Root locus

The given TFs are:

$$G_{p1}(s)=\frac{1}{s(s+1)(s+2)},\ G_{p2}(s)=\frac{1}{(s+0.8+3j)(s+0.8-3j)}$$

```
>> Gp1=tf(1,poly([0 -1 -2]))

Transfer function:
        1
-----------------
s^3 + 3 s^2 + 2 s

>> Gp2=tf(1,poly([-0.8-3*i -0.8+3*i]))

Transfer function:
        1
------------------
s^2 + 1.6 s + 9.64

>> G=Gp1*Gp2

Transfer function:
                       1
-----------------------------------------------
s^5 + 4.6 s^4 + 16.44 s^3 + 32.12 s^2 + 19.28 s
```

Using Matlab:

```
>> rlocus(G)
```

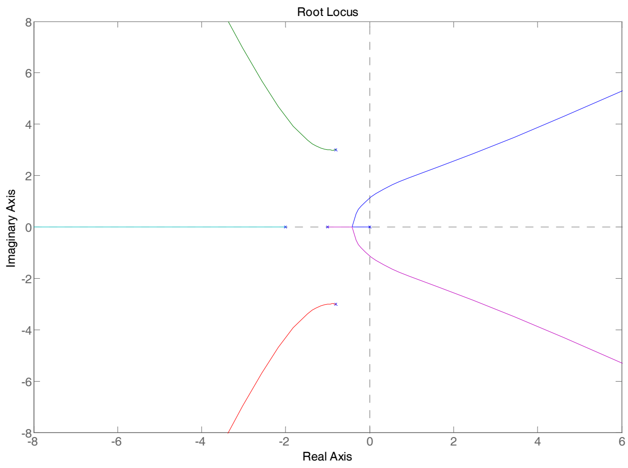

Figure 1 Root locus produced using rlocus(.)

The graphical method:

The open loops poles (there are no zeros) are obviously 0, -1 , -2 from the 1st TF and -0.8±3j from the 2nd TF but it can also be found using Matlab:

```
>> [p,z]=pzmap(G)
p =
                 0
 -0.80000000000000 + 3.00000000000000i
 -0.80000000000000 - 3.00000000000000i
 -2.00000000000001
 -1.00000000000000
z =
   Empty matrix: 0-by-1
```

Note: The steps shown here are at a different sequence that those presented in your handout (chapter 4, last page).

Step 1: Draw the OL poles and zeros:

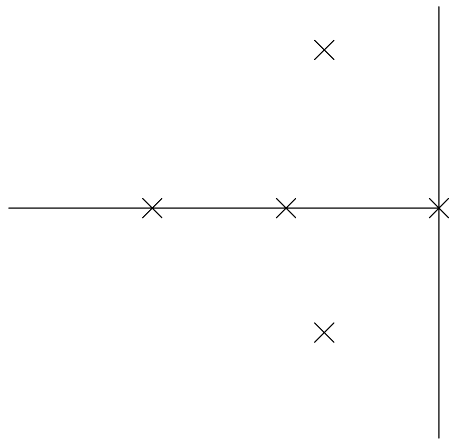

Figure 2 OL poles and zeros

Step 2: Determine which parts of the real axis belong to the RL:

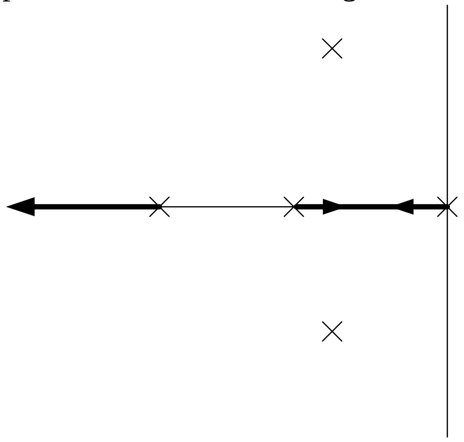

Figure 3 Real axis part

Step 3: The Break in/out points can be found from $\frac{dk}{ds}=0$ where $k(s)$ is found using the CL CE: $k=-s(s+1)(s+2)(s+0.8+3j)(s+0.8-3j)$

It will be easier if we first do the multiplication. This product is already calculated from:

```
>> G

Transfer function:
                       1
-----------------------------------------------
s^5 + 4.6 s^4 + 16.44 s^3 + 32.12 s^2 + 19.28 s
```

To isolate the OL denominator:

```
>> [n, d]=tfdata(G,'v')
n =
     0     0     0     0     0     1
d =
  Columns 1 through 4
   1.00000000000000   4.60000000000000  16.44000000000000
32.12000000000001
  Columns 5 through 6
  19.28000000000000                  0
```

To speed up the calculations (for our system) we have that k=-OL Denominator and hence:

```
>> dd=[5*d(1) 4*d(2) 3*d(3) 2*d(4) d(5)]
dd =
  Columns 1 through 4
   5.00000000000000  18.40000000000000  49.32000000000001
64.24000000000001
  Column 5
  19.28000000000000
```

Note: Be **very** careful when you use the above trick!!!

