Chapter #2

EEE8013-3001

State Space Analysis and Controller Design

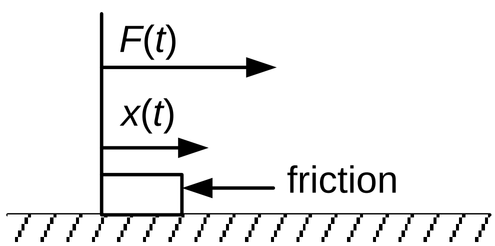
- **Introduction to state space**
- **Observability/Controllability**

1. Introduction

Assume an n^th order system: $x^{(n)} = f\left(x, x', x'', ..x^{(n-1)}\right)$. Very difficult to be studied (theoretically we can use geometric and/or analytic methods) => so we use computers. Computers are better with 1^st order ODE => break the n_{th} order to a system of n 1^st order. Also by using matrices we can use powerful tools from the linear algebra!

Example:

Assume the simple mass, spring system:



Using Newtonian mechanics we get:

$$\frac{d^2x}{dt^2} = F - B\frac{dx}{dt} - kx = m\ddot{x} = F - B\dot{x} - kx$$

By choosing as $x_1 = x,\, x_2 = \dot{x}$ we have:

$$\left.\begin{aligned}\dot{x}_1 &= \dot{x} = x_2 \\ \dot{x}_2 &= \ddot{x} = \frac{1}{m}\left(F - Bx_2 - kx_1\right)\end{aligned}\right\} \Rightarrow \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -\frac{k}{m} & -\frac{B}{m}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ F/m\end{bmatrix}$$

Or $\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -\frac{k}{m} & -\frac{B}{m}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ 1/m\end{bmatrix}F \Leftrightarrow \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$

Now, in order to "monitor" the system we need sensors to measure various variables like the displacement and velocity of the mass.

Let's assume that we can buy both sensors, then we define the output of the system to be:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Leftrightarrow \mathbf{y} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{x} \Leftrightarrow \mathbf{y} = \mathbf{C}\mathbf{x}$$

Let's assume that we can buy only one sensor, that measures the displacement, then the output is:

$$y = x_1 \Leftrightarrow \mathbf{y} = [1 \quad 0]\mathbf{x} \Leftrightarrow \mathbf{y} = \mathbf{C}\mathbf{x}$$

Let's assume that we can buy only one sensor, that measures the velocity, then the output is:

$$y = x_2 \Leftrightarrow \mathbf{y} = [0 \quad 1]\mathbf{x} \Leftrightarrow \mathbf{y} = \mathbf{C}\mathbf{x}$$

Let's assume that we have only one sensor that measures a linear combination of the displacement and velocity:

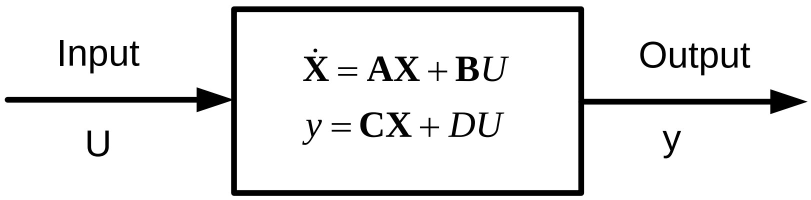
$$y = a_1 x_1 + a_2 x_2 \Leftrightarrow \mathbf{y} = [a_1 \quad a_2]\mathbf{x} \Leftrightarrow \mathbf{y} = \mathbf{C}\mathbf{x}$$

Hence, the most general case (for the above example):

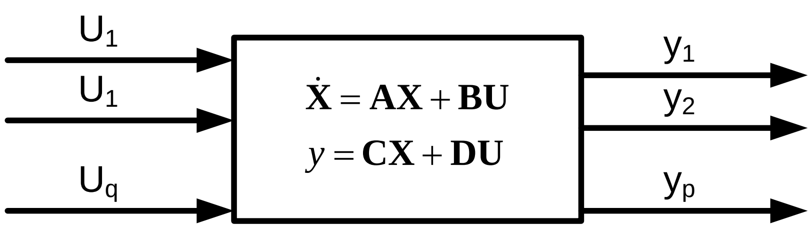
$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_1 x_1 + b_1 x_2 \\ a_2 x_1 + b_2 x_2 \\ \cdots \\ a_n x_1 + b_n x_2 \end{bmatrix} \Leftrightarrow \mathbf{y} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{bmatrix}\mathbf{x} \Leftrightarrow \mathbf{y} = \mathbf{C}\mathbf{x}$$

Finally let's assume that (in a rather artificial case) that the input can directly influence the output, then we have: $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}u$, For some matrix **D**.

