Chapter #3

EEE3001 & EEE8013

State Space Analysis and Controller Design

1. General Solution of 2nd Order State Space Models

1.1 Case 1: Real and unequal eigenvalues

Assume that $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Not assessed material

Then if we go back to a 2nd order DE:

$$\left\{ \begin{aligned} \dot{x}_1 &= -2x_1 + 2x_2 \\ \dot{x}_2 &= 2x_1 - 5x_2 \end{aligned} \right\} \Rightarrow$$

$$x_1 = \frac{1}{2}\left(\dot{x}_2 + 5x_2\right) \Leftrightarrow$$

$$\dot{x}_1 = \frac{1}{2}\left(\ddot{x}_2 + 5\dot{x}_2\right)$$

But $\dot{x}_1 = -2x_1 + 2x_2$:

$$-2x_1 + 2x_2 = \frac{1}{2}\left(\ddot{x}_2 + 5\dot{x}_2\right)$$

But we have found that $x_1 = \frac{1}{2}\left(\dot{x}_2 + 5x_2\right)$:

$$-2\left(\frac{1}{2}\left(\dot{x}_2 + 5x_2\right)\right) + 2x_2 = \frac{1}{2}\left(\ddot{x}_2 + 5\dot{x}_2\right) \Leftrightarrow$$

$$\frac{1}{2}\ddot{x}_2 + \frac{5}{2}\dot{x}_2 = -\dot{x}_2 - 5x_2 + 2x_2 \Leftrightarrow$$

$$\ddot{x}_2 + 5\dot{x}_2 = -2\dot{x}_2 - 10x_2 + 4x_2 \Leftrightarrow$$

$$\ddot{x}_2 + 7\dot{x}_2 + 6x_2 = 0$$

Similarly for x_1:

$$\left.\begin{cases}\dot{x}_1 = -2x_1 + 2x_2 \\ \dot{x}_2 = 2x_1 - 5x_2\end{cases}\right\} \Rightarrow$$

$$x_2 = \frac{1}{2}(\dot{x}_1 + 2x_1) \Leftrightarrow$$

$$\dot{x}_2 = \frac{1}{2}(\ddot{x}_1 + 2\dot{x}_1) \Leftrightarrow$$

$$2x_1 - 5x_2 = \frac{1}{2}(\ddot{x}_1 + 2\dot{x}_1) \Leftrightarrow$$

$$2x_1 - 5\left(\frac{1}{2}(\dot{x}_1 + 2x_1)\right) = \frac{1}{2}(\ddot{x}_1 + 2\dot{x}_1) \Leftrightarrow$$

$$2x_1 - \frac{5}{2}\dot{x}_1 - 5x_1 = \frac{1}{2}\ddot{x}_1 + \dot{x}_1 \Leftrightarrow$$

$$4x_1 - 5\dot{x}_1 - 10x_1 = \ddot{x}_1 + 2\dot{x}_1 \Leftrightarrow$$

$$\ddot{x}_1 + 7\dot{x}_1 + 6x_1 = 0$$

Thus the homogeneous state space model $\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-2 & 2 \\ 2 & -5\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$ has been transformed to $\ddot{x}_2 + 7\dot{x}_2 + 6x_2 = 0$ (and $\ddot{x}_1 + 7\dot{x}_1 + 6x_1 = 0$)

Then we have a common CE is: $r^2 + 7r + 6 = 0$

This will give two solutions: $x_{2a} = C_2 e^{-t}$ and $x_{2b} = D_2 e^{-6t}$

Similarly the ODE for x_1 will give me $x_{1a} = C_1 e^{-t}$ and $x_{1b} = D_1 e^{-6t}$

Thus a solution to our state space model is:

$$\mathbf{x} = \begin{bmatrix}C_1 \\ C_2\end{bmatrix} e^{-t} \text{ and } \mathbf{x} = \begin{bmatrix}D_1 \\ D_2\end{bmatrix} e^{-6t} \text{ and thus } \mathbf{x} = \begin{bmatrix}C_1 \\ C_2\end{bmatrix} e^{-t} + \begin{bmatrix}D_1 \\ D_2\end{bmatrix} e^{-6t}.$$

Obviously the choice of C_1 (or D_1) will influence the choice of C_2 (or D_2) and thus the vectors $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ and $\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$ are not "completely arbitrary".

Thus we should say that our solution is:

$\mathbf{x} = A \times \mathbf{e}_1 \times e^{-t} + B \times \mathbf{e}_2 \times e^{-6t}$, where $\mathbf{e}_1 = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$ and A, B are arbitrary constants that depend on the initial conditions.

But this approach is rather cumbersome and of course we cannot (easily) find the values of the vectors $\mathbf{e}_1, \mathbf{e}_2$.

But, the important point here is the solution will be a linear combination of two vectors that are multiplied by an exponentials and the 2 exponents are the 2 eigenvalues.

Now back to our system: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Let's try $x = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t}$

So $\dot{\mathbf{x}} = \lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t} \Rightarrow \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \end{bmatrix} = \begin{bmatrix} -2a_1 + 2a_2 \\ 2a_1 - 5a_2 \end{bmatrix}$.

How can we solve that? It is a nonlinear system with 2 equations and 3 unknowns!

Assume λ is a parameter => A homogeneous linear system:

$\begin{Bmatrix} (-2-\lambda)a_1 + 2a_2 = 0 \\ 2a_1 + (-5-\lambda)a_2 = 0 \end{Bmatrix}$ Always a trivial solution a_1=a_2=0.

For a nontrivial solution (see Cramer's rule from Linear Algebra):

$$\begin{vmatrix} -2-\lambda & 2 \\ 2 & -5-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 + 7\lambda + 6 = 0$$

(This last equation is the characteristic equation of the system).

$\lambda^2 + 7\lambda + 6 = 0 \Rightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = -6 \end{cases}$. Hence for each of these I have to find a_1, a_2:

For $\lambda_1 = -1$

$$\left.\begin{cases} -a_1 + 2a_2 = 0 \\ 2a_1 - 4a_2 = 0 \end{cases}\right\} \text{the same}$$

PBL1: Explain why we should have expected the same equation twice.

I assume that a_2=1 so a_1=2 so for that value of λ_1, $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and one solution to the system is $\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$.