The solution of $\frac{dk}{ds}=0$:

```
>> roots(dd)
ans =
 -0.83563600935911 + 2.26917322324496i
 -0.83563600935911 - 2.26917322324496i
 -1.59539068314996
 -0.41333729813182
```

Hence the accepted solution (break out point) is the last one as the -1.59 is a point on the real that is not in the RL (use your mouse pointer on the Matlab figure to crosscheck your answer).

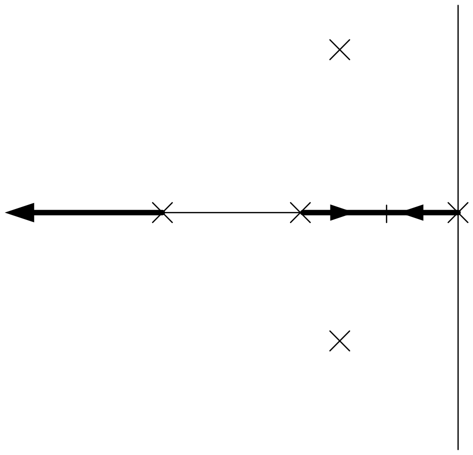

Figure 4 Break out point

Step 4: There are $n_p - n_z = 5$ asymptotes and the point of intersection with the real axis:

$$s = -\frac{\sum_{i=1}^{n_p} p_i - \sum_{i=1}^{n_z} z_i}{n_p - n_z}$$

The poles are:

```
>> pj=-p
pj =
                 0
  0.80000000000000 - 3.00000000000000i
  0.80000000000000 + 3.00000000000000i
  2.00000000000001
  1.00000000000000
```

And hence:

```
>> sum(pj)/5
ans =
   0.92000000000000
```

Hence s=-0.92

The angles of the asymptotes with the real axis are:

# asymptote	Angle
1	$180^o / 5 = 36^o$
2	$3 \times 180^o / 5 = 108^o$
3	$5 \times 180^o / 5 = 180^o$
4	$7 \times 180^o / 5 = 252^o$ or -108^o
5	$9 \times 180^o / 5 = 324^o$ or -36^o

The last 2 can also be found using the symmetry of the RL.

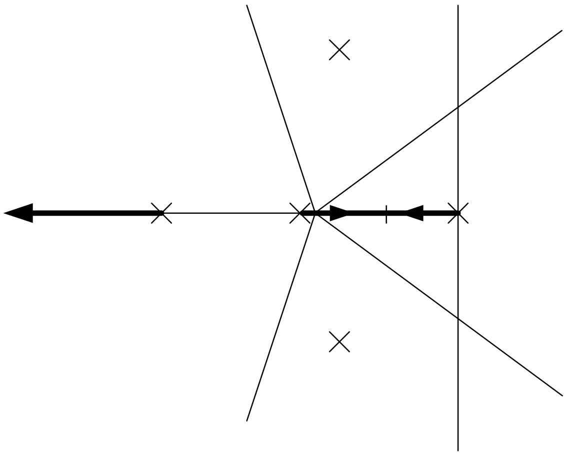

Figure 5 Asymptotes

<u>Step 5:</u> The angle of departure from the complex poles is found using the following drawing:

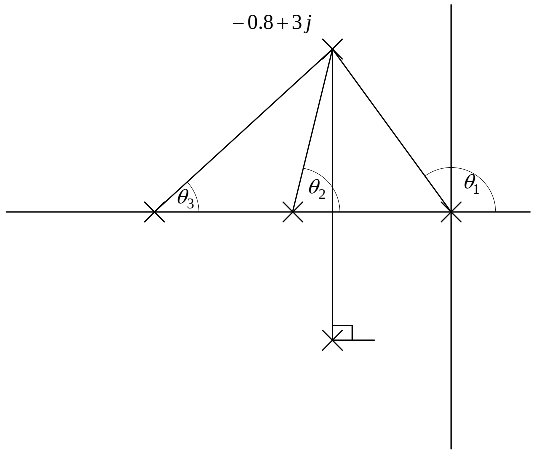

Figure 6 Calculation of departure angle

$$\theta_3 = \tan^{-1}\left(\frac{3}{1.2}\right) = 68.2^o$$

$$\theta_2 = \tan^{-1}\left(\frac{3}{0.2}\right) = 86.2^o$$

$$\tan^{-1}\left(\frac{3}{0.8}\right) = 75^o \Rightarrow \theta_2 = 180^o - 75^o = 105^o$$