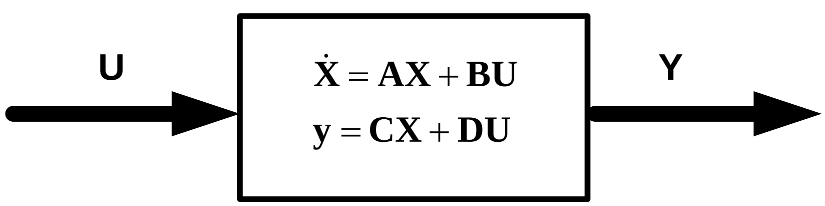
So the system is described by $\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}u \end{aligned}$:



Generally I can have more than one inputs and/or outputs:



Or in a vector form:



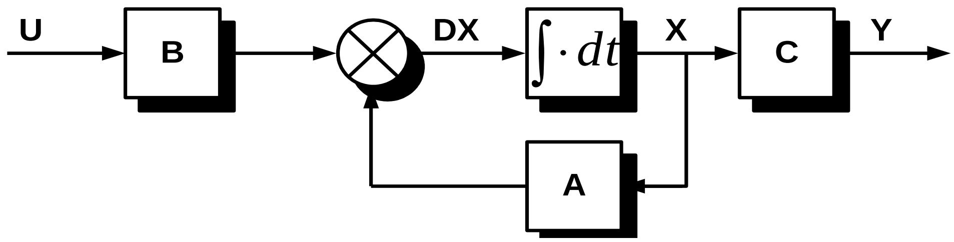
Where: $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_q \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_p \end{bmatrix}$

In general: $$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

Where

- **x** is an *n* x 1 state vector
- **u** is an *q* x 1 input vector
- **y** is an *p* x 1 output vector
- **A** is an *n* x n state matrix
- **B** is an *n* x q input matrix
- **C** is an *p* x n output matrix
- **D** is an *p* x q feed forward matrix (usually zero)

If the system is Linear Time Invariant (LTI): $$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$



The state vector describes the system => Gives its state => The **state** of a system is a complete summary of the system at a particular point in time. If the current state of the system and the future input signals are known then it is possible to define the future states and outputs of the system.

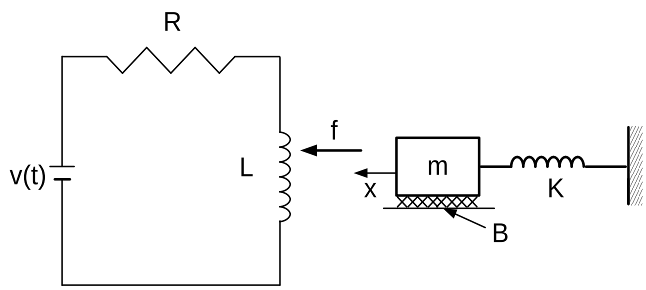
The choice of the state space variables is free as long as some rules are followed:

- They must be linearly independent.
- They must specify completely the dynamic behaviour of the system.
- Finally they must not be input of the system.

Examples of state space models (NOT ASSESSED MATERIAL)

Example 1

Assume the following simple electromechanical that consists of an electromagnet and



The force of the magnetic field is directly related to the current in the *RL* network. The force that is exerted on the object is $f = k_A \frac{i^2}{x^2}$, where k_A is a positive constant. To simplify the analysis we assume that the displacement x is very small and in that small area the current has a linear relationship with the force: $f = k_A i$

Using circuit theory: $\frac{di}{dt} = \frac{1}{L}(v - iR)$

Using Newton's 2nd law: $f - kx - B\dot{x} = m\ddot{x} \Leftrightarrow k_A i - kx - B\dot{x} = m\ddot{x}$

Now, we can define $x_1 = x$, $x_2 = \dot{x}$ and $x_3 = i$. Thus:

$$\dot{x}_3 = \frac{1}{L}(v - x_3 R) \Leftrightarrow \dot{x}_3 = -x_3 \frac{R}{L} + \frac{v}{L}$$

