Chapter #4

EEE8013

State Space Analysis and Controller Design

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Module Leader: Dr Damian Giaouris - damian.giaouris@ncl.ac.uk

State space transformations

State space representations are not unique. We can have equivalent forms with the same input/output properties, for example same eigenvalues.

$\dot{\mathbf{x}}(t) = \mathbf{Ax}(t) + \mathbf{Bu}(t)$, $\mathbf{y}(t) = \mathbf{Cx}(t) + \mathbf{Du}(t)$

Assume that $\mathbf{x} = \mathbf{Tz}$. $\mathbf{T}$ is an invertible matrix (not singular) and $\mathbf{z}$ is the new state vector:

So $\mathbf{z} = \mathbf{T}^{-1}\mathbf{x} \Rightarrow \dot{\mathbf{z}} = \mathbf{T}^{-1}\dot{x} \Rightarrow \dot{\mathbf{z}} = \mathbf{T}^{-1}\mathbf{Ax}(t) + \mathbf{T}^{-1}\mathbf{Bu}(t)$

$\dot{\mathbf{z}} = \mathbf{T}^{-1}\mathbf{ATz} + \mathbf{T}^{-1}\mathbf{Bu}(t) \Rightarrow \dot{\mathbf{z}} = \tilde{\mathbf{A}}\mathbf{z} + \tilde{\mathbf{B}}\mathbf{u}(t)$, where $\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{AT}$ is the state matrix and $\tilde{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}$ is the new input matrix.

And $\mathbf{y}(t) = \mathbf{CTz} + \mathbf{Du}(t) \Rightarrow \mathbf{y}(t) = \tilde{\mathbf{C}}\mathbf{z} + \tilde{\mathbf{D}}\mathbf{u}(t)$, where $\tilde{\mathbf{C}} = \mathbf{CT}$ is the new output matrix and $\tilde{\mathbf{D}} = \mathbf{D}$ is the new input/output coupling matrix.

Do these two systems have the TF?

$\mathbf{G}_1(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ and $\mathbf{G}_2(s) = \tilde{\mathbf{C}}(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}} + \tilde{\mathbf{D}}$

$$\mathbf{G}_1(s) = \mathbf{C}\left(\mathbf{TT}^{-1}\right)(s\mathbf{I} - \mathbf{A})^{-1}\left(\mathbf{TT}^{-1}\right)\mathbf{B} + \tilde{\mathbf{D}} \Leftrightarrow$$
$$\mathbf{G}_1(s) = \mathbf{CT}\left(\mathbf{T}(s\mathbf{I} - \mathbf{A})\mathbf{T}^{-1}\right)^{-1}\mathbf{T}^{-1}\mathbf{B} + \tilde{\mathbf{D}} \Leftrightarrow$$
$$\mathbf{G}_1(s) = \tilde{\mathbf{C}}\left(\left(s\mathbf{I} - \mathbf{TAT}^{-1}\right)\right)^{-1}\tilde{\mathbf{B}} + \tilde{\mathbf{D}} \Leftrightarrow$$
$$\mathbf{G}_1(s) = \tilde{\mathbf{C}}\left(s\mathbf{I} - \tilde{\mathbf{A}}\right)^{-1}\tilde{\mathbf{B}} + \tilde{\mathbf{D}} = \mathbf{G}_2(s)$$

Normal or Canonical Form

We have seen that a similarity transformation transforms a state space model into another "equivalent" form using the formula: $\mathbf{x} = \mathbf{T}\mathbf{z}$, where $\mathbf{x}$, $\mathbf{z}$ are $n \times 1$ vectors and $\mathbf{T}$ is an invertible $n \times n$ matrix. This means that the new system will have the same eigenvalues but different eigenvectors. The only requirement is that $\mathbf{T}$ is invertible. But what will happen if $\mathbf{T}$ has a specific form? For simplicity we will only see homogenous systems:

$$\mathbf{x} = \mathbf{T}\mathbf{z} \Leftrightarrow \dot{\mathbf{x}} = \mathbf{T}\dot{\mathbf{z}} \Leftrightarrow \mathbf{A}\mathbf{x} = \mathbf{T}\dot{\mathbf{z}} \Leftrightarrow \mathbf{A}\mathbf{T}\mathbf{z} = \mathbf{T}\dot{\mathbf{z}} \Leftrightarrow \dot{\mathbf{z}} = \underbrace{\mathbf{T}^{-1}\mathbf{A}\mathbf{T}}_{\tilde{\mathbf{A}}}\mathbf{z}$$

Hence the new "equivalent" is $\dot{\mathbf{z}} = \tilde{\mathbf{A}}\mathbf{z}$ with $\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$.