So the angle of departure is $180^o - 68.2^o - 86.2^o - 105^o - 90^o = -169.4^o$

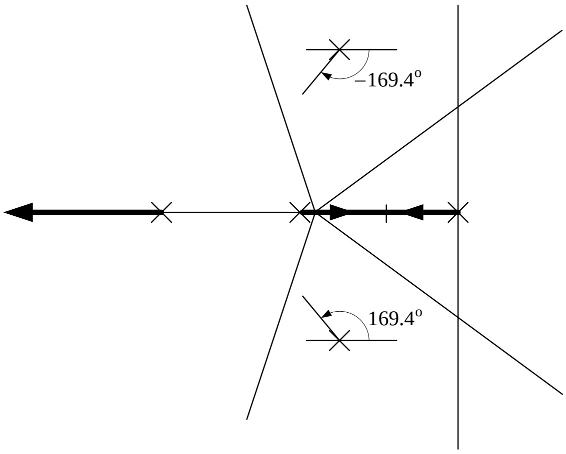

Figure 7 Angle of departure

<u>Step 6:</u> The last point is the point of intersection with the imaginary axis. To do that we place at the CL CE *s*=*jω*:

$$s^5 + 4.6s^4 + 16.44s^3 + 32.12s^2 + 19.28s + k = 0 \Leftrightarrow$$

$$j\omega^5 + 4.6\omega^4 - 16.44j\omega^3 - 32.12\omega^2 + 19.28j\omega + k = 0 \Rightarrow$$

$$\left\{\begin{matrix} \omega^5 - 16.44\omega^3 + 19.28\omega = 0 \\ 4.6\omega^4 - 32.12\omega^2 + k \end{matrix}\right\} \Rightarrow$$

$$\left\{\begin{matrix} \omega^4 - 16.44\omega^2 + 19.28 = 0 \\ 4.6\omega^4 - 32.12\omega^2 + k \end{matrix}\right\}$$

```
>> roots([1 0 -16.44 0 19.28])
ans =
   3.89473818599817
  -3.89473818599818
   1.12739277207531
  -1.12739277207531
```

To find the accepted value:

```
>> w=3.89473818599817; k=-4.6*w^4+32.12*w^2
k =
  -5.712235468221927e+002
>> w=1.12739277207531; k=-4.6*w^4+32.12*w^2
k =
  33.39378682219768
```

Hence the accepted value is 1.12739277207531 and hence the gain is 33.4

Now we have all that we need to sketch the root locus:

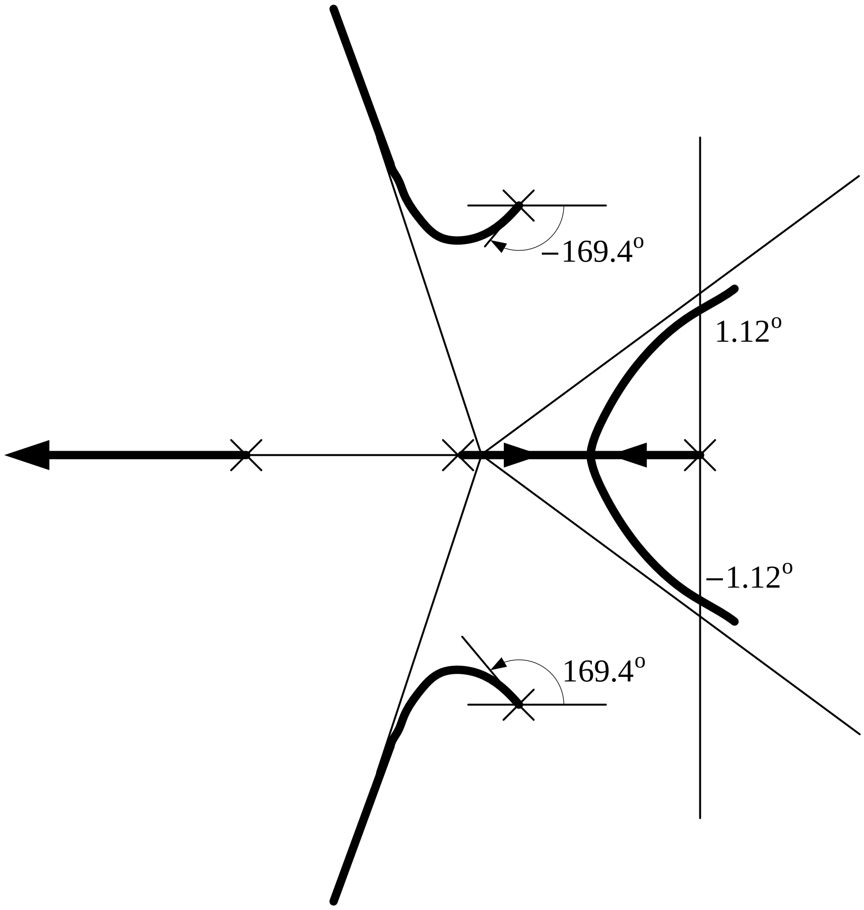

Figure 8

The given TFs are:

$$G_{p1}(s)=\frac{1}{s(s+1)(s+2)},\ G_{p2}(s)=\frac{(s+2+5j)(s+2-5j)}{(s+0.8+3j)(s+0.8-3j)}$$

Note: This is a difficult one

```
>> gp1=tf(1,poly([0 -1 -2]))

Transfer function:
        1
-----------------
s^3 + 3 s^2 + 2 s

>> gp2=tf(poly([-0.8-3*i -0.8+3*i]),poly([-0.8-3*i -0.8+3*i]))

Transfer function:
s^2 + 1.6 s + 9.64
------------------
s^2 + 1.6 s + 9.64

>> gp2=tf(poly([-2-5*i -2+5*i]),poly([-0.8-3*i -0.8+3*i]))

Transfer function:
  s^2 + 4 s + 29
------------------
s^2 + 1.6 s + 9.64

>> g=gp1*gp2

Transfer function:
                s^2 + 4 s + 29
-----------------------------------------------
s^5 + 4.6 s^4 + 16.44 s^3 + 32.12 s^2 + 19.28 s
```

Using Matlab:

```
>> rlocus(G)
```

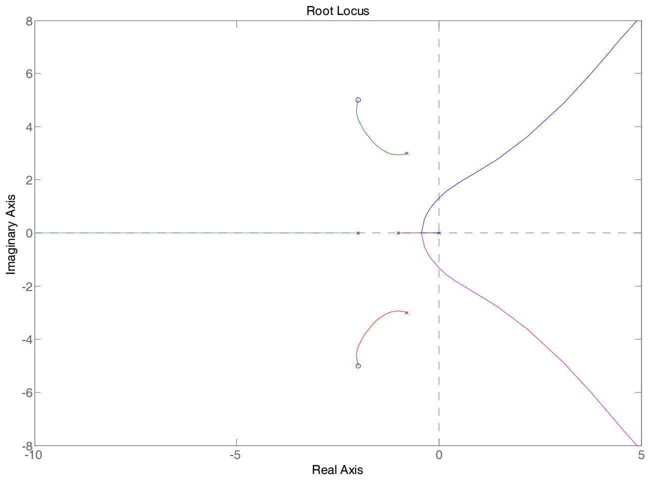

Figure 9 Root locus produced using rlocus(.)

The graphical method:

The open loops poles are obviously 0, -1 , -2 from the 1st TF and -0.8±3j from the 2nd TF. The zeros are -2±5j but it can also be found using Matlab:

```
>> [p,z]=pzmap(G)
>> [p,z]=pzmap(g)
p =
                 0
 -0.80000000000000 + 3.00000000000000i
 -0.80000000000000 - 3.00000000000000i
 -2.00000000000001
 -1.00000000000000
z =
 -2.00000000000000 + 5.00000000000000i
 -2.00000000000000 - 5.00000000000000i
```

Note: The steps shown here are at a different sequence that those presented in your handout (chapter 4, last page).