For $\lambda_2 = -6$ $\left.\begin{cases} 4a_1 + 2a_2 = 0 \\ 2a_1 + a_2 = 0 \end{cases}\right\}$, I assume that a_1=1 so a_2=-2

So a second solution is $\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-6t}$

From superposition principle: $\mathbf{x}(t) = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-6t}$

Example:

Find the response of the previous system when $\mathbf{x}(0)=\begin{bmatrix}1\\0\end{bmatrix}$.

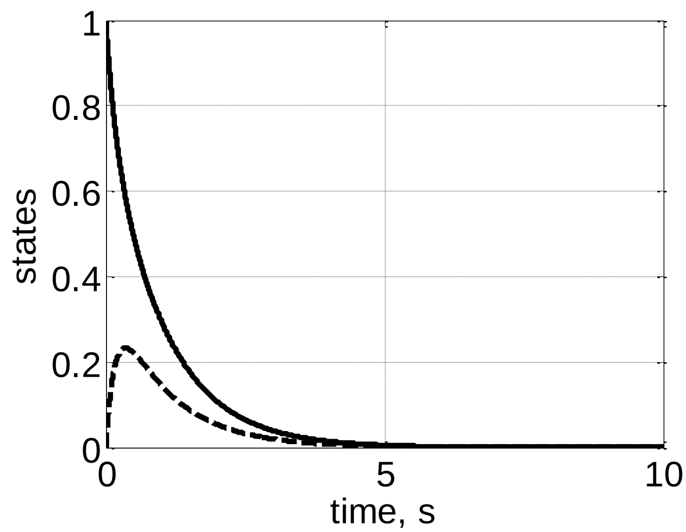
Analytical approach:

Using $\mathbf{x}(t)=C_1\begin{bmatrix}2\\1\end{bmatrix}e^{-t}+C_2\begin{bmatrix}1\\-2\end{bmatrix}e^{-6t}$ we have that:

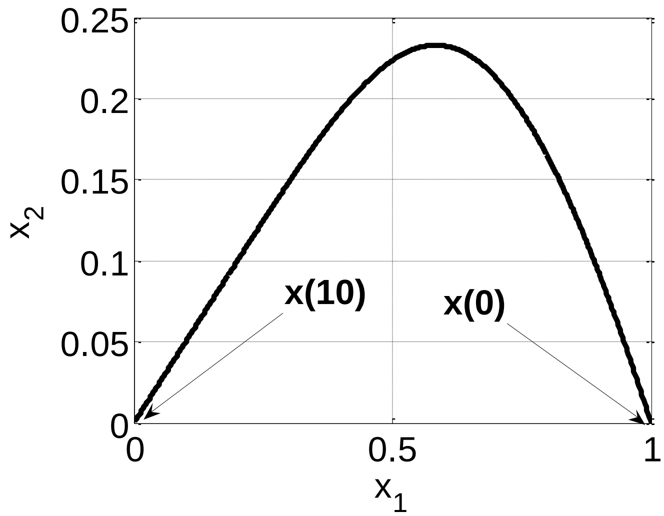
$$\mathbf{x}(0)=C_1\begin{bmatrix}2\\1\end{bmatrix}+C_2\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\Rightarrow\begin{bmatrix}2C_1+C_2\\C_1-2C_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

This last system can be solved with various methods like substitution… an easier way is:

$$\begin{bmatrix}2C_1+C_2\\C_1-2C_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\Leftrightarrow\begin{bmatrix}2&1\\1&-2\end{bmatrix}\begin{bmatrix}C_1\\C_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\Leftrightarrow\begin{bmatrix}C_1\\C_2\end{bmatrix}=\begin{bmatrix}2&1\\1&-2\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0.4\\0.2\end{bmatrix}$$



In the state space:



1.2 Case 2: Repeated Eigenvalues

Now it is possible to have 2 sub-cases:

- I can find 2 LI vectors (a rather artificial case)

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda_{1,2} = 2 \Rightarrow \begin{cases} \mathbf{e}_1 = [1 \quad 0]^T \\ \mathbf{e}_2 = [0 \quad 1]^T \end{cases}$$

  Hence we have 2 uncoupled 1st order ODEs which can be solved separately (a rather artificial case)

- I **cannot** find 2 LI vectors

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow \lambda_{1,2} = 2 \Rightarrow \mathbf{e}_{1,2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

In that case I have that $(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{b} = \mathbf{0}$ (**b** is called the generalised eigenvector of **A**), which can be written as $(\mathbf{A} - \lambda\mathbf{I})(\mathbf{A} - \lambda\mathbf{I})\mathbf{b} = 0$. Now I substitute $\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{b}$ and I have $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$, i.e. $\mathbf{v}$ is one eigenvector of $\mathbf{A}$ for the eigenvalue λ. Now it can be proved that the solution is $\mathbf{x}(t) = C_1(\mathbf{v}t + \mathbf{b})e^{\lambda t} + C_2\mathbf{v}e^{\lambda t}$.

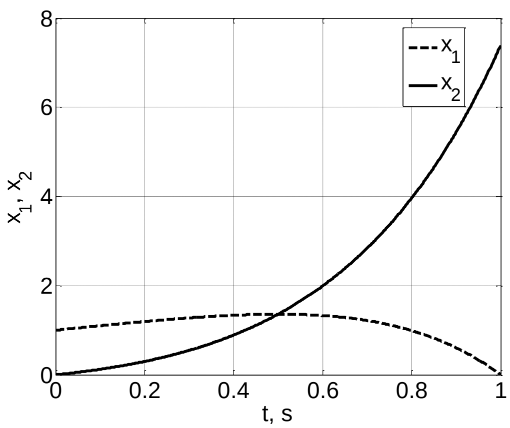
So in that case

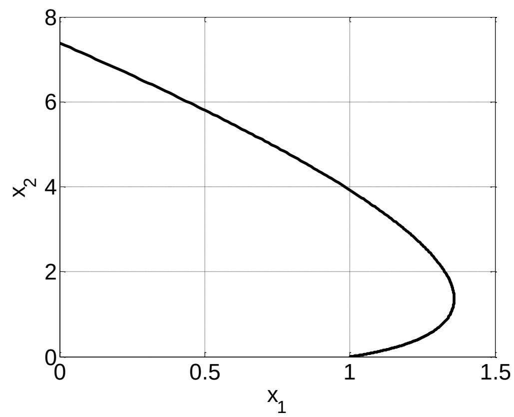
$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (\mathbf{A} - 2\mathbf{I})\mathbf{b} \Leftrightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -b_1 - b_2 \\ b_1 + b_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Hence the solution is: $\mathbf{x}(t) = C_1\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)e^{\lambda t} + C_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{\lambda t}$