We have seen $e^{\mathbf{A}t} = \mathbf{T}e^{\tilde{\mathbf{A}}t}\mathbf{T}^{-1}$

So it is natural to ask what will be the form of the new state transition matrices, how they can be used to find the STM of the original system and the form of the new eigenvectors, as before we have 3 cases, for 2 dimensional systems:

Case 1: Real and distinct eigenvalues λ_1, λ_2

In that case we choose $\mathbf{T} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix}$ ($\mathbf{e}_1, \mathbf{e}_2$ are the 2 LI eigenvectors) we have the new state matrix $\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

PBL: Prove that formula for the new state matrix

Obviously if we have an n[th] order system with only real and distinct eigenvalues then the new state matrix is:

$$\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}, \text{ with } \mathbf{T} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix}$$

The eigenvectors of the new 2x2 system are found to be:

$$\lambda = \lambda_1 \Rightarrow \left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} - \lambda_1 \mathbf{I} \right) \mathbf{e} = 0 \Leftrightarrow \begin{bmatrix} 0 & 0 \\ 0 & \lambda_2 - \lambda_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{cases} x = anything = 1 \\ y = 0 \end{cases}$$

$$\lambda = \lambda_2 \Rightarrow \left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} - \lambda_2 \mathbf{I} \right) \mathbf{e} = 0 \Leftrightarrow \begin{bmatrix} \lambda_1 - \lambda_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{cases} x = 0 \\ y = anything = 1 \end{cases}$$

Thus we see that the new eigenvectors are $[1 \quad 0]^T, [0 \quad 1]^T$.

We know that new state transition matrix is $e^{\tilde{\mathbf{A}}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$. Thus the state transition matrix of the original system is: $e^{\mathbf{A}t} = \mathbf{T} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \mathbf{T}^{-1}$

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Example: The state matrix of a system is $\mathbf{A}=\begin{bmatrix}11 & -10\\ 10 & -14\end{bmatrix}$. Find the state transition matrix.

Method 1: The eigenvalues and the eigenvectors are $\lambda_1=-9, \lambda_2=6$ and $\mathbf{e}_1=[1 \quad 2]^T$ and $\mathbf{e}_2=[2 \quad 1]^T$. Thus the FSM is:

$$\mathbf{X}(t)=\begin{bmatrix} e^{-9t}\begin{bmatrix}1\\2\end{bmatrix} & e^{6t}\begin{bmatrix}2\\1\end{bmatrix}\end{bmatrix}=\begin{bmatrix} e^{-9t} & 2e^{6t}\\ 2e^{-9t} & e^{6t}\end{bmatrix}$$

$$\mathbf{X}(0)=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\Rightarrow \mathbf{X}^{-1}(0)=\begin{bmatrix}-1/3 & 2/3\\ 2/3 & -1/3\end{bmatrix}$$

Hence the STM is:

$$\begin{bmatrix} e^{-9t} & 2e^{6t}\\ 2e^{-9t} & e^{6t}\end{bmatrix}\begin{bmatrix}-1/3 & 2/3\\ 2/3 & -1/3\end{bmatrix}=\begin{bmatrix} -\frac{1}{3}e^{-9t}+\frac{4}{3}e^{6t} & \frac{2}{3}e^{-9t}-\frac{2}{3}e^{6t}\\ -\frac{2}{3}e^{-9t}+\frac{2}{3}e^{6t} & \frac{4}{3}e^{-9t}-\frac{1}{3}e^{6t}\end{bmatrix}$$

Method 2:

The eigenmatrix is $\mathbf{T}=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$ and hence $\mathbf{T}^{-1}=\begin{bmatrix}-1/3 & 2/3\\ 2/3 & -1/3\end{bmatrix}$, and the new state matrix: $\tilde{\mathbf{A}}=\begin{bmatrix}-9 & 0\\ 0 & 6\end{bmatrix}$. Now the state transition matrix of the new system (called canonical system) is:

$e^{\tilde{\mathbf{A}}t} = \begin{bmatrix} e^{-9t} & 0 \\ 0 & e^{6t} \end{bmatrix}$ and hence the state transition matrix of the original system is:

$$e^{\mathbf{A}t} = \mathbf{T}e^{\tilde{\mathbf{A}}t}\mathbf{T}^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} e^{-9t} & 0 \\ 0 & e^{6t} \end{bmatrix}\begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \Rightarrow$$

$$e^{\mathbf{A}t} = \begin{bmatrix} -\frac{1}{3}e^{-9t} + \frac{4}{3}e^{6t} & \frac{2}{3}e^{-9t} - \frac{2}{3}e^{6t} \\ -\frac{2}{3}e^{-9t} + \frac{2}{3}e^{6t} & \frac{4}{3}e^{-9t} - \frac{1}{3}e^{6t} \end{bmatrix}$$

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Case 2: Complex Eigenvalues

In that case we have 2 complex eigenvectors and we choose $\mathbf{T} = [\mathrm{Re}(\mathbf{e}) \quad \mathrm{Im}(\mathbf{e})]$ and $\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} \mathrm{Re}(\lambda) & \mathrm{Im}(\lambda) \\ -\mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{bmatrix}$.