Step 1: Draw the OL poles and zeros:

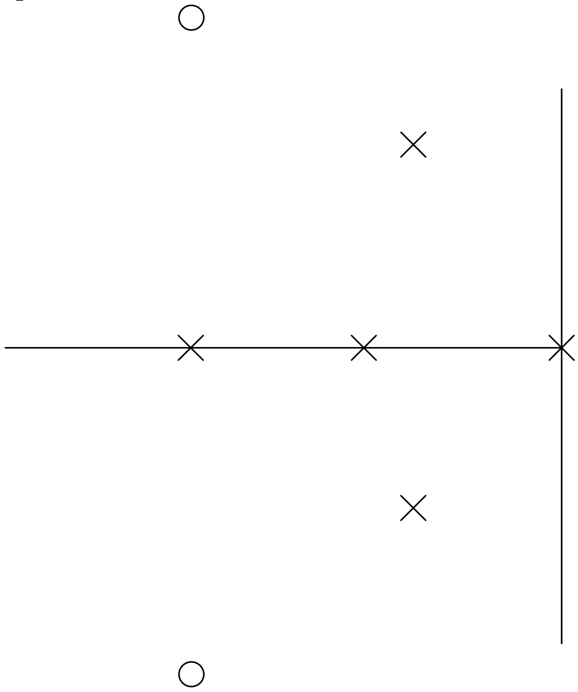

Figure 10 OL poles and zeros

Step 2: Determine which parts of the real axis belong to the RL:

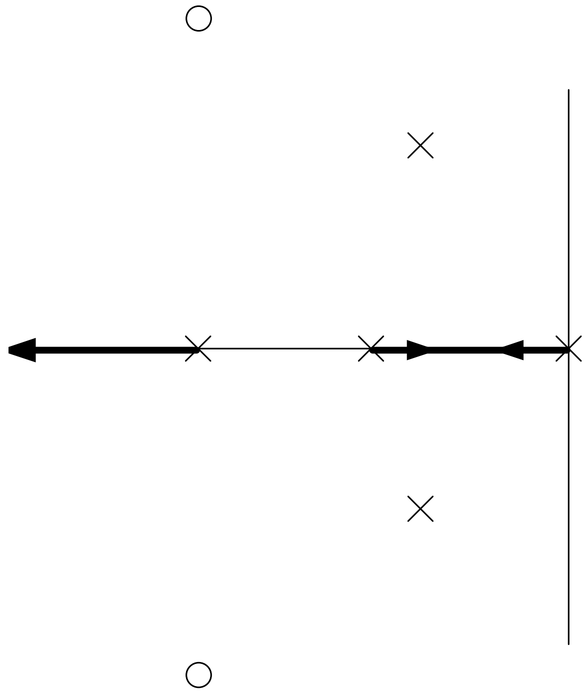

Figure 11 Real axis part

<u>Step 3:</u> The Break in/out points can be found from $\frac{dk}{ds}=0$ where $k(s)$ is found using the CL CE: $k=\frac{-s(s+1)(s+2)(s+0.8+3j)(s+0.8-3j)}{(s+2+5j)(s+2-5j)}$

(note: try to clearly understand the following steps)

```
>> [n, d]=tfdata(g,'v')
dd=[5*d(1) 4*d(2) 3*d(3) 2*d(4) d(5)]
nd=[5*d(1) 4*d(2) 3*d(3) 2*d(4) d(5)]

roots(conv(dd,n)-conv(d,nd))
n =
     0     0     0     1     4    29
d =
  Columns 1 through 4
   1.00000000000000   4.60000000000000  16.44000000000000
32.12000000000001
  Columns 5 through 6
  19.28000000000000                  0
dd =
  Columns 1 through 4
   5.00000000000000  18.40000000000000  49.32000000000001
64.24000000000001
  Column 5
  19.28000000000000
nd =
  Columns 1 through 4
   5.00000000000000  18.40000000000000  49.32000000000001
64.24000000000001
  Column 5
  19.28000000000000
ans =
 -0.72486921028210 + 3.06190964590823i
 -0.72486921028210 - 3.06190964590823i
 -0.83563600935911 + 2.26917322324497i
```

```
 -0.83563600935911 - 2.26917322324497i
 -1.90295033612183 + 0.91994346786326i
 -1.90295033612183 - 0.91994346786326i
 -1.59539068314997
  0.65563909280787
 -0.41333729813182
```

Hence the accepted solution (break out point) is -0.41 (use your mouse pointer on the Matlab figure to crosscheck your answer).