If we are given the same initial conditions:

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ -C_1 \end{bmatrix} + \begin{bmatrix} C_2 \\ -C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} C_2 = 1 \\ C_1 = -1 \end{cases}$$





1.3 Case 3: Complex Eigenvalues

If I have complex eigenvalues then $\lambda_1 = \overline{\lambda_2}$ and the corresponding eigenvectors are $\mathbf{e}_1 = \overline{\mathbf{e}_2}$. In that case the general solution is given by:

$$\mathbf{x}(t) = A_1\mathbf{e}_1 e^{\lambda_1 t} + A_2\mathbf{e}_2 e^{\lambda_2 t} \Leftrightarrow \mathbf{x}(t) = A_1(\mathbf{a}+\mathbf{b}j)e^{(\mu+\nu j)t} + A_2(\mathbf{a}-\mathbf{b}j)e^{(\mu-\nu j)t}$$

Now we can use the Euler's formula $e^{njt} = (\cos(nt) + j\sin(nt))$ we have a complex solution to a real problem and hence we have seen that its real and imaginary parts are also real solutions of the state equation:

$$\begin{aligned}\mathrm{Re}(\mathbf{X}(t)) &= C_1 e^{\mu t}(\mathbf{a}\cos(\nu t) - \mathbf{b}\sin(\nu t))\\ \mathrm{Im}(\mathbf{X}(t)) &= C_2 e^{\mu t}(\mathbf{a}\sin(\nu t) + \mathbf{b}\cos(\nu t))\end{aligned}$$ for some constants C_1, C_2.

So the overall solution is:

$$\begin{aligned}\mathbf{x}(t) &= e^{\mu t}(C_1(\mathbf{a}\cos(\nu t) - \mathbf{b}\sin(\nu t)) + C_2(\mathbf{a}\sin(\nu t) + \mathbf{b}\cos(\nu t))) \Leftrightarrow\\ \mathbf{x}(t) &= e^{\mu t}(\mathbf{a}(C_1\cos(\nu t) + C_2\sin(\nu t)) + \mathbf{b}(C_2\cos(\nu t) - C_1\sin(\nu t)))\end{aligned}$$

If we do not want to use the Euler's formula then we can simply write:

$$\mathbf{x}(t) = A_1\mathrm{Re}\left(\mathbf{e}e^{\lambda t}\right) + A_2\,\mathrm{Im}\left(\mathbf{e}e^{\lambda t}\right)$$

Example:

$$\mathbf{A}=\begin{bmatrix}-1/2 & 1\\ -1 & -1/2\end{bmatrix}\Rightarrow \lambda=-0.5\pm j\Rightarrow \mathbf{e}=\begin{bmatrix}1\\ j\end{bmatrix}\Rightarrow \mathbf{x}(t)=e^{-0.5t}\left(C_1\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}+C_2\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}\right)$$

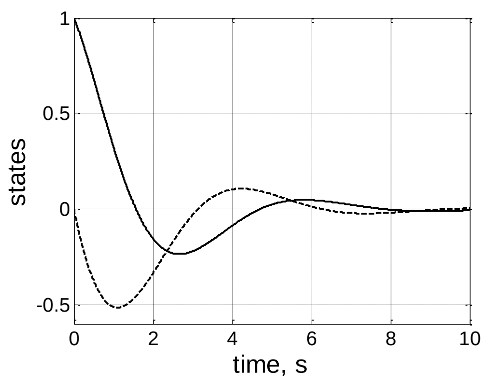
$$\Rightarrow \mathbf{x}(t)=e^{-0.5t}\begin{bmatrix}C_1\cos t+C_2\sin t\\ -C_1\sin t+C_2\cos t\end{bmatrix}$$

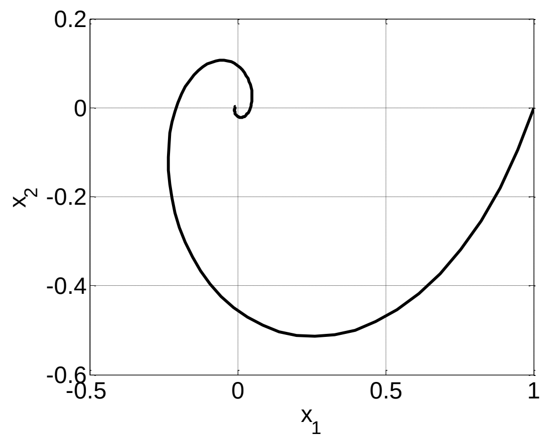
Assuming $\mathbf{x}(0)=\begin{bmatrix}1\\ 0\end{bmatrix}$

We have that:

$$\begin{bmatrix}1\\ 0\end{bmatrix}=C_1\begin{bmatrix}1\\ 0\end{bmatrix}+C_2\begin{bmatrix}0\\ 1\end{bmatrix}\Leftrightarrow\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}C_1\\ C_2\end{bmatrix}\Leftrightarrow\begin{bmatrix}C_1\\ C_2\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}$$

Thus the solution is: $\mathbf{x}(t)=e^{-0.5t}\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}$





PBL2: If we see the set of all solutions of an ODE $\dot{x} = Ax$ as a vector space (VS) V, then one way to express all possible solutions of the ODE is to use a linear combination of a basis in this space.

If we see the ODE as a linear transformation in this VS then one basis can be the generalised eigenbasis and hence the solutions are:

$x(t) = c_1(t)e_1 + c_2(t)e_2$ where e_i are the generalised eigenvectors.

Using this approach prove the 3 formulas that we saw before.

2. Uniqueness, Existence, Independent Solutions

Up to this point we assumed that a solution always exists. This is not always true but for the purposes of this module we can assume that there is one and only one solution of the IVP $\dot{x} = Ax, x(0) = x_0$.

PBL3: Find and state the theorem that the above statement is based (you do not have to know its proof).

