Chapter #1

EEE8013-3001

State Space Analysis and Controller Design

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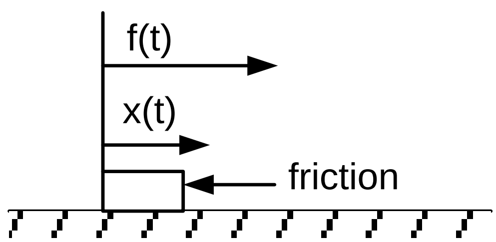
1. Introduction

System: is a set of objects/elements that are connected or related to each other in such a way that they create and hence define a unity that performs a certain objective.

Control: means regulate, guide or give a command**.**

Task: To study, analyse and ultimately to control the system to produce a "satisfactory" performance.

Model: Ordinary Differential Equations (ODE):



$$\Sigma F = ma \Leftrightarrow f - f_{friction} = ma \Leftrightarrow f - B\omega = ma \Leftrightarrow f - B\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

Dynamics: Properties of the system, we have to solve/study the ODE.

2. First order ODEs: $\frac{dx}{dt} = f(x,t)$

Analytical solution: Explicit formula for $x(t)$ (a solution which can be found using various methods) which satisfies $\frac{dx}{dt} = f(x,t)$

Example: The ODE $\frac{dx}{dt} = a$ has as **a** solution $x(t) = at$ since $\frac{dx}{dt} = a$.

Note: We write x but we mean $x(t)$

First order Initial Value Problem : $\frac{dx}{dt} = f(x,t), \quad x(t_0) = x_0$

Analytical solution: Explicit formula for $x(t)$ which satisfies $\frac{dx}{dt} = f(x,t)$ and passes through x_0 when $t = t_0$

Example: $\frac{dx}{dt} = a \Leftrightarrow \int dx = \int a dt \Leftrightarrow x(t) = at + C \Rightarrow x(0) = C \Rightarrow x(t) = at + x_0$

For that reason some books use a different symbol for the above solution: $\phi(t, t_0, x_0)$.

You must be clear about the difference between an ODE and the solution to an IVP! From now on we will just study IVP unless otherwise explicitly mentioned.

PBL 1. What is the Integral Version of an IVP?

First order linear equations - (linear in *x* and *x'*)

General form: $\begin{cases} a(t)x'+b(t)x=c(t), a(t)\neq 0 & \textit{Non autonomous} \\ ax'+bx=c, a\neq 0 & \textit{Autonomous} \end{cases}$

Another form:

$$a(t)x'+b(t)x=c(t)\Leftrightarrow \frac{a(t)}{a(t)}x'+\frac{b(t)}{a(t)}x=\frac{c(t)}{a(t)}\Leftrightarrow x'+k(t)x=u(t)$$

In order to solve this **LINEAR** ODE we can use the method of the Integrating factor:

- $p(t)=e^{\int k(t)dt}$ (no need to worry about ICs)
- $h(t)=\int p(t)u(t)dt$ (be careful, you need to use the ICs)
- $x=\dfrac{h(t)}{p(t)}$

Example: $x'+kx=u(t)$, i.e. $k=const$

$$p(t)=e^{\int kdt}\overset{k=const}{=}e^{kt}$$

$$h(t)=\int p(t)u(t)dt=\int e^{kt}u(t)dt$$

So $x=\dfrac{h(t)}{p(t)}=e^{-kt}\int e^{kt}u(t)dt$

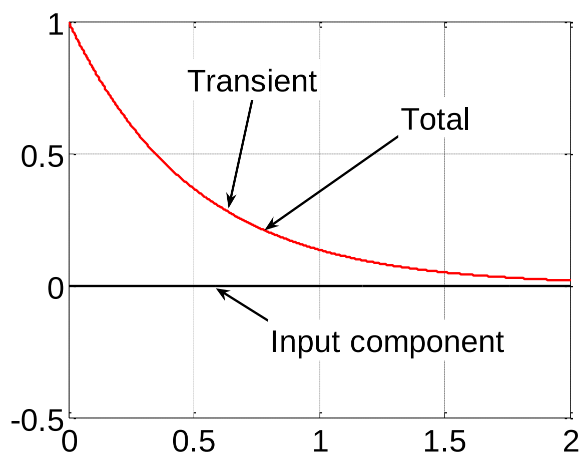
Now, we can transform the above indefinite integral to definite:

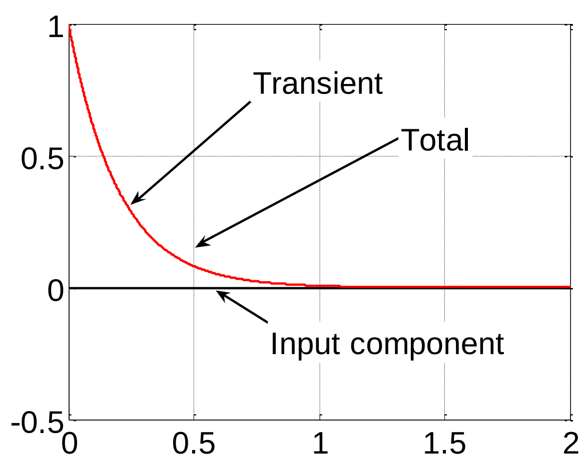
$$x = e^{-kt}\int e^{kt}u(t)dt = e^{-kt}\left(\int_{t_0}^{t} e^{kt_1}u(t_1)dt_1 + x(t_0)\right) = e^{-kt}\int_{t_0}^{t} e^{kt_1}u(t_1)dt_1 + e^{-kt}x(t_0)$$

In most of our cases t_0=0 so: $x = e^{-kt}x(0) + e^{-kt}\int_{0}^{t} e^{kt_1}u\,dt_1$

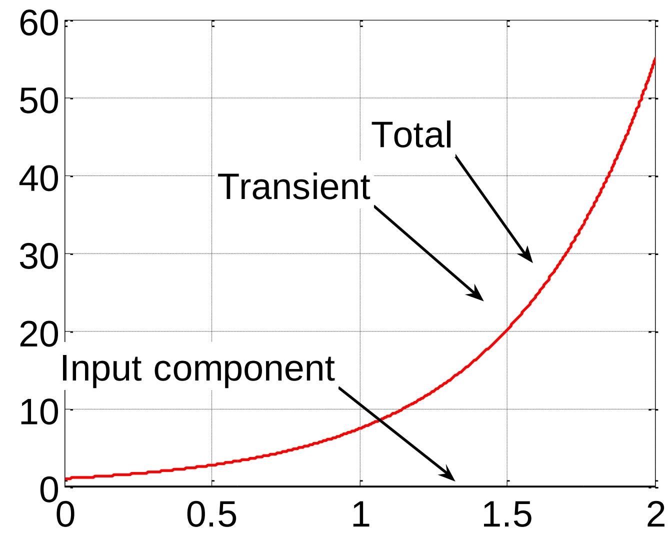
Assuming that k>0 the first part is called transient and the second is called steady state solution.