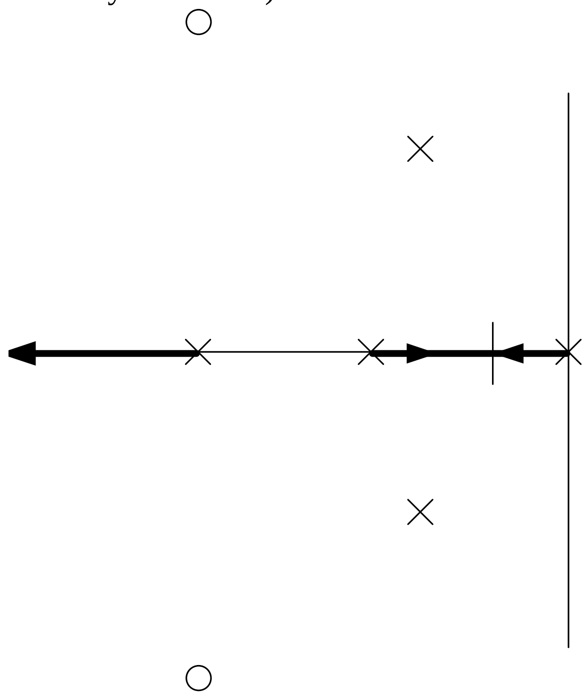

Figure 12 Break out point

<u>Step 4:</u> There are $n_p - n_z = 5 - 2 = 3$ asymptotes and the point of intersection with the real axis:

$$s = -\frac{\sum_{i=1}^{n_p} p_i - \sum_{i=1}^{n_z} z_i}{n_p - n_z}$$

```
>> pj=-p
pj =
                 0
  0.80000000000000 - 3.00000000000000i
  0.80000000000000 + 3.00000000000000i
  2.00000000000001
  1.00000000000000
>> zj=-z
zj =
  2.00000000000000 - 5.00000000000000i
  2.00000000000000 + 5.00000000000000i
>> -sum(pj)/sum(pi)
ans =
  -1.46422547644544
```

Hence s=-1.46

The angles of the asymptotes with the real axis are:

# asymptote	Angle
1	$180^o / 3 = 60^o$
2	$3 \times 180^o / 3 = 180^o$
3	$5 \times 180^o / 3 = 300^o$ or -60^o

The last 2 can also be found using the symmetry of the RL.

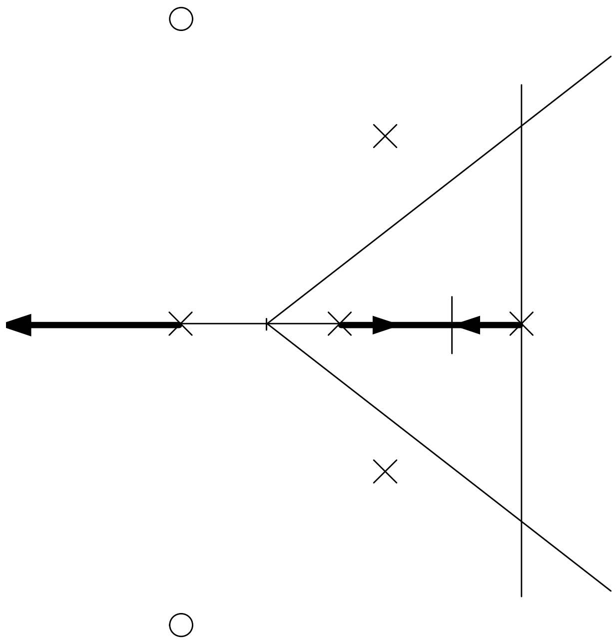

Figure 13 Asymptotes

<u>Step 5:</u> The angle of departure from the complex poles is found using the following drawing:

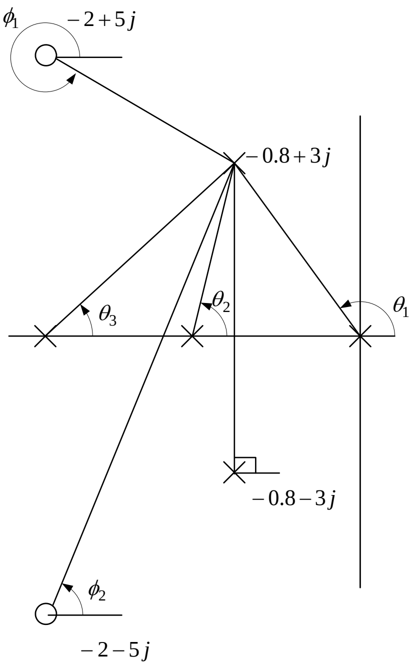

Figure 14 Calculation of departure angle

$$\theta_3 = \tan^{-1}\left(\frac{3}{1.2}\right) = 68.2^{\mathrm{o}}$$

$$\theta_2 = \tan^{-1}\left(\frac{3}{0.2}\right) = 86.2^{\mathrm{o}}$$

$$\tan^{-1}\left(\frac{3}{0.8}\right) = 75^{\mathrm{o}} \Rightarrow \theta_2 = 180^{\mathrm{o}} - 75^{\mathrm{o}} = 105^{\mathrm{o}}$$

$$\phi_2 = \tan^{-1}\left(\frac{3+5}{1.2}\right) = 81.5^{\mathrm{o}}$$

$$\tan^{-1}\left(\frac{5-3}{1.2}\right) = 59^{\mathrm{o}} \Rightarrow \phi_1 = 270 + 59 = 329$$