Previously we said that for a 2nd order ODE we need 2 and only 2 "good solutions", this means that the 2 solutions that are linear independent. Since any 2nd ODE can be written as a 2nd order system of first order ODEs this means that we need 2 and only 2 solutions for the systems that we have studied until now. This is the reason behind using 2 components in the solution in the 3 previous cases. But one question that remains unanswered is why we need 2 and only 2 solutions to describe any other solution of a second order system. This is what we will prove here:

Assume that we have 2 solutions $x_1(t)=\begin{bmatrix} x_{1A}(t) \\ x_{1B}(t) \end{bmatrix}$ and $x_2=\begin{bmatrix} x_{2A}(t) \\ x_{2B}(t) \end{bmatrix}$.

Then $x(t)=C_1\begin{bmatrix} x_{1A}(t) \\ x_{1B}(t) \end{bmatrix}+C_2\begin{bmatrix} x_{2A}(t) \\ x_{2B}(t) \end{bmatrix}$ is also a solution. At t=0 we have that $x(0)=\begin{bmatrix} x_{0A} \\ x_{0B} \end{bmatrix}$ which implies that:

$$\begin{bmatrix} x_{0A} \\ x_{0B} \end{bmatrix}=C_1\begin{bmatrix} x_{1A}(0) \\ x_{1B}(0) \end{bmatrix}+C_2\begin{bmatrix} x_{2A}(0) \\ x_{2B}(0) \end{bmatrix}=\begin{bmatrix} x_{1A}(0) & x_{2A}(0) \\ x_{1B}(0) & x_{2B}(0) \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

In order to have a unique solution the above system (with unknowns the 2 constants C_1 and C_2) must have a unique solution and therefore the matrix $\begin{bmatrix} x_{1A}(0) & x_{2A}(0) \\ x_{1B}(0) & x_{2B}(0) \end{bmatrix}$ must be invertible. This means that the solutions x_1 and x_2 must be linear independent.

If we had 3 solutions then the system (2 equations with 3 unknowns) would have infinite solutions and hence this would violate the uniqueness condition. Similarly if we had 1 solution we would have a system (2 equations, 1 unknown) that probably would not have a solution and hence we would violate the existence condition.

PBL4: What is the Wronskian of a system and how does it connect with the above analysis?

3. High order systems

If we have a higher order system then the same analysis would apply:

Let's assume a generic homogeneous system: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$.

Trial $\mathbf{x}(t) = \mathbf{e}e^{\lambda t} \Rightarrow \lambda\,\mathbf{e}e^{\lambda t} = \mathbf{A}\mathbf{e}e^{\lambda t} \Leftrightarrow \lambda\,\mathbf{e} = \mathbf{A}\mathbf{e} \Leftrightarrow (\lambda\mathbf{I} - \mathbf{A})\mathbf{e} = 0$

For this system to have a nontrivial solution: $|\lambda\mathbf{I} - \mathbf{A}| = 0$ (for a 2^{nd} order system this can also be written as $\lambda^2 - \text{trace}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$)

The roots of this equation are called eigenvalues (in German it means characteristic values) of the system because they satisfy: $\lambda\mathbf{e} = \mathbf{A}\mathbf{e}$

The vectors $\mathbf{e}_i$ are called eigenvectors of the system corresponding to the eigenvalues λ_i.

And thus we would have a combination of the previous cases.

Example:

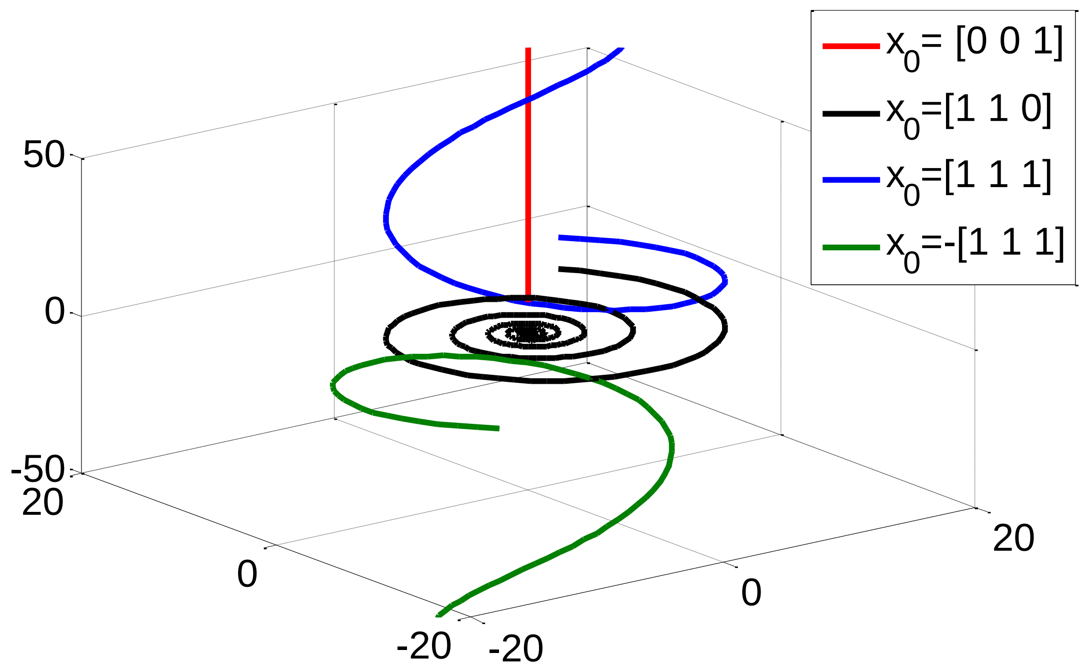
The system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$, with $\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ has 3 eigenvalues $\lambda_c = -1 + j$ and $\lambda_R = 3$ this will give us one complex $\mathbf{e}_c = \begin{bmatrix} 1 \\ j \\ 0 \end{bmatrix}$ and one real $\mathbf{e}_R = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ eigenvector and therefore the general solution is:

$$\mathbf{x} = C_1 \mathbf{e}_R e^{\lambda_R t} + C_2 \operatorname{Re}\left(\mathbf{e}_c e^{\lambda_c t}\right) + C_3 \operatorname{Im}\left(\mathbf{e}_c e^{\lambda_c t}\right)$$

If we are given the ICs:

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow C_1 \mathbf{e}_R + C_2 \operatorname{Re}(\mathbf{e}_c) + C_3 \operatorname{Im}(\mathbf{e}_c) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow$$

$$C_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow C_1 = C_2 = C_3 = 1$$