$\dot{x}_1 = \dot{x} = x_2$

$$m\dot{x}_2 = k_A x_3 - kx_1 - Bx_2 \Leftrightarrow \dot{x}_2 = -\frac{k}{m}x_1 - \frac{B}{m}x_2 + \frac{k_A}{m}x_3$$

Hence the state space model is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -k/m & -B/m & -k_A/m \\ 0 & 0 & -R/L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L \end{bmatrix} v \Leftrightarrow \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

Now let's assume that we have only one sensor that will return the displacement x:

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

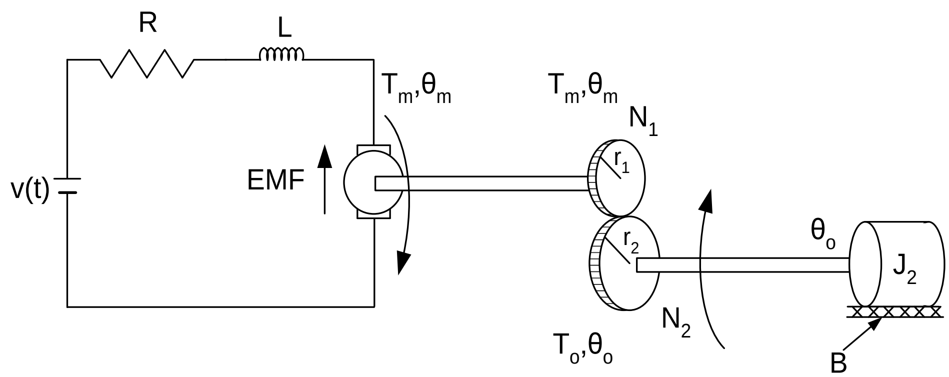
Thus the state space model is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -k/m & -B/m & -k_A/m \\ 0 & 0 & -R/L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L \end{bmatrix} v$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 2

Another example is shown in the next figure.



The shaft of the separately excited DC motor is connected to the load J_2 through a gear box.

$$\left.\begin{array}{l} J\ddot{\theta}_0 = T_0 - B\dot{\theta}_0 \\ T_0 = \frac{n_2}{n_1} T_m \\ T_m = K_T \varphi i_a \end{array}\right\} \Rightarrow J\ddot{\theta}_0 = \frac{n_1}{n_2} K_T \varphi i_a - B\dot{\theta}_0$$

$$v_a = i_a R_a + L_a \frac{di_a}{dt} + K_T\, \varphi\, \dot{\theta}_m \Leftrightarrow v_a = i_a R_a + L_a \frac{di_a}{dt} + K_T\, \varphi \frac{n_2}{n_1} \dot{\theta}_0 \quad \Rightarrow$$

$$\left.\begin{array}{l} J\ddot{\theta}_0 = \frac{n_1}{n_2} K_T \varphi i_a - B\dot{\theta}_0 \\ v_a = i_a R_a + L_a \frac{di_a}{dt} + K_T\, \varphi \frac{n_2}{n_1} \dot{\theta}_0 \end{array}\right\} \overset{K_2 = K_T \varphi \frac{n_1}{n_2}}{\underset{K_1 = K_T \varphi \frac{n_2}{n_1},}{\Rightarrow}}$$

$$\left.\begin{array}{l} J\ddot{\theta}_0 = K_2 i_a - B\dot{\theta}_0 \\ v_a = i_a R_a + L_a \frac{di_a}{dt} + K_1 \dot{\theta}_0 \end{array}\right\}$$

I define $\dot{\theta}_0 = x_1, i_a = x_2$:

$$\left.\begin{aligned} J\dot{x}_1 &= K_2 x_2 - B x_1 \\ v_a &= x_2 R_a + L_a \dot{x}_2 + K_1 x_1 \end{aligned}\right\} \Leftrightarrow \left.\begin{aligned} \dot{x}_1 &= -\frac{B}{J}x_1 + \frac{K_2}{J}x_2 \\ \dot{x}_2 &= -\frac{K_1}{L_a}x_1 - x_2\frac{R_a}{L_a} + \frac{1}{L_a}v_a \end{aligned}\right\} \Leftrightarrow$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{B}{J} & \frac{K_2}{J} \\ -\frac{K_1}{L_a} & \frac{R_a}{L_a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L_a} \end{bmatrix} v_a$$