So the angle of departure is $180^{\mathrm{o}} - 68.2^{\mathrm{o}} - 86.2^{\mathrm{o}} - 105^{\mathrm{o}} - 90^{\mathrm{o}} + 81.5 + 329 = 241^{\mathrm{o}}$

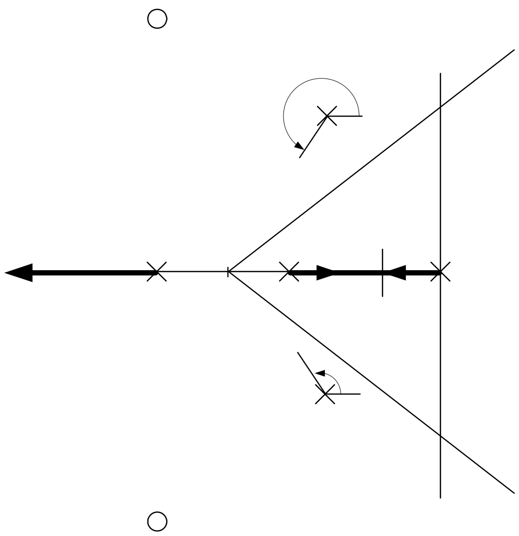

Figure 15 Angle of departure

Step 6: Angle of arrival at complex zero:

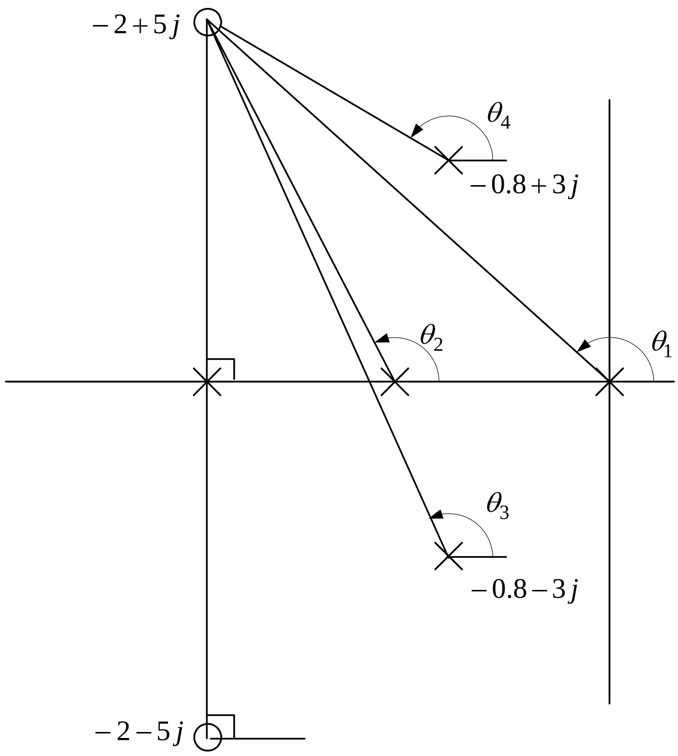

Figure 16

$$\tan^{-1}\left(\frac{5-3}{1.2}\right)=59^{o}\Rightarrow\theta_4=180-59=121^{o}$$

$$\tan^{-1}\left(\frac{5+3}{1.2}\right)=81.5^{o}\Rightarrow\theta_3=98.5^{o}$$

$$\tan^{-1}\left(\frac{5}{1}\right)=78.7^{o}\Rightarrow\theta_2=101.3^{o}$$

$$\tan^{-1}\left(\frac{5}{2}\right)=68.2^{o}\Rightarrow\theta_1=111.8^{o}$$

$$180^{o}-90^{o}+90^{o}+121^{o}+98.5^{o}+101.3^{o}+111.8^{o}=612.6^{o}\text{ or }252.6^{o}$$