4. Solution Matrices and Solution of Linear Systems

4.1 2nd order systems

For second order systems $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ we have seen that we have 2 solutions (x_1, x_2) depending on the eigenvalues of **A** (real and distinct, repeated and complex):

$$\mathbf{x}_1 = \mathbf{e}_1 e^{\lambda_1 t}, \mathbf{x}_2 = \mathbf{e}_2 e^{\lambda_2 t} \text{ if } \lambda_1 \neq \lambda_2, \lambda_1, \lambda_2 \in \mathbb{R}.$$

$$\mathbf{x}_1 = \mathbf{e}e^{\lambda t}, x_2 = (\mathbf{e}t + \mathbf{b})e^{\lambda t} \text{ if } \lambda_1 = \lambda_2 = \lambda, \lambda \in \mathbb{R}.$$

$$\mathbf{x}_2 = \mathrm{Re}\left(\mathbf{e}e^{\lambda t}\right), \mathbf{x}_2 = \mathrm{Im}\left(\mathbf{e}e^{\lambda t}\right) \text{ if } \lambda_1 = \overline{\lambda_2} = \lambda, \lambda \in \mathbb{C}.$$

Now any combination $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is also a solution (principle of superposition) and also any other solution can be expressed by the above combination. This effectively means that to describe the behaviour of a 2nd order system we just need $\mathbf{x}_1$ and $\mathbf{x}_2$. When we are given an initial condition $\mathbf{x}_0$ effectively we are asked to find a specific solution that passes (starts) through $\mathbf{x}_0$, and this can be done by finding the appropriate values of c_1, c_2 (this is what we have done before).

Now, $\mathbf{x}_1$ and $\mathbf{x}_2$ are 2 2by1 column vectors. If we put them together in one matrix (this matrix is called "**<u>the fundamental solution matrix</u>**") we have $\mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2]$ which is 2by2. It will be better if we write as: $\mathbf{X}(t) = [\mathbf{x}_1(t) \quad \mathbf{x}_2(t)]$.

Thus $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ can be written as: $\mathbf{x}(t) = \mathbf{X}(t) \times \mathbf{c}$, where $\mathbf{c} = [c_1 \quad c_2]^T$

We are given the value at t=0 as $\mathbf{x}_0$: $\mathbf{x}_0 = \mathbf{X}(0)\times\mathbf{c}$ or $\mathbf{c} = \mathbf{X}^{-1}(0)\mathbf{x}_0$. Hence going back to $\mathbf{x}(t) = \mathbf{X}(t)\times\mathbf{c}$ we have: $\boxed{\mathbf{x}(t) = \mathbf{X}(t)\times\mathbf{X}^{-1}(0)\mathbf{x}_0}$

The product $\mathbf{X}(t)\times\mathbf{X}^{-1}(0)$ is called the State Transition Matrix.

This means that if found or we are given $\mathbf{X}(t)$ we can easily find $\mathbf{X}(0)$ and $\mathbf{X}^{-1}(0)$. Then using $\mathbf{x}(t) = \mathbf{X}(t)\times\mathbf{X}^{-1}(0)\mathbf{x}_0$ we can find any solution given the initial conditions. This is effectively what we previously did but now it is in a more compact form, it can easily be extended to high order systems and above all it can be used in time varying systems (i.e. where the state matrix $\mathbf{A}$ is not constant). Before we see how it can be used for time varying systems let's see how it can be used for the systems that we previously studied:

Example 1:

$\mathbf{x}_1 = \mathbf{e}_1 e^{\lambda_1 t}, \mathbf{x}_2 = \mathbf{e}_2 e^{\lambda_2 t} \Rightarrow \mathbf{X}(t) = \begin{bmatrix}\mathbf{e}_1 e^{\lambda_1 t} & \mathbf{e}_2 e^{\lambda_2 t}\end{bmatrix} \Rightarrow \mathbf{X}(0) = \begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{bmatrix}$

Hence $\mathbf{x} = \begin{bmatrix}\mathbf{e}_1 e^{\lambda_1 t} & \mathbf{e}_2 e^{\lambda_2 t}\end{bmatrix}\times\begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{bmatrix}^{-1}\mathbf{x}_0$

Example 2:

$\mathbf{x}_1 = \mathbf{e}e^{\lambda t}, \mathbf{x}_2 = (\mathbf{e}t+\mathbf{b})e^{\lambda t} \Rightarrow \mathbf{X}(t) = \begin{bmatrix}\mathbf{e}e^{\lambda t} & (\mathbf{e}t+\mathbf{b})e^{\lambda t}\end{bmatrix} \Rightarrow \mathbf{X}(0) = \begin{bmatrix}\mathbf{e} & \mathbf{b}\end{bmatrix}$

Hence $\mathbf{x} = \begin{bmatrix}\mathbf{e}e^{\lambda t} & (\mathbf{e}t+\mathbf{b})e^{\lambda t}\end{bmatrix}\times\begin{bmatrix}\mathbf{e} & \mathbf{b}\end{bmatrix}^{-1}\mathbf{x}_0$

Example 3:

$\mathbf{x}_2 = \mathrm{Re}\left(\mathbf{e}e^{\lambda t}\right), \mathbf{x}_2 = \mathrm{Im}\left(\mathbf{e}e^{\lambda t}\right) \Rightarrow \mathbf{X}(t) = \begin{bmatrix}\mathrm{Re}\left(\mathbf{e}e^{\lambda t}\right) & \mathrm{Im}\left(\mathbf{e}e^{\lambda t}\right)\end{bmatrix} \Rightarrow \mathbf{X}(0) = \begin{bmatrix}\mathrm{Re}(\mathbf{e}) & \mathrm{Im}(\mathbf{e})\end{bmatrix}$

Hence $\mathbf{x} = \begin{bmatrix}\mathrm{Re}\left(\mathbf{e}e^{\lambda t}\right) & \mathrm{Im}\left(\mathbf{e}e^{\lambda t}\right)\end{bmatrix}\times\begin{bmatrix}\mathrm{Re}(\mathbf{e}) & \mathrm{Im}(\mathbf{e})\end{bmatrix}^{-1}\mathbf{x}_0$

Example 4:

We know that for $\mathbf{A}=\begin{bmatrix}-8 & -4\\ 1.5 & -3\end{bmatrix}$ we have $\begin{cases}\lambda_1=-6, \mathbf{e}_1=[-2 \quad 1]^T\\ \lambda_2=-5, \mathbf{e}_2=[1 \quad -3/4]^T\end{cases}$.