PBL: Prove that formula for the new state matrix

Obviously if we have an nth order system with only complex eigenvalues the new state matrix is:

$$\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} C(a_1, b_1) & 0 & 0 & 0 \\ 0 & C(a_2, b_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & C(a_p, b_p) \end{bmatrix},$$

with $C(a_i, b_i) = \begin{bmatrix} a_i & b_i \\ -b_i & a_i \end{bmatrix}$ and $\mathbf{T} = [\mathrm{Re}(\mathbf{e}_1) \quad \mathrm{Im}(\mathbf{e}_1) \quad \cdots \quad \mathrm{Re}(\mathbf{e}_p) \quad \mathrm{Im}(\mathbf{e}_p)]$, $p=n/2$.

Again the new eigenvectors of the 2x2 system are given by: $\left(\begin{bmatrix} \mathrm{Re}(\lambda) & \mathrm{Im}(\lambda) \\ -\mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{bmatrix} - \lambda \mathbf{I} \right) \mathbf{e}$

This can be written as: $\begin{bmatrix} \mathrm{Re}(\lambda) - \lambda & \mathrm{Im}(\lambda) \\ -\mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) - \lambda \end{bmatrix} \mathbf{e} = 0$

Now if we assume $\lambda = a + bj$ we have:

$$\begin{bmatrix} a-(a+bj) & b \\ -b & a-(a+bj) \end{bmatrix} e = 0 \Leftrightarrow \begin{bmatrix} -bj & b \\ -b & -bj \end{bmatrix} e = 0$$

Since b is not zero:

$$\begin{bmatrix} -j & 1 \\ -1 & -j \end{bmatrix} e = 0 \Leftrightarrow \begin{bmatrix} -j & 1 \\ -1 & -j \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{cases} -jx + y = 0 \\ -x - jy = 0 \end{cases} \Rightarrow$$

$$\begin{cases} -jx + y = 0 \\ -jx - j^2 y = 0 \end{cases} \Rightarrow \begin{cases} -jx + y = 0 \\ -jx + y = 0 \end{cases}$$

Thus we have $-jx + y = 0 \Leftrightarrow y = jx$

If we assume that x=1 then y=j: $\mathbf{e}_1 = [1 \quad j]^T$ and obviously $\mathbf{e}_2 = \overline{\mathbf{e}_1} = [1 \quad -j]^T$.

In that case it can be proved that: $e^{\tilde{\mathbf{A}}t} = e^{\mathrm{Re}(\lambda)t} \begin{bmatrix} \cos(\mathrm{Im}(\lambda)t) & \sin(\mathrm{Im}(\lambda)t) \\ -\sin(\mathrm{Im}(\lambda)t) & \cos(\mathrm{Im}(\lambda)t) \end{bmatrix}$.

The state transition matrix is of the original system is

$$e^{\mathbf{A}t} = \mathbf{T}\left(e^{\mathrm{Re}(\lambda)t} \begin{bmatrix} \cos(\mathrm{Im}(\lambda)t) & \sin(\mathrm{Im}(\lambda)t) \\ -\sin(\mathrm{Im}(\lambda)t) & \cos(\mathrm{Im}(\lambda)t) \end{bmatrix} \right) \mathbf{T}^{-1}$$

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Case 3: Repeated Eigenvalues

In that case we have only one LI eigenvector and to solve the system we have seen that we can find another LI (generalised) eigenvector by solving $\mathbf{e} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{b}$. Hence $\mathbf{T} = [\mathbf{e} \quad \mathbf{b}]$ and we have as before: $\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ but now the result is $\tilde{\mathbf{A}} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

PBL: Prove that formula for the new state matrix

If we have an nth order system with one eigenvalue:

$$\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

The new eigenvectors of the 2x2 system are: $\left(\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} - \lambda I\right)\mathbf{e} = 0$. Which can be written as: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{e} = 0 \Leftrightarrow y = 0, x = anything = 1$ and hence the only eigenvector is $\mathbf{e} = [1 \quad 0]$.

The corresponding exponential matrix is: $e^{\tilde{\mathbf{A}}t} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$. Thus

$$e^{\mathbf{A}t} = \mathbf{T}\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}\mathbf{T}^{-1}$$

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