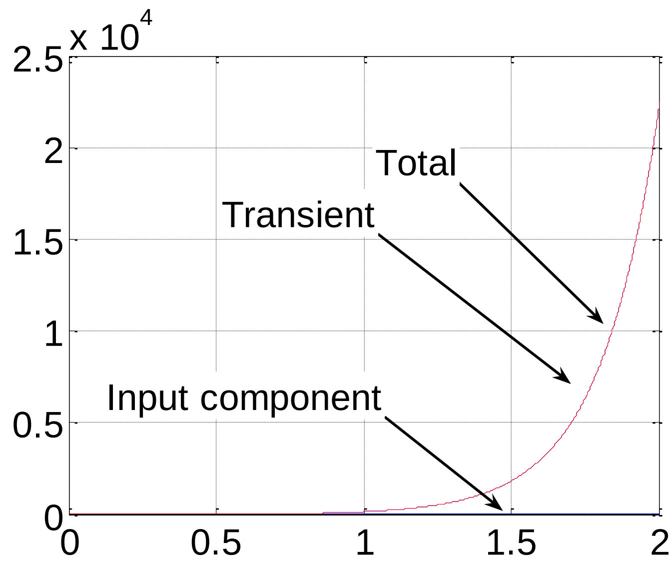
Examples: u=0 and k=2 & 5, x_0=1



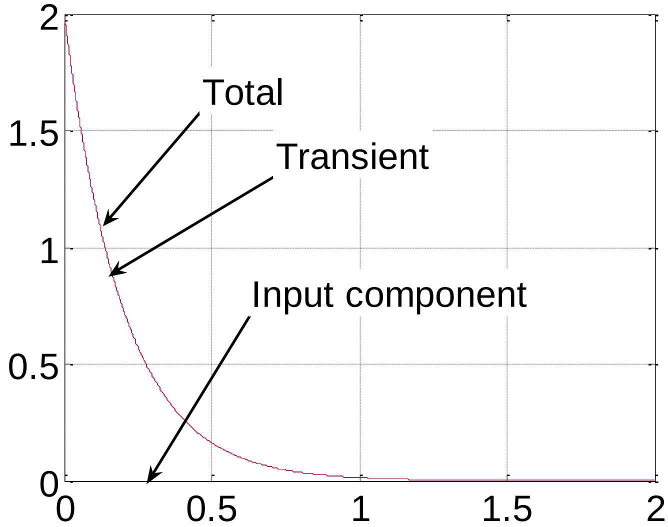


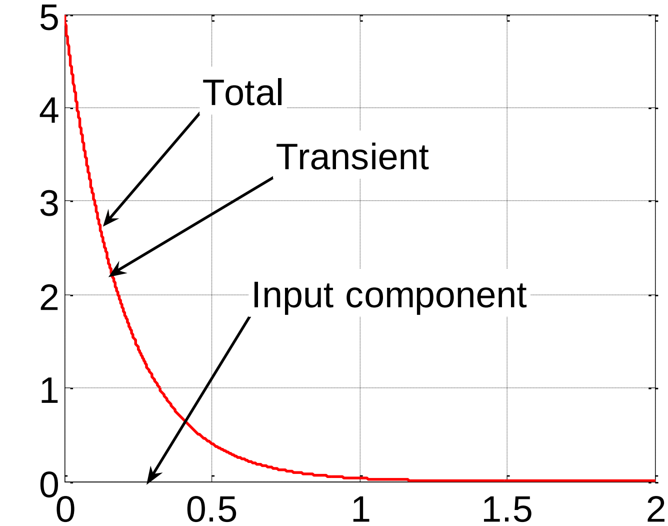
Examples: u=0 and k=-2 & 5, x_0=1



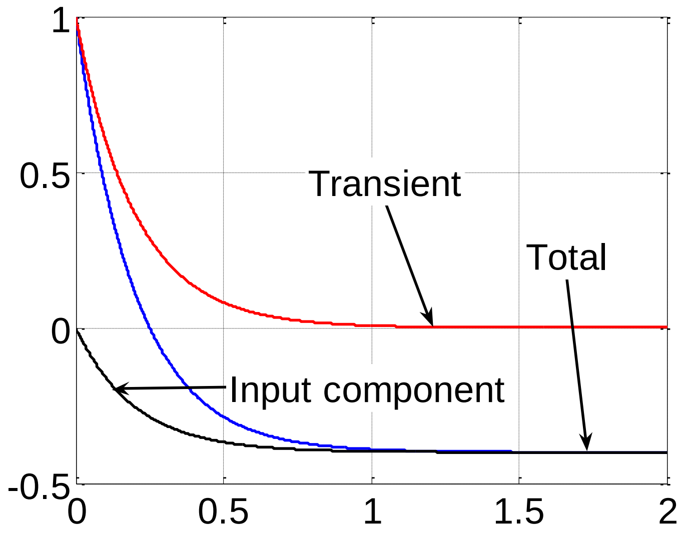


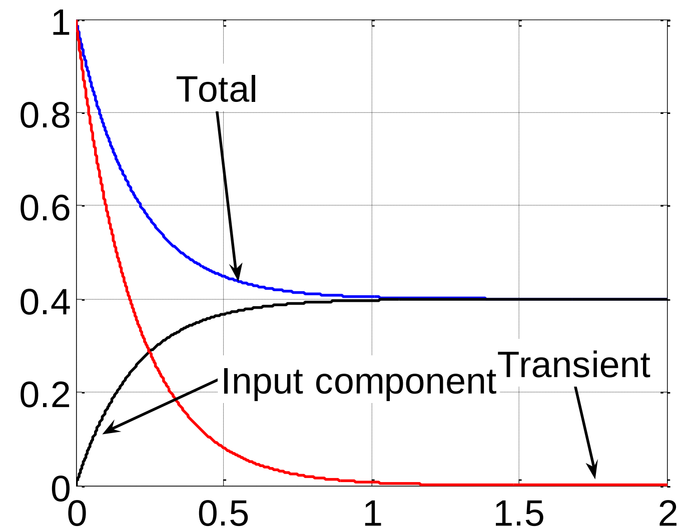
Examples: u=0 and k=5, x_0=1 & 5





Examples: u=-2 & 2 and k=5, x_0=1





Response to a sinusoidal input (Not assessed)