Hence the FSM is: $\mathbf{X}(t)=\begin{bmatrix}\mathbf{e}_1 e^{\lambda_1 t} & \mathbf{e}_2 e^{\lambda_2 t}\end{bmatrix}=\begin{bmatrix}-2e^{-6t} & e^{-5t}\\ e^{-6t} & -\frac{3}{4}e^{-5t}\end{bmatrix}$

$$\Rightarrow \mathbf{X}(0)=\begin{bmatrix}-2 & 1\\ 1 & -\frac{3}{4}\end{bmatrix} \Rightarrow \mathbf{X}^{-1}(0)=\begin{bmatrix}-1.5 & -2\\ -2 & -4\end{bmatrix}$$

Thus if $\mathbf{x}_0=\begin{bmatrix}1\\2\end{bmatrix}$:

$$\mathbf{x}(t)=\mathbf{X}(t)\mathbf{X}^{-1}(0)\mathbf{x}_0=\begin{bmatrix}-2e^{-6t} & e^{-5t}\\ e^{-6t} & -\frac{3}{4}e^{-5t}\end{bmatrix}\begin{bmatrix}-1.5 & -2\\ -2 & -4\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=-5.5\begin{bmatrix}-2\\1\end{bmatrix}e^{-6t}-10\begin{bmatrix}1\\-3/4\end{bmatrix}e^{-5t}$$

Unfortunately we cannot follow a similar strategy when A is time varying, for example $\mathbf{A}(t)=\begin{bmatrix}-t & 1\\ -e^{-t} & -e^{-2t}\end{bmatrix}$. In these cases we have to rely on numerical solutions.

Even though we cannot find $\mathbf{x}_1$ and $\mathbf{x}_2$ we know that they exist. Hence we know that the FSM exists as well: $\mathbf{X}(t)=[\mathbf{x}_1(t) \quad \mathbf{x}_2(t)]=\begin{bmatrix}x_{1_A}(t) & x_{2_A}(t)\\ x_{1_B}(t) & x_{2_B}(t)\end{bmatrix}$

and of course at t=0 we have a constant matrix $\mathbf{X}(0)=\begin{bmatrix}x_{1_A}(0) & x_{2_A}(0)\\ x_{1_B}(0) & x_{2_B}(0)\end{bmatrix}$

with the inverse $\mathbf{X}^{-1}(0)=\begin{bmatrix}x_{1_A}(0) & x_{2_A}(0)\\ x_{1_B}(0) & x_{2_B}(0)\end{bmatrix}^{-1}$ also being constant. Let's assume that for our case $\mathbf{X}^{-1}(0)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for some constants a, b, c, d.

Then:

$$\mathbf{X}(t)\times\mathbf{X}^{-1}(0)=\begin{bmatrix} x_{1_A}(t) & x_{2_A}(t) \\ x_{1_B}(t) & x_{2_B}(t) \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} ax_{1_A}(t)+cx_{2_A}(t) & bx_{1_A}(t)+dx_{2_A}(t) \\ ax_{1_B}(t)+cx_{2_B}(t) & bx_{1_B}(t)+dx_{2_B}(t) \end{bmatrix}$$

Or: $[a\mathbf{x}_1(t)+c\mathbf{x}_2(t) \quad b\mathbf{x}_1(t)+d\mathbf{x}_2(t)]$

Now, since $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$ then so must be $\mathbf{x}_3=a\mathbf{x}_1(t)+c\mathbf{x}_2(t)$ and $\mathbf{x}_4=b\mathbf{x}_1(t)+d\mathbf{x}_2(t)$. This means that $\dot{\mathbf{x}}_3=\mathbf{A}\mathbf{x}_3$ and $\dot{\mathbf{x}}_4=\mathbf{A}\mathbf{x}_4$.

Also $\mathbf{X}(0)\times\mathbf{X}^{-1}(0)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and hence $[\mathbf{x}_3(0) \quad \mathbf{x}_4(0)]=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\mathbf{x}_3(0)=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}_4(0)=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. **Be careful** we do not yet know the functions $\mathbf{x}_3(t)$ and $\mathbf{x}_4(t)$ since we do not know $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$.

In order for us to find $\mathbf{x}_3(t)$ and $\mathbf{x}_4(t)$ we simply have to numerically solve $\dot{\mathbf{x}}_3=\mathbf{A}\mathbf{x}_3$ and $\dot{\mathbf{x}}_4=\mathbf{A}\mathbf{x}_4$ for $x_3(0)=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x_4(0)=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Now, in general our solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ also depend on t_0 which may not be zero as in the previous case, hence we should have written $\mathbf{x}_1(t,t_0)$ and $\mathbf{x}_2(t,t_0)$. To avoid confusion and to comply with various other authors we will use $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ for most cases and $\boldsymbol{\varphi}_1(t,t_0)$ and $\boldsymbol{\varphi}_2(t,t_0)$ when we want to say that our solutions also depend on the initial time. Hence the FSM is $\mathbf{\Phi}(t,t_0)$ and not $\mathbf{X}(t)$. Also since $\mathbf{x}(t)=\mathbf{\Phi}(t,t_0)\times\mathbf{\Phi}^{-1}(t_0,t_0)\mathbf{x}_0$, i.e. our solution to the IVP also depend on the initial condition: $\boldsymbol{\varphi}(t,t_0,x_0)=\mathbf{\Phi}(t,t_0)\times\mathbf{\Phi}^{-1}(t_0,t_0)\mathbf{x}_0$.

4.2 High Order Systems

For 1st and 2nd order systems we may be able (if we are lucky) to use some other methods like direct integration, exponential factor… Here we want to concentrate on higher order systems and hence a more "universal" method is needed. Since we have linear systems it is always easier to start with homogeneous systems.

$$\dot{\mathbf{x}}(t)=\mathbf{A}(t)\mathbf{x}(t),\ \text{where }\ \mathbf{x}(t)\in\mathbb{R}^n\ \text{ and }\ \mathbf{A}(t)\in\mathbb{R}^{n\times n} \tag{1}$$

Note that this is just a Differential Equation (DE) and not an Initial Value Problem (IVP).