Example 3

It can be proved that a model of the Induction Machine is:

$$\left.\begin{aligned} \frac{d\psi_{sD}}{dt} &= -R_s i_{sD} + u_{sD} \\ \frac{d\psi_{sQ}}{dt} &= -R_s i_{sQ} + u_{sQ} \\ \frac{di_{sD}}{dt} &= \frac{-R_r}{\sigma_1}\psi_{sD} + \frac{-\omega_r L_r}{\sigma_1}\psi_{sQ} + i_{sD}\frac{(L_s R_r + L_r R_s)}{\sigma_1} - i_{sQ}\omega_r - \frac{L_r}{\sigma_1}u_{sD} \\ \frac{di_{sQ}}{dt} &= \frac{-R_r}{\sigma_1}\psi_{sQ} + \frac{\omega_r L_r}{\sigma_1}\psi_{sD} + i_{sQ}\frac{(L_s R_r + L_r R_s)}{\sigma_1} + i_{sD}\omega_r - \frac{L_r}{\sigma_1}u_{sQ} \end{aligned}\right\} \Leftrightarrow$$

$$\overset{\bullet}{\begin{bmatrix} \psi_{sD} \\ \psi_{sQ} \\ i_{sD} \\ i_{sQ} \end{bmatrix}} = \begin{bmatrix} 0 & 0 & -R_s & 0 \\ 0 & 0 & 0 & -R_s \\ \frac{-R_r}{\sigma_1} & \frac{-\omega_r L_r}{\sigma_1} & \frac{(L_s R_r + L_r R_s)}{\sigma_1} & -\omega_r \\ \frac{\omega_r L_r}{\sigma_1} & \frac{-R_r}{\sigma_1} & \omega_r & \frac{(L_s R_r + L_r R_s)}{\sigma_1} \end{bmatrix} \begin{bmatrix} \psi_{sD} \\ \psi_{sQ} \\ i_{sD} \\ i_{sQ} \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{L_r}{\sigma_1} & 0 \\ 0 & -\frac{L_r}{\sigma_1} \end{bmatrix} \begin{bmatrix} u_{sD} \\ u_{sQ} \end{bmatrix}$$

Or:

$$\left.\begin{aligned}
\frac{di_{sD}}{dt} &= -\frac{R_s}{\sigma L_s} i_{sD} + \frac{\omega_r L_m^2}{L_r \sigma L_s} i_{sQ} + \frac{L_m R_r}{L_r \sigma L_s} i_{rd} + \frac{\omega_r L_m}{\sigma L_s} i_{rq} + \frac{1}{\sigma L_s} u_{sD} \\
\frac{di_{sQ}}{dt} &= -\frac{\omega_r L_m^2}{L_r \sigma L_s} i_{sD} - \frac{R_s}{\sigma L_s} i_{sQ} - \frac{\omega_r L_m}{\sigma L_s} i_{rd} + \frac{L_m R_r}{L_r \sigma L_s} i_{rq} + \frac{1}{\sigma L_s} u_{sQ} \\
\frac{di_{rd}}{dt} &= \frac{L_m R_s}{L_r \sigma L_s} i_{sD} - \frac{\omega_r L_m}{\sigma L_r} i_{sQ} - \frac{R_r}{\sigma L_r} i_{rd} - \frac{\omega_r}{\sigma} i_{rq} - \frac{L_m}{L_s \sigma L_r} u_{sD} \\
\frac{di_{rq}}{dt} &= \frac{\omega_r L_m}{\sigma L_r} i_{sD} + \frac{L_m R_s}{L_r \sigma L_s} i_{sQ} + \frac{\omega_r}{\sigma} i_{rd} - \frac{R_r}{\sigma L_r} i_{rq} - \frac{L_m}{L_s \sigma L_r} u_{sQ}
\end{aligned}\right\} \Leftrightarrow$$