$$y_1' + ky_1 = k\cos(\omega t)$$

$$y_2' + ky_2 = k\sin(\omega t) \Rightarrow jy_2' + kjy_2 = kj\sin(\omega t)$$

$$jy_2' + kjy_2 + y_1' + ky_1 = kj\sin(\omega t) + k\cos(\omega t)$$

$$\tilde{y}' + k\tilde{y} = ke^{j\omega t}$$

Integrating factor: $e^{kt} \Rightarrow \left(\tilde{y}e^{kt}\right)' = ke^{(k+j\omega t)}$

$$\tilde{y}e^{kt} = \frac{k}{k + j\omega t} e^{(k+j\omega t)} \Rightarrow$$

$$\tilde{y} = \frac{k}{k + j\omega t} e^{j\omega t} \Rightarrow$$

$$\tilde{y} = \frac{1}{\sqrt{1 + \omega^2 / k^2}} e^{j\left(\omega t - \tan^{-1}(\omega / k)\right)}$$

$$\mathrm{Re}(\tilde{y}) = \mathrm{Re}\left(\frac{1}{\sqrt{1 + \omega^2 / k^2}} e^{j\left(\omega t - \tan^{-1}(\omega / k)\right)}\right) = \frac{1}{\sqrt{1 + \omega^2 / k^2}} \cos\left(\omega t - \tan^{-1}(\omega / k)\right)$$

$$\frac{1}{\sqrt{1 + \omega^2 / k^2}} \cos\left(\omega t - \tan^{-1}(\omega / k)\right)$$

= magnified/attenuated amplitude and phase shifted.

Exercise: Find the response (analytically) for k=0.5, u=1 and u=-1, initial conditions: 0, 1, -1. Describe the system's behaviour.

PBL 2. Which 2 other methods can we use to solve 1st order ODEs?

Autonomous 1st order ODEs => Liner Time Invariant (LTI) systems

$$\frac{dx}{dt} = f(x)$$ (not t on RHS).

Analytic solution: Can be solved as before: Transient and steady state part.

3. Second order ODEs: $\frac{d^2x}{dt^2} = f(x', x, t)$

Second order linear ODEs with constant coefficients: $x'' + Ax' + Bx = u$

u=0 => Homogeneous ODE; I need two "representative solutions" x_1 and x_2. Then all solutions can be written as a linear combination of these two solution: $x = c_1x_1 + c_2x_2$ with c_1, c_2 being arbitrary constants.

PBL 3. What do we mean when we say "representative solutions"?

$x'' + Ax' + Bx = 0$, assume $x = e^{rt}$ => $x' = re^{rt}$ & $x'' = r^2e^{rt}$ =>

$x'' + Ax' + Bx = 0 \Leftrightarrow r^2e^{rt} + Are^{rt} + Be^{rt} = 0 \Leftrightarrow$

$r^2 + Ar + B = 0$; Characteristic or Eigenvalue equation => Check its roots.

$r = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$, these are the Characteristic values or Eigenvalues.

So we have $r_1 = \frac{-A + \sqrt{A^2 - 4B}}{2}$ and $r_2 = \frac{-A - \sqrt{A^2 - 4B}}{2}$

Thus, using elementary calculus: $A = -r_1 - r_2$ and $B = r_1r_2$:

$x'' + (-r_1 - r_2)x' + r_1r_2x = 0$

- Roots are real and unequal: r_1 and r_2 ($A^2 > 4B$ =>Overdamped system)

$x_1 = e^{r_1 t}$ and $x_2 = e^{r_2 t}$ are solutions of the ODE =>

$x = C_1 x_1 + C_2 x_2 = C_1 e^{r_1 t} + C_2 e^{r_2 t}$. If r_1 and r_2<0 then $x \to 0$.

PBL 4. Why if x_1 and x_2 is a solution then $C_1x_1+C_2x_2$ is also a solution?

PBL 5. By using the fact that if x(t) is a solution then x(t)u(t) is also a solution find the function $u(t)$ when we have 2 real and distinct solutions.