Then we can find *n* linear independent solutions. For example if we have an LTI system then:

- If we have distinct eigenvalues then we can simple use the eigenvectors/eigenvalues.
- If we have complex eigenvalues then we take the real and imaginary part.
- If we have repeated eigenvalues then we can use generalised eigenvectors like $(\mathbf{v}t+\mathbf{b}), \mathbf{v}$).

These LI solutions have 2 basic properties:

- That any linear combination of these vectors is also a solution
- Any other solution can be written as a linear combination of these solutions (the proof of this is rather convoluted).

So let's say that our (fundamental) solutions are:

$$\{\boldsymbol{\varphi}_1(t,t_0), \boldsymbol{\varphi}_2(t,t_0), \cdots \boldsymbol{\varphi}_n(t,t_0)\} \tag{2}$$

Then from the theory of superposition a function of the form

$$\boldsymbol{\varphi}(t,t_0) = \sum_{i=1}^{n} c_i \boldsymbol{\varphi}_i(t,t_0), \tag{3}$$

where c_i are arbitrary constants, is also a solution.

By taking these n-LI solutions and putting them in one matrix we have the **Fundamental Solution Matrix** (FSM):

$$\boldsymbol{\Phi}(t,t_0) = [\boldsymbol{\varphi}_1(t,t_0) \quad \boldsymbol{\varphi}_2(t,t_0) \quad \cdots \quad \boldsymbol{\varphi}_n(t,t_0)] \tag{4}$$

Note that by definition $\det(\boldsymbol{\Phi}(t,t_0)) \neq 0$.

Hence (3) can be written as:

$$\boldsymbol{\varphi}(t,t_0) = \sum_{i=1}^{n} c_i \boldsymbol{\varphi}_i(t,t_0) = \boldsymbol{\Phi}(t,t_0)[c_1 \quad c_2 \quad \cdots \quad c_n]^T = \boldsymbol{\Phi}(t,t_0)\mathbf{c}$$

If we are given the IVP $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \quad \mathbf{x}_0 = \mathbf{x}(t_0)$ (5)

then we have that:

$$\boldsymbol{\varphi}(t_0, t_0, x_0) = \boldsymbol{\Phi}(t_0, t_0)\mathbf{c} \Leftrightarrow \mathbf{x}_0 = \boldsymbol{\Phi}(t_0, t_0)\mathbf{c} \Leftrightarrow \mathbf{c} = \boldsymbol{\Phi}^{-1}(t_0, t_0)\mathbf{x}_0$$

where $\mathbf{c}$ now depends on the ICs. The last step is allowed since $\det(\boldsymbol{\Phi}(t, t_0)) \neq 0$.

Now, the generic solution can be written as:

$$\boldsymbol{\varphi}(t, t_0, x_0) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{\Phi}^{-1}(t_0, t_0)\mathbf{x}_0 \tag{6}$$

The matrix $\boldsymbol{\Phi}_{STM}(t, t_0) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{\Phi}^{-1}(t_0, t_0)$ (7)

is called **State Transition Matrix** (STM) as it takes the orbit from a given location when $t = t_0$ to another location when $t = t$.

One property of this matrix is that:

$$\boldsymbol{\Phi}_{STM}(t_0, t_0) = \boldsymbol{\Phi}(t_0, t_0)\boldsymbol{\Phi}^{-1}(t_0, t_0) = \mathbf{I}_{n \times n} \tag{8}$$

For that reason the STM is also called the normalised (at $t = t_0$) FSM.

The STM satisfies the following IVP:

$$\frac{d}{dt}\boldsymbol{\Phi}_{STM}(t, t_0) = \mathbf{A}(t)\boldsymbol{\Phi}_{STM}(t, t_0),\ \boldsymbol{\Phi}_{STM}(t_0, t_0) = \mathbf{I}_{n \times n} \tag{9}$$

Unfortunately if (9) describes a Linear Time Varying (LTV) system and (unless we are very lucky) we cannot find an analytic formula for

$\mathbf{\Phi}_{STM}(t,t_0)$. In this case we have to use numerical methods and we have to solve n vector DEs:

$$\frac{d}{dt}\mathbf{\Phi}_{STM1}(t,t_0)=\mathbf{A}(t)\mathbf{\Phi}_{STM1}(t,t_0),\ \mathbf{\Phi}_{STM1}(t_0,t_0)=\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}^T$$
$$\frac{d}{dt}\mathbf{\Phi}_{STM2}(t,t_0)=\mathbf{A}(t)\mathbf{\Phi}_{STM2}(t,t_0),\ \mathbf{\Phi}_{STM2}(t_0,t_0)=\begin{bmatrix}0 & 1 & \cdots & 0\end{bmatrix}^T$$
$$\cdots$$
$$\frac{d}{dt}\mathbf{\Phi}_{STMn}(t,t_0)=\mathbf{A}(t)\mathbf{\Phi}_{STMn}(t,t_0),\ \mathbf{\Phi}_{STMn}(t_0,t_0)=\begin{bmatrix}0 & 0 & \cdots & 1\end{bmatrix}^T$$

4.3 State transition matrix for LTI systems

For a scalar ODE: $\dot{x}=ax$ the solution was $x(t)=e^{at}x(0)$ (no special cases) so can we do the same with $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$, i.e. $\mathbf{x}(t)=e^{\mathbf{A}t}\mathbf{x}(0)$?

If only we knew how to calculate $e^{\mathbf{A}t}$ => No special cases are needed then.

It can be proved that $e^{\mathbf{A}t}=\mathrm{I}+\mathbf{A}t+\frac{1}{2!}(\mathbf{A}t)^2+\frac{1}{3!}(\mathbf{A}t)^3+\ldots$

PBL5: Prove that the matrix $e^{\mathbf{A}t}$ is the state transition matrix of $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$

How $\mathbf{x}(t)=e^{\mathbf{A}t}\mathbf{x}(0)$ is associated with

$\mathbf{x}(t)=C_1\mathbf{e}_1e^{\lambda_1 t}+C_2\mathbf{e}_2e^{\lambda_2 t}+\ldots+C_n\mathbf{e}_ne^{\lambda_n t}$?