$$\begin{bmatrix} \frac{di_{sD}}{dt} \\ \frac{di_{sQ}}{dt} \\ \frac{di_{rd}}{dt} \\ \frac{di_{rq}}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_s}{\sigma L_s} & \frac{\omega_r L_m^2}{L_r \sigma L_s} & \frac{L_m R_r}{L_r \sigma L_s} & \frac{\omega_r L_m}{\sigma L_s} \\ -\frac{\omega_r L_m^2}{L_r \sigma L_s} & -\frac{R_s}{\sigma L_s} & -\frac{\omega_r L_m}{\sigma L_s} & \frac{L_m R_r}{L_r \sigma L_s} \\ \frac{L_m R_s}{L_r \sigma L_s} & -\frac{\omega_r L_m}{\sigma L_r} & -\frac{R_r}{\sigma L_r} & -\frac{\omega_r}{\sigma} \\ \frac{\omega_r L_m}{\sigma L_r} & \frac{L_m R_s}{L_r \sigma L_s} & \frac{\omega_r}{\sigma} & -\frac{R_r}{\sigma L_r} \end{bmatrix} \begin{bmatrix} i_{sD} \\ i_{sQ} \\ i_{rd} \\ i_{rq} \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{1}{\sigma L_s} & 0 \\ 0 & \frac{1}{\sigma L_s} \\ -\frac{L_m}{L_s \sigma L_r} & 0 \\ 0 & -\frac{L_m}{L_s \sigma L_r} \end{bmatrix} \begin{bmatrix} u_{sD} \\ u_{sQ} \end{bmatrix}$$

State space

The system's states can be written in a vector form as:

$$\mathbf{x}_1 = [x_1, 0, \cdots, 0]^T, \ \mathbf{x}_2 = [0, x_2, \cdots, 0]^T, \ldots, \ \mathbf{x}_n = [0, 0, \cdots, x_n]^T$$

=> A standard orthogonal basis (since they are linear independent) for an *n*-dimensional vector space called state space.

PBL: What is a vector space?

What is a linear combination of a set of vectors?

What is a span of a set of vectors?

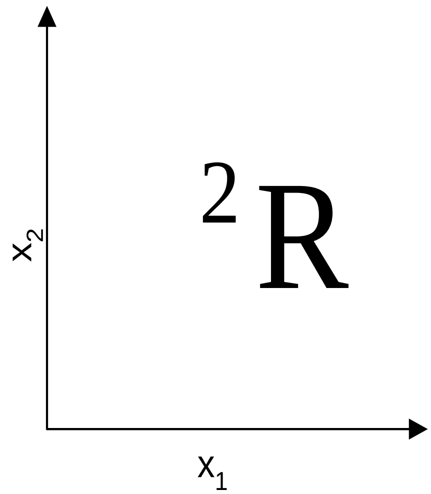
What do we mean when we say that 2 vectors are linearly independent?

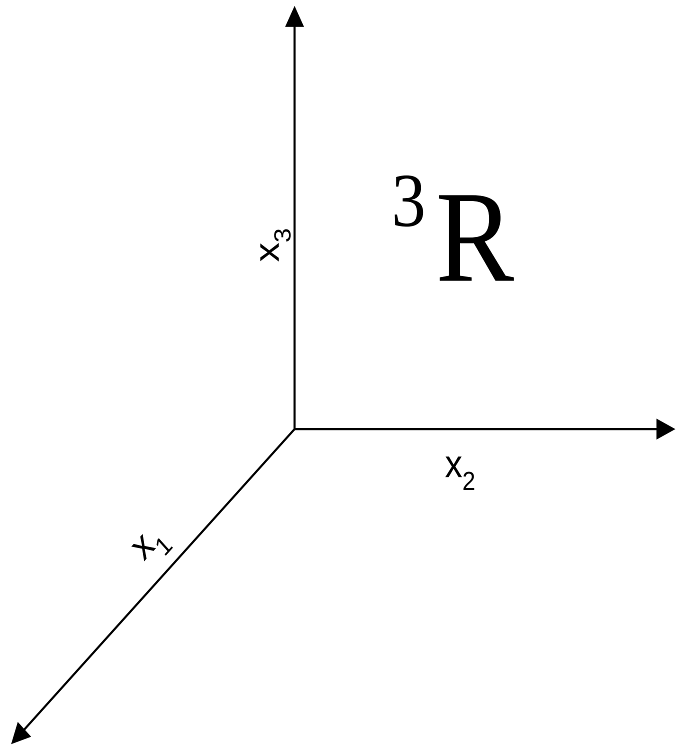
What is a basis in a vector space?

What is a linear transformation from a vector space *V* to a vector space *W*?