Example:

$x''+4x'+3x = 0 \Leftrightarrow r^2 + 4r + 3 = 0 \Leftrightarrow (r+3)(r+1) = 0$

$x = C_1 e^{-3t} + C_2 e^{-t}$. Assume that $x(0) = 1$ and $x'(0) = 0$:

$x(0) = C_1 + C_2 = 1$ and $x' = -3C_1 e^{-3t} - C_2 e^{-t} \Rightarrow$ $x'(0) = -3C_1 - C_2 = 0$ =>
C_1=-0.5, C_2=3/2 => $x = -0.5e^{-3t} + \frac{3}{2} e^{-t}$:

- Roots are real and equal: r_1=r_2 ($A^2 = 4B$ Critically damped system)

$x_1 = e^{rt}$ and $x_2 = te^{rt}$ => $x = C_1 x_1 + C_2 x_2 = C_1 e^{rt} + C_2 te^{rt}$

PBL 6. By using the fact that if x(t) is a solution then x(t)u(t) is also a solution prove why the second "representative" solution is $x_2 = te^{rt}$. Can it be $x_2 = (C + Dt)e^{rt}$, where C, D are arbitrary constants?

Example:

A=2, B=1, x(0)=1, x'(0)=0 => c_1=c_2=1

- Roots are complex: r=a+bj $A^2 < 4B$
  - Underdamped system $A \neq 0$

So $x = e^{rt} = e^{(a+bj)t} = e^{at+bjt} = e^{at}e^{jbt} = e^{at}(\cos(bt) + j\sin(bt))$=Re+jIm.

Theorem: If x is a complex solution to a real ODE then Re(x) and Im(x) are the real solutions of the ODE:

$x_1 = e^{at}\cos(bt), x_2 = e^{at}\sin(bt)$ =>

$$x = c_1x_1 + c_2x_2 = c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt)$$
$$= e^{at}(c_1\cos(bt) + c_2\sin(bt)) = e^{at}G\cos(bt - \phi)$$

where $G = \dfrac{c_1}{\cos\left(\tan^{-1}\left(c_2/c_1\right)\right)}$, & $\phi = \tan^{-1}\left(c_2/c_1\right)$

PBL 7. Prove the above theorem

Example:

A=1, B=1, x(0)=1, x'(0)=0 => c_1=1, c_2=1/$\sqrt{3}$

Undamped system $A = 0$

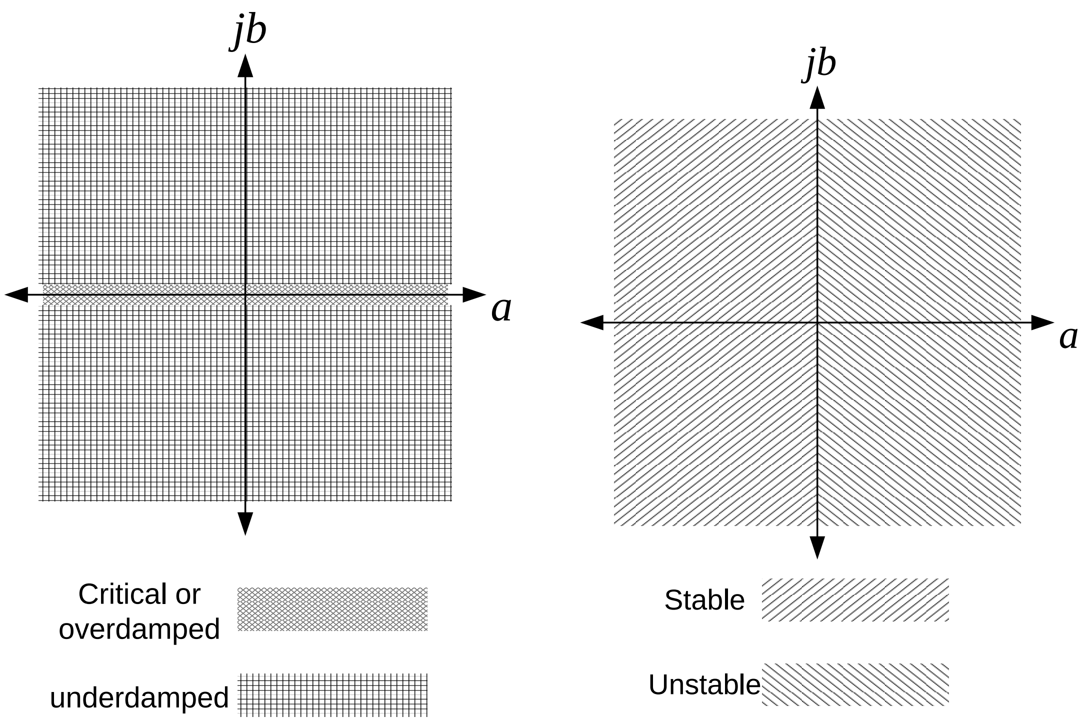
$x'' + 0 + Bx = 0 \Leftrightarrow r^2 e^{rt} + 0 + Be^{rt} = 0 \Rightarrow r^2 = -B$ => Imaginary roots (If B<0 then I would have two equal real roots).

So $r = jb$ => $x = c_1 \cos(bt) + c_2 \sin(bt) = G\cos(bt - \phi)$

A=0, B=1, x(0)=1, x'(0)=0 => c_1=1, c_2=0:

In all previous cases if the real part is positive then the solution will diverge to infinity and the ODE (and hence the system) is called unstable.

Root Space



Name	Oscillations?	Components of solution
Overdamped	No	Two exponentials: $e^{k_1 t}, e^{k_2 t},\ k_1, k_2 < 0$
Critically damped	No	Two exponentials: $e^{kt}, te^{kt},\ k < 0$
Underdamped	Yes	One exponential and one cosine $e^{kt}, \cos(\omega t),\ k < 0$
Undamped	Yes	one cosine $\cos(\omega t)$

Natural frequency, damping frequency, damping factor

2nd order systems very important with rich dynamic behaviour

So $A = 2\zeta\omega_n, \quad B = \omega_n^2$ => $x'' + 2\zeta\omega_n x' + \omega_n^2 x = 0$

ζ is the damping factor and ω_n is the natural frequency of the system.

$$r = \frac{-A \pm \sqrt{A^2 - 4B}}{2} = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2}$$

$$r = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$$

1. Real and unequal $\zeta^2\omega_n^2 > \omega_n^2 \Leftrightarrow \zeta^2 > 1 \overset{\zeta>0}{\Rightarrow} \zeta > 1$ => Overdamped system implies that $\zeta > 1$; $r_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$ => replace at $x = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.
2. Real and equal $\zeta^2\omega_n^2 = \omega_n^2 \Leftrightarrow \zeta^2 = 1 \Rightarrow \zeta = 1$ => Critically damped system implies that $\zeta = 1$; $r = -\omega_n$ => $x = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t}$.
3. Complex $\zeta^2\omega_n^2 < \omega_n^2 \Leftrightarrow \zeta^2 < 1 \overset{\zeta>0}{\Rightarrow} \zeta < 1$ => Underdamped systems implies $\zeta < 1$; $r_{1,2} = -\zeta\omega_n \pm j\sqrt{\omega_n^2 - \zeta^2\omega_n^2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ $\overset{\omega_d = \omega_n\sqrt{1-\zeta^2}}{=} -\zeta\omega_n \pm j\omega_d$ => $x = e^{-\zeta\omega_n t} G\cos(\omega_d t - \phi)$; ω_d is called damped frequency or pseudo-frequency.
4. Imaginary roots $\zeta = 0$ and therefore the solution is $r = \pm j\omega_n$ => $x = G\cos(\omega_n t - \phi)$; so when there is no damping the frequency of the oscillations = natural frequency. $x = G\cos(\omega_n t - \phi)$
5. In all the previous cases if $\zeta > 0$ then the transient part tends to zero. If $\zeta < 0$ then the system will diverge to infinity with or without oscillations.

4. NonHomogeneous (NH) differential equations

$x'' + Ax' + Bx = u,\ u = const$

- u=0 => Homogeneous => x_1 & x_2.
- Assume a particular solution of the nonhomogeneous ODE: x_p
    - If $u(t)$=R=cosnt => $x_P = \frac{R}{B}$
- Then all the solutions of the NHODE are $x = x_P + c_1 x_1 + c_2 x_2$
- So we have all the previous cases for under/over/un/critically damped systems plus a constant R/B.
- If complementary solution is stable then the particular solution is called steady state.

Example:

$$x'' + x' + x = 2 \Rightarrow x_P = 2$$

$$x = 2 + c_1 x_1 + c_2 x_2 = 2 + e^{at}\left(c_1 \cos(bt) + c_2 \sin(bt)\right)$$

x(0)=1, x'(0)=0 => c_1=-1, c_2=-1/$\sqrt{3}$