Remember: $\mathbf{A}\mathbf{e}_i=\mathbf{e}_i\lambda_i$ or in a matrix notation:

$$\underbrace{\mathbf{A}}_{n\times n}\underbrace{\begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n\end{bmatrix}}_{n\times n}=\underbrace{\begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n\end{bmatrix}}_{n\times n}\underbrace{\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}}_{n\times n}$$

$$\underbrace{\mathbf{A}}_{n\times n}\underbrace{\mathbf{T}}_{n\times n}=\underbrace{\mathbf{T}}_{n\times n}\underbrace{\mathbf{\Lambda}}_{n\times n} \text{ or } \mathbf{A}=\mathbf{T\Lambda}\mathrm{T}^{-1}$$

So:

$$e^{\mathbf{A}t}=\mathrm{I}+\mathbf{A}t+\frac{1}{2!}(\mathbf{A}t)^2+\frac{1}{3!}(\mathbf{A}t)^3+\ldots=\mathrm{I}+\left(\mathbf{T\Lambda}\mathrm{T}^{-1}\right)t+\frac{1}{2!}\left(\left(\mathbf{T\Lambda}\mathrm{T}^{-1}\right)t\right)^2+\frac{1}{3!}\left(\left(\mathbf{T\Lambda}\mathrm{T}^{-1}\right)t\right)^3+\ldots$$

But $\left(\mathbf{T\Lambda}\mathrm{T}^{-1}\right)^2=\left(\mathbf{T\Lambda}\mathrm{T}^{-1}\right)\left(\mathbf{T\Lambda}\mathrm{T}^{-1}\right)=\mathbf{T\Lambda}^2\mathrm{T}^{-1}$

so:

$$e^{\mathbf{A}t}=\mathbf{TI}\mathrm{T}^{-1}+\left(\mathbf{T\Lambda}\mathrm{T}^{-1}\right)t+\frac{1}{2!}\left(\mathbf{T\Lambda}^2\mathrm{T}^{-1}\right)t^2+\frac{1}{3!}\left(\mathbf{T\Lambda}^3\mathrm{T}^{-1}\right)t^3+\ldots=$$

$$=\mathbf{T}\left(\mathrm{I}+\mathbf{\Lambda}t+\frac{1}{2!}(\mathbf{\Lambda}t)^2+\frac{1}{3!}(\mathbf{\Lambda}t)^3+\ldots\right)\mathrm{T}^{-1}=\mathbf{T}e^{\mathbf{\Lambda}t}\mathrm{T}^{-1}$$

And $e^{\mathbf{\Lambda}t}=\begin{bmatrix}e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{bmatrix}$

PBL6: Prove the above formula (for a 2by2 system)

Hence: $\underbrace{\mathbf{x}(t)}_{n\times 1}=\underbrace{e^{\mathbf{A}t}}_{n\times x}\underbrace{\mathbf{x}(0)}_{n\times 1}=\underbrace{\mathbf{T}}_{n\times x}\underbrace{e^{\mathbf{\Lambda}t}}_{n\times x}\underbrace{\mathbf{T}^{-1}}_{n\times x}\underbrace{\mathbf{x}(0)}_{n\times 1}=$

$$\underbrace{\mathbf{x}(t)}_{n\times 1}=\underbrace{[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n]}_{n\times x}\underbrace{\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{bmatrix}}_{n\times x}\underbrace{[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n]^{-1}}_{n\times x}\underbrace{\mathbf{x}(0)}_{n\times 1}$$

$$\underbrace{\mathbf{x}(t)}_{n\times 1}=\underbrace{[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n]}_{n\times x}\underbrace{\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{bmatrix}}_{n\times x}\underbrace{\begin{bmatrix}\mathbf{w}_1\\ \mathbf{w}_2\\ \vdots\\ \mathbf{w}_n\end{bmatrix}}_{n\times x}\underbrace{\mathbf{x}(0)}_{n\times 1}$$

Note: the vectors **e** are nx1 vectors, and the vectors **w** are 1xn vectors.

$$\mathbf{x}(t)=\mathbf{e}_1 e^{\lambda_1 t}\mathbf{w}_1\mathbf{x}(0)+\mathbf{e}_2 e^{\lambda_2 t}\mathbf{w}_2\mathbf{x}(0)+\ldots=\sum_{i=1}^{n}\mathbf{e}_i e^{\lambda_i t}(\mathbf{w}_i\mathbf{x}(0))$$

$$\mathbf{x}(t)=\sum_{i=1}^{n}\mathbf{e}_i e^{\lambda_i t} b_i(0)$$

This is similar to $\mathbf{x}(t)=C_1\mathbf{e}_1 e^{\lambda_1 t}+C_2\mathbf{e}_2 e^{\lambda_2 t}+\ldots+C_n\mathbf{e}_n e^{\lambda_n t}$.

PBL7: Calculate the formula of $e^{\mathbf{\Lambda} t}$ for a 2by2 system when there is only one independent eigenvalue.

PBL8 (Harder): Calculate the formula of $e^{\mathbf{\Lambda} t}$ for a 2by2 system when there 2 complex eigenvalues.

Hint: Use Taylor series only for one element on the matrix and keep up to 3rd order components.

PBL9: Prove that if the initial condition $\mathbf{x}_0$ is on an eigenvector $\mathbf{e}_1$ with a corresponding eigenvalue λ_1 then $e^{\mathbf{A}t}\mathbf{e}_1 = e^{\lambda_1 t}\mathbf{e}_1$, i.e. that we stay forever on that eigenvector.

PBL10: Using the result of the previous exercise and assuming that we have a 2by2 system and $\mathbf{x}_0 = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$, with c_1, c_2 being arbitrary constants and $\mathbf{e}_2$ being the second LI eigenvector prove the classical formula $\mathbf{x} = c_1\mathbf{e}_1 e^{\lambda_1 t} + c_2\mathbf{e}_2 e^{\lambda_2 t}$.

PBL11: Using the same methodology as in the previous PBL exercise prove the formulae $\mathbf{x}(t) = C_1 e^{\mu t}(\mathbf{a}\cos(\nu t) - \mathbf{b}\sin(\nu t)) + C_2 e^{\mu t}(\mathbf{a}\sin(\nu t) + \mathbf{b}\cos(\nu t))$ and $\mathbf{x}(t) = C_1(\mathbf{v}t + \mathbf{b})e^{\lambda t} + C_2\mathbf{v}e^{\lambda t}$

4.4 Nonhomogeneous systems

As before see the scalar first: $\dot{x} = ax + bu \Rightarrow \dot{x} - ax = bu$

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)} bu\, d\tau$$

Similarly for the vector case: $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{Bu}\, d\tau$.