What is a generalised eigenvector and how can it be used to form a basis in a VS?

Examples of state spaces are the state plane (*n*=2) and state 3D space (*n*=3),





Relation of state space and TF

If we have an LTI state space (ss) system, how can we find its TF?

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \overset{LT}{\Rightarrow} s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \Rightarrow$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) + \mathbf{x}(0) \Rightarrow$$
$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)$$

And from the 2nd equation of the ss system:

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) =>$$
$$\mathbf{Y}(s) = \mathbf{C}\left((s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)\right) + \mathbf{D}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \left(\mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right)\mathbf{U}(s) + \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{x}(0)$$

By definition TF: $\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ and $\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)$ the response to the IC.

Also: $\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{X}(0) \overset{ILT}{\Rightarrow}$

$$\mathbf{x}(t) = L^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)\right\} + L^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\}\mathbf{x}(0)$$

If **U**=0 => $\mathbf{X}(t) = L^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\}\mathbf{X}(0)$

So $\mathbf{G}(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$ is the TF. From linear algebra: $\mathbf{G}_{i,j}(s)=\frac{\begin{vmatrix} s\mathbf{I}-\mathbf{A} & -\mathbf{B}_i \\ \mathbf{C}_j & \mathbf{D}\end{vmatrix}}{|s\mathbf{I}-\mathbf{A}|}$, where $\mathbf{B}_i$ is the ith column of the matrix **B** and $\mathbf{C}_j$ is the jth row of **C**.

Hence $|sI-\mathbf{A}|$ is the CE of the TF!!!

So: $\mathbf{G}_{i,j}(s)=\begin{bmatrix} G_{11}(s) & G_{12}(s) & \dots & G_{1q}(s) \\ G_{21}(s) & G_{22}(s) & \dots & G_{2q}(s) \\ \dots & \dots & \dots & \dots \\ G_{p1}(s) & G_{p2}(s) & \dots & G_{pq}(s)\end{bmatrix}$

$\frac{Y_1}{U_1}=G_{11},\quad \frac{Y_1}{U_2}=G_{12},\quad \frac{Y_2}{U_1}G_{21},\quad \frac{Y_2}{U_2}=G_{22}\dots$

Example:

Find the TF of $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & -0.5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}2$ and $y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

$$\left.\begin{array}{l} |s\mathbf{I}-\mathbf{A}|=\begin{vmatrix} s & -1\\ 1 & s+0.5\end{vmatrix}=s(s+0.5)+1 \\ \begin{vmatrix} s & -1 & 0\\ 1 & s+0.5 & -1\\ 1 & 0 & 0\end{vmatrix}=1 \end{array}\right\}\Rightarrow G(s)=\frac{1}{s(s+0.5)+1}$$

Observability

Assume that we have the following system:

$$\left.\begin{array}{l}\dot{\mathbf{x}}=\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}\mathbf{x}+\begin{bmatrix}2\\ 1\end{bmatrix}\mathbf{u}\\ \mathbf{y}=\begin{bmatrix}3 & 0\end{bmatrix}\mathbf{x}\end{array}\right\}.$$

Notice that the model is uncoupled and since C is 1x2 it is impossible to see how x_2 behaves (no problem if A was not diagonal or C was 2x2). This implies that we cannot monitor x_2, for example it can diverge to infinity with catastrophic results for our system.

Assume that we have another system: $\left.\begin{array}{l}\dot{x}=-2x+2u\\ y=3x\end{array}\right\}$

Clearly these two models are different. In that case it can be proved that the 2 systems have the same transfer function as there is a pole-zero cancelation:

$$\left.\begin{array}{l}\dot{x}=-2x+2u\\ y=3x\end{array}\right\}\Rightarrow G(s)=\frac{\begin{vmatrix}s\mathbf{I}-\mathbf{A} & -\mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{vmatrix}}{|s\mathbf{I}-\mathbf{A}|}=\frac{6}{s+2}$$

$$\left.\begin{array}{l}\dot{\mathbf{x}}=\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}\mathbf{x}+\begin{bmatrix}2\\ 1\end{bmatrix}u\\ \mathbf{y}=\begin{bmatrix}3 & 0\end{bmatrix}\mathbf{x}\end{array}\right\}\Rightarrow G(s)=\frac{\begin{vmatrix}s\mathbf{I}-\mathbf{A} & -\mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{vmatrix}}{|s\mathbf{I}-\mathbf{A}|}=\frac{6(s+1)}{(s+2)(s+1)}=\frac{6}{(s+2)}$$

which is exactly the same as the TF of the first system, what is wrong? There is a pole zero cancellation at the second model