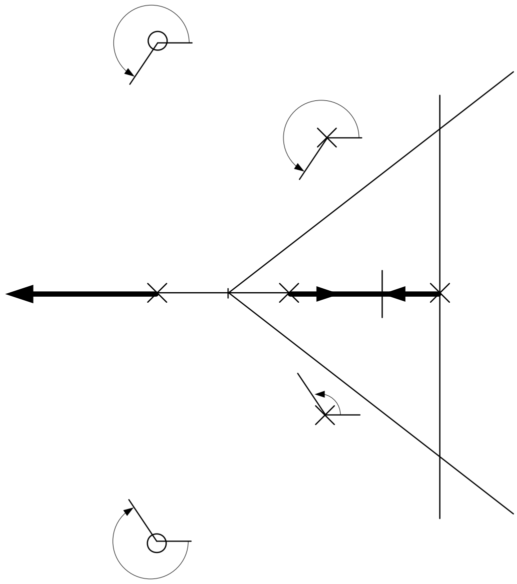

Figure 17 Angle of arrival

<u>Step 7:</u> The last point is the point of intersection with the imaginary axis. To do that we place at the CL CE $s=j\omega$:

$$s^5+4.6\,s^4+16.44s^3+32.12s^2+19.28s+K\left(s^2+4s+29\right)=0\Leftrightarrow$$
$$j\omega^5+4.6\,\omega^4-16.44j\omega^3-32.12\omega^2+19.28j\omega+K\left(-\omega^2+4j\omega+29\right)=0\Leftrightarrow$$
$$j\omega^5+4.6\,\omega^4-16.44j\omega^3-32.12\omega^2+19.28j\omega-K\omega^2+4Kj\omega+29K=0\Rightarrow$$
$$\begin{Bmatrix}\omega^5-16.44\omega^3+19.28\omega+4K\omega=0\\ 4.6\omega^4-32.12\omega^2-K\omega^2+29K=0\end{Bmatrix}\Rightarrow$$
$$\begin{Bmatrix}\omega^4-16.44\omega^2+19.28+4K=0\\ 4.6\omega^4-32.12\omega^2-K\omega^2+29K=0\end{Bmatrix}\Rightarrow$$

To solve that I will use the command *solve*(.)

```
>> f=solve('w^4-16.44*w^2+19.28+4*k','4.6*w^4-32.12*w^2-w^2*k+29*k')
f =
    k: [6x1 sym]
    w: [6x1 sym]
>> f.k
ans =
                                     1.5305102953532435416275289415472
                                     1.5305102953532435416275289415472
 48.066744852323378229186235529226+28.380111761239544420282550946110*i
 48.066744852323378229186235529226+28.380111761239544420282550946110*i
 48.066744852323378229186235529226-28.380111761239544420282550946110*i
 48.066744852323378229186235529226-28.380111761239544420282550946110*i
>> f.w
ans =
                                     -1.3139395436854506015568077305538
                                      1.3139395436854506015568077305538
 -3.9148964819517940077720670409119+1.6339011679500992572044216409525*i
  3.9148964819517940077720670409119-1.6339011679500992572044216409525*i
 -3.9148964819517940077720670409119-1.6339011679500992572044216409525*i
  3.9148964819517940077720670409119+1.6339011679500992572044216409525*i
```

Hence the gain in 1.53 and the point is 1.31 rad/s

Now we have all that we need to sketch the root locus:

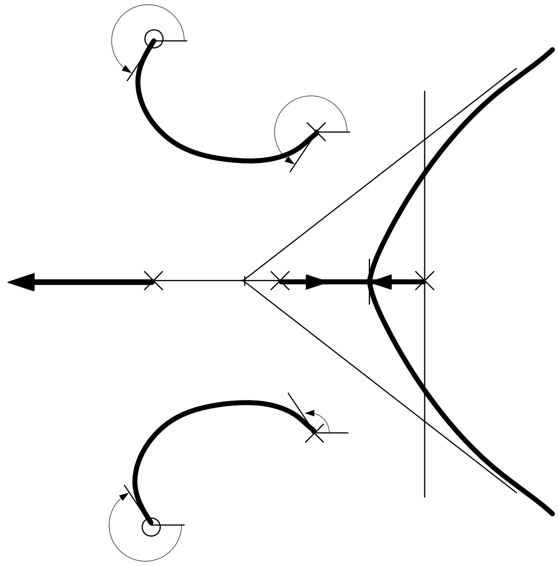

Figure 18

Further exercises

1) Repeat the original exercise but with $G_{p2}(s)=(s+2+5j)(s+2-5j)$.
2) Repeat the original exercise but with $G_{p2}(s)=\dfrac{1}{(s+0.5+0.3j)(s+0.5-0.3j)}$.
3) Repeat the original exercise but with $G_{p2}(s)=\dfrac{(s+1.3+j)(s+1.3-j)}{(s+0.8+3j)(s+0.8-3j)}$.

Note: it is EXTREMELY important that you do and fully UNDERSTAND these exercises.