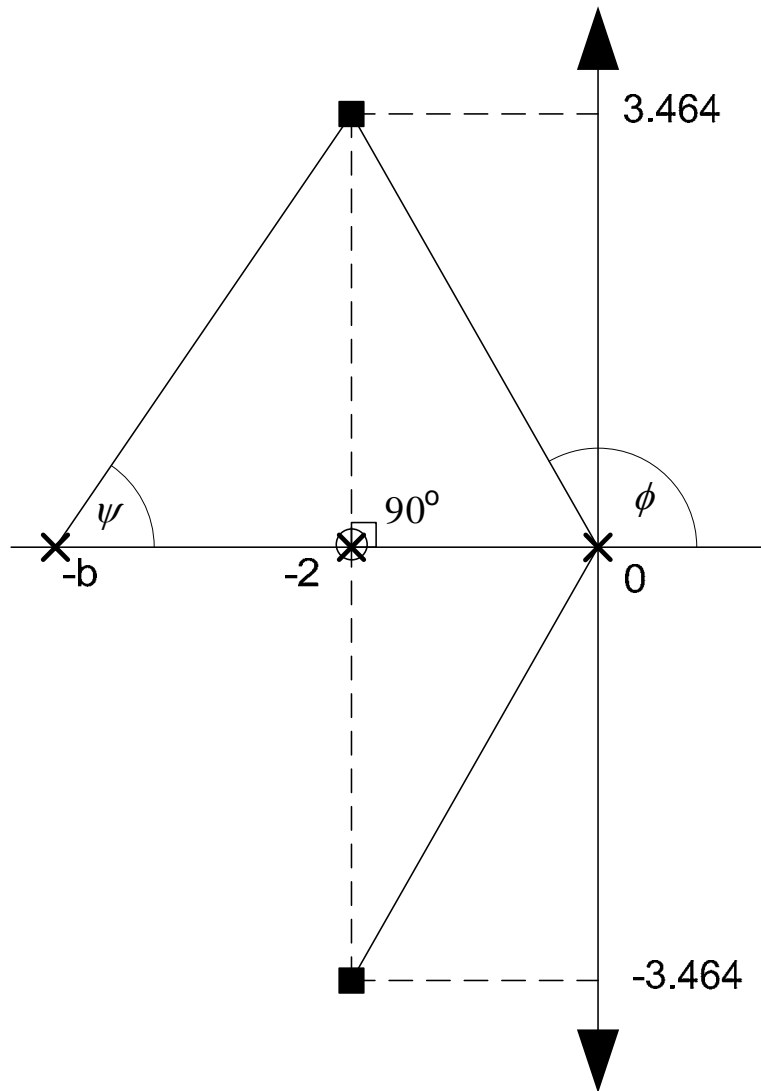
In order for the closed loop pole to be on the desired point the new characteristic equation must satisfy the angle condition:

$\phi + \psi + 90 - 90 = 180 \Rightarrow \psi + 120 = 180 \Rightarrow \psi = 30$ and hence b=4.

So the OLTF is:

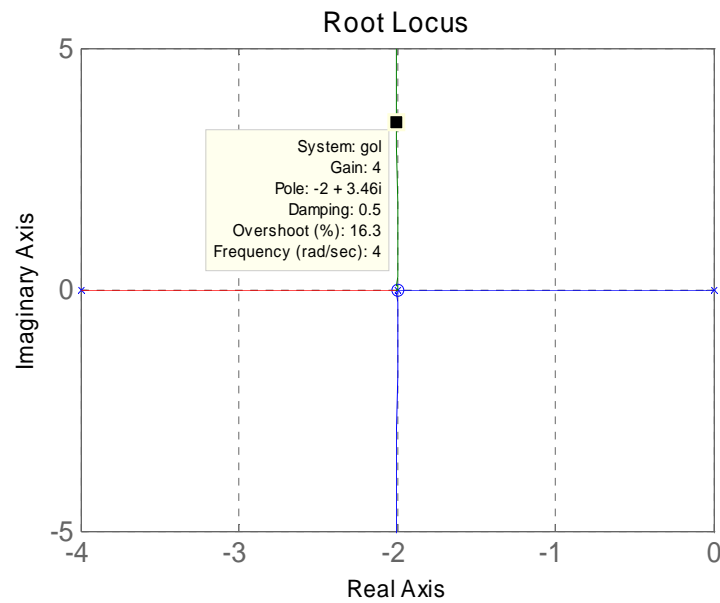$$G = K \frac{s+2}{s+4} \frac{4}{s(s+2)} = K \frac{4}{s(s+4)}$$