Controllability

Assume that we have the following system:

$$\left.\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}\mathbf{x} + \begin{bmatrix}2\\ 0\end{bmatrix}\mathbf{u}\\ \mathbf{y} &= \begin{bmatrix}3 & 2\end{bmatrix}\mathbf{x}\end{aligned}\right\}$$

In this case we can see how both states behave but we cannot change **u** in any way so that we can influence x_2 due to the form of **B**. If A was not diagonal we would be able to control x_2 through x_1.

Similarly we have a pole-zero cancellation in:

$$\left.\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}\mathbf{x} + \begin{bmatrix}2\\ 0\end{bmatrix}u\\ y &= \begin{bmatrix}3 & 2\end{bmatrix}\mathbf{x}\end{aligned}\right\} \Rightarrow G(s) = \frac{\begin{vmatrix}s\mathbf{I}-\mathbf{A} & -\mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{vmatrix}}{\left|s\mathbf{I}-\mathbf{A}\right|} = \frac{6(s+1)}{(s+2)(s+1)} = \frac{6}{(s+2)}$$

Hence in the first case by properly defining **u** we can control both states but we cannot see the second state, while in the second case we can see both states but we cannot control the second state. The first system is called **unobservable** and the second **uncontrollable**. The loss of the controllability and/or observability is due to a pole/zero cancellation. These systems are unacceptable and the solution to that problem is to re-model the system.

The systems that are both controllable and observable are called **minimal realisation**.

We need to develop tests to determine the controllability and observability properties of the system. Difficult task if the system is nonlinear. In our case we simply have to find the rank (the number of Linear Independent (LI) rows or columns) of two matrices.

For observability:

$\mathbf{M}_O = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$. If the rank of this matrix is less than n then the system is unobservable.

Example:

$$\left.\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}u \\ \mathbf{y} &= \begin{bmatrix} 3 & 0 \end{bmatrix}\mathbf{x} \end{aligned}\right\} \Rightarrow \mathbf{CA} = \begin{bmatrix} 3 & 0 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -6 & 0 \end{bmatrix}$$

$$\mathbf{M}_O = \begin{bmatrix} 3 & 0 \\ -6 & 0 \end{bmatrix}$$

And obviously there is only one LI column/row

Example:

$$\left.\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}\mathbf{x} + \begin{bmatrix}2\\ 1\end{bmatrix}\mathbf{u}\\ \mathbf{y} &= \begin{bmatrix}3 & 2\end{bmatrix}\mathbf{x}\end{aligned}\right\} \Rightarrow \mathbf{CA} = \begin{bmatrix}3 & 2\end{bmatrix}\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}-6 & -2\end{bmatrix}$$

$$\mathbf{M}_O = \begin{bmatrix}3 & 2\\ -6 & -2\end{bmatrix}$$

And obviously there are 2 LI column/rows

For controllability:

$\mathbf{M}_C = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}$. If the rank of this matrix is less than n then the system is uncontrollable.

Example

$$\left.\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}\mathbf{x} + \begin{bmatrix}2\\ 0\end{bmatrix}\mathbf{u}\\ \mathbf{y} &= \begin{bmatrix}3 & 2\end{bmatrix}\mathbf{x}\end{aligned}\right\} \Rightarrow \mathbf{AB} = \begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}2\\ 0\end{bmatrix} = \begin{bmatrix}-4\\ 0\end{bmatrix}$$

$$\mathbf{M}_C = \begin{bmatrix}2 & -4\\ 0 & 0\end{bmatrix}$$

And obviously there is only one LI column/row

Example

$$\left.\begin{aligned}\mathbf{x} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\mathbf{u} \\ \mathbf{y} &= \begin{bmatrix} 3 & 2 \end{bmatrix}\mathbf{x}\end{aligned}\right\} \Rightarrow \mathbf{AB} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

$$\mathbf{M}_C = \begin{bmatrix} 2 & -4 \\ 1 & -1 \end{bmatrix}$$

And obviously there are 2 LI column/rows.