Now we can use the magnitude criterion to find the gain:

$$\left| \frac{K4}{s(s+4)} \right|_{s=-2+j2\sqrt{3}} = 1 \text{ (as } 1+GH = 0)$$

$$\frac{4K}{\left| -2 + j2\sqrt{3} \right| \left| 2 + j2\sqrt{3} \right|} = 1 \Leftrightarrow$$

$$\frac{4K}{4 \times 4} = 16 \Leftrightarrow$$

$$\frac{4K}{16} = 1 \Rightarrow K = 4$$

Lag Control

Used when the steady state properties of the system are not satisfactory but the pole location is ok.

General form: $G_c(s) = K_c \dfrac{s+a}{s+b}, \quad a > b$

Other forms of Lead controller: $G_c(s) = K_c \beta \dfrac{Ts+1}{Ts\beta+1} = K_c \dfrac{s + \frac{1}{T}}{s + \frac{1}{\beta T}}, \; \beta > 1,$ so:

$a = 1/T, \quad 1/\beta T = b$

Design strategy:

1. Express the design requirements as gain (usually as K_v), pole location (damping factor and natural frequency).

2. Check if it is possible to achieve these with a simple proportional controller.

3. If the location is possible but at that point K_v is too small (i.e. big error for a ramp input) then add a lag controller such as the controller pole will be close to the origin (for example at -0.001) and the zero to the left of the pole as many times as it is required to achieve the desired K_v.

4. Use the magnitude criterion to find the necessary controller gain.

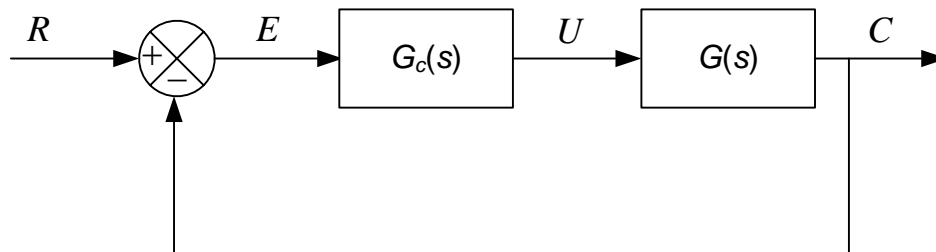5. Check to see if you have achieved the design specifications.

Example – Lag Control

An plant transfer function is $G(s) = \dfrac{1.06}{s(s+1)(s+2)}$. Create a control structure that will achieve the following:

- $\zeta = 0.491, \quad \omega_n = 0.673$
- $K_v = 5$

Solution

If a simple proportional controller is used then the CLTF is:

$$G_{CL}(s) = \frac{K \times 1.06}{s(s+1)(s+2) + K \times 1.06} = \frac{K'}{s(s+1)(s+2) + K'}$$

Hence the CE is:

$$s(s+1)(s+2) + K' = 0 \Leftrightarrow$$
$$s^3 + 3s^2 + 2s + K' = 0$$

As this is a 3rd order system the generic CE ca be written as:

$$(s+a)(s^2 + 2\zeta\omega_n s + \omega_n^2) = 0 \Leftrightarrow$$
$$s^3 + (2\zeta\omega_n + a)s^2 + (2\zeta\omega_n a + \omega_n^2)s + a\omega_n^2 = 0$$

By equating the corresponding coefficients:

$$\left.\begin{array}{l} 2\zeta\omega_n + a = 3 \\ 2\zeta\omega_n a + \omega_n^2 = 2 \\ a\omega_n^2 = K' \end{array}\right\} \begin{array}{c} \omega_n = 0.673 \\ \Rightarrow \\ \zeta = 0.491 \end{array} K' = 1.06, \quad a = 2.34$$

Hence remarkably the desired pole location is achieved even without any extra proportional controller.

But:

$$K_v = \lim_{s \to 0} s \frac{1.06}{s(s+1)(s+2)} = 0.53$$

Hence we need to increase K_v 10 times (more or less).

We use a Lag controller:

We place the pole at $b = -0.005$ and hence the zero at $a = -0.05$ (as we need to increase the gain 10 times).:

$$G_{OL}(s) = K_c \frac{s + 0.05}{s + 0.005} \frac{1.06}{s(s+1)(s+2)} \Rightarrow$$

$$K_c \left| \frac{s + 0.05}{s + 0.005} \frac{1.06}{s(s+1)(s+2)} \right|_{s = -0.33 + 0.58j} = 1 \Rightarrow$$

$$K_c = 1.0181$$

The new K_v is:

$$K_v = \lim_{s \to 0} s \frac{s + 0.05}{s + 0.005} \frac{1.0792}{s(s+1)(s+2)} = 5.396$$

And the new pole location is:

```
num=1.0792*[1 0.05]
den=conv(conv(conv([1 0.005],[1 0]),[1 1]),[1 2])
gcl=feedback(g,1); damp(gcl)
        Eigenvalue              Damping        Freq. (rad/s)
 -5.46e-002                     1.00e+000       5.46e-002
 -3.06e-001 + 5.73e-001i        4.71e-001       6.50e-001
 -3.06e-001 - 5.73e-001i        4.71e-001       6.50e-001
 -2.34e+000                     1.00e+000       2.34e+000
```

Lead - Lag Control

This is a combination of a Lead and a lag controller (similar to PID).

General form:
$$G_c(s) = K_c \frac{s + 1/T_1}{s + c/T_1} \frac{s + 1/T_2}{s + 1/cT_2}$$

$\dfrac{s+1/T_1}{s+c/T_1}$ is the lead controller and $\dfrac{s+1/T_2}{s+1/cT_2}$ is the lag controller.

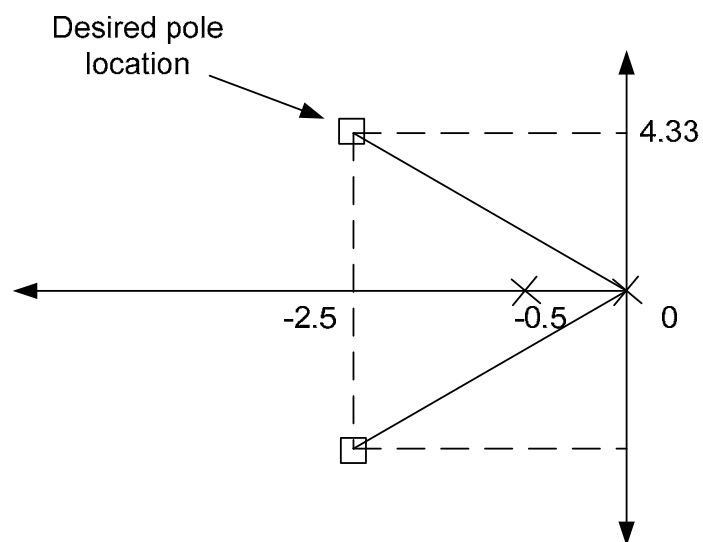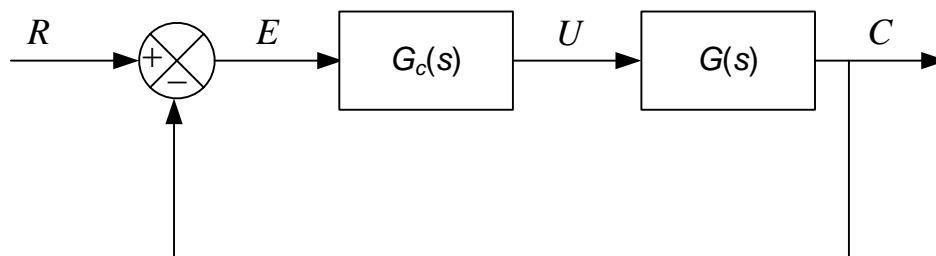**Example – Lead - Lag Control**

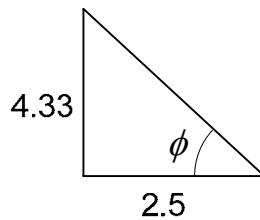An plant transfer function is $G(s)=\dfrac{4}{s(s+0.5)}$. Create a control structure that will achieve the following:
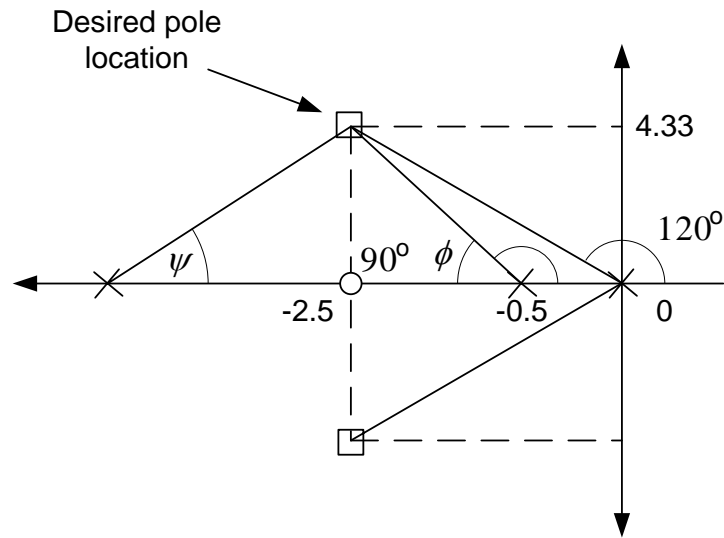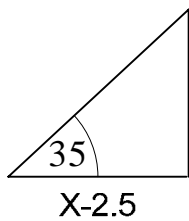
- $\zeta=0.5,\quad \omega_n=5$
- $K_v=80$

Solution

By using the lead controller design steps:

Desired pole location

$$\phi = \tan^{-1}\left(\frac{4.33}{2}\right) \Rightarrow \phi = 65^{\circ}$$

$$\psi + (180 - 65) + 120 - 90 = 180 \Rightarrow \psi = 35^{\circ}$$

$$\tan(35) = \frac{4.33}{x - 2.5} \Rightarrow x = 8.68$$

Hence $\dfrac{c}{T_1} = 8.68 \Rightarrow c = 3.47$

By choosing now $T_2=10$ so that the second pole will be close to the origin:

$$K_v = \lim_{s \to 0} s K_c \frac{s+1/T_1}{s+c/T_1} \frac{s+1/T_2}{s+1/cT_2} \frac{4}{s(s+0.5)} = 80$$

$$\frac{4}{0.5} K_c = 80 \Leftrightarrow$$

$$K_c = 10$$

And hence:

$$G_c(s) = 10 \frac{s+2.5}{s+8.68} \frac{s+0.1}{s+0.0288}, \quad G_{OL}(s) = 40 \frac{s+2.5}{s+8.68} \frac{s+0.1}{s+0.0288} \frac{1}{s(s+0.5)}$$

```
>>num=40*conv([1 2.5],[1 0.1]);
den=conv(conv(conv([1 8.68], [1 .0288]),[1 0]),[1
0.5]);
>> g=tf(num,den)
Transfer function:
        40 s^2 + 104 s + 10
   -----------------------------------
   s^4 + 9.209 s^3 + 4.604 s^2 + 0.125 s
>> gcl=feedback(g,1)
Transfer function:
          40 s^2 + 104 s + 10
   ------------------------------------------
   s^4 + 9.209 s^3 + 44.6 s^2 + 104.1 s + 10
>> damp(gcl)
Eigenvalue               Damping        Freq. (rad/s)
 -1.00e-001               1.00e+000        1.00e-001
 -4.34e+000               1.00e+000        4.34e+000
 -2.39e+000+4.16e+000i   4.97e-001        4.80e+000
 -2.39e+000-4.16e+000i   4.97e-001        4.80e+000
```

$$Kv = \lim_{s \to 0} s 40 \frac{s+2.5}{s+8.68} \frac{s+0.1}{s+0.288} \frac{1}{s(s+0.5)} = 40 \frac{2.5}{8.68} \frac{0.1}{0.0288} \frac{1}{0.5} \approx